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Circular Geodesics Stability in a Static Black Hole in New Massive Gravity

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Abstract: We study the existence and stability of circular geodesics in a family of asymptotically AdS static black holes in New Massive Gravity theory. We show that the mathematical sign of the hair parameter determines the existence of such geodesics. For a positive hair parameter, the stability regions follow the usual pattern, with the innermost geodesic being null, unstable, and separated from the horizon, followed by a region of unstable timelike geodesics and then a region of stable timelike geodesics, which extends in the asymptotic region.

Keywords: new massive gravity; black holes; circular geodesics

1. Introduction

The New Massive Gravity (NMG) theory, proposed in 2009 by Bergshoeff, Hohm, and Townsend, describes gravity in a vacuum (2 + 1) spacetime with a massive graviton [1]. It is an interacting and generally covariant extension of the Pauli–Fierz theory for massive spin 2 particles in dimension 3. NMG is a quadratic gravity theory developed as a candidate to produce a fully consistent quantum gravity theory by including higher-derivative terms to the framework. In 1950, Pais made one of the first suggestions to include quadratic terms as a generalization of Einstein field equations, to eliminate the divergent features of General Relativity (GR) [2]. The Einstein–Hilbert theory of gravity contains non-renormalizable ultraviolet divergences in four dimensions [3,4]. A non-renormalizable theory means that perturbatively is not well-defined at short distances and high energies. To address this problem, a new UV-complete theory must replace GR at these scales, being quadratic gravity theories promising candidates. Although it was not part of the main motivation, a positive feature of these theories is that there are black hole solutions that contain both a black hole horizon and a cosmological horizon, with a static region between them for a certain range of the positive cosmological constant [5]. Furthermore, generically the gravitational field in non-Schwarzschild black holes decays similarly to the Schwarzschild black hole in GR [6,7]. Recently, de Rham, Gabadadze, and Tolley (dRGT) proposed a nonlinear gravity theory [8,9], which seems to be the most promising massive gravity theory to date. dRGT is a ghost-free theory [10,11], and there are indications that this is the only existing ghost-free massive gravity theory in dimension 4 [12]. Notwithstanding the similarities in the construction of the action, NMG is not a particular case of dRGT, as NMG is not a bimetric theory of gravity.

NMG has received a lot of attention due to its remarkable properties, particularly in the context of the AdS/CFT correspondence conjecture and because a variety of exact solutions have been found (see [13–17]). In this paper, we focus on the asymptotically AdS rotating black hole solution found in [17]. This solution has a hair parameter and the rotational parameter satisfies $|a| < l$; where, the parameter l is related with the cosmological constant as $\Lambda = -l^{-2}$. The extreme rotating case of this NMG black hole can be included after making a change in the hair parameter as suggested in [18]. The extreme case is obtained when $|a| = l$. We study the existence and stability of circular geodesics in the static case of this spacetime. Geodesics in solutions to NMG have been studied profusely, as they provide important information on the properties of the spacetimes. For instance, such studies have been carried out for the BTZ black hole [19], for Lifschitz black holes [20], and from the perspective of deviation angles of null geodesics in the same spacetime as this paper focuses on [21]. We are also interested in seeking relationships between circular geodesics and isoperimetric surfaces studied in [22], although the obtained results are discouraging. We follow [23] for analyzing the stability, obtaining the corresponding principal Lyapunov exponent for the geodesics.

The paper is organized as follows. In Section 2, the family of rotating black holes in NMG is presented. Then, conditions for the existence and stability of circular geodesics are discussed in Section 3. We restrict ourselves to the static case in Section 4, finding the timelike and null circular geodesics and determining their stability. There, we also briefly consider the non-existent relationship with isoperimetric surfaces. Finally, the conclusions are presented in Section 5.

2. New Massive Gravity and Black Hole Solutions

The action for NMG is

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R - 2\lambda - \frac{K}{m^2}\right), \tag{1}$$

where

$$K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2, \tag{2}$$

and G is the gravitational constant in a (2 + 1)-spacetime, whereas m and λ are parameters related with the cosmological constant [1,17]. In the limit $m^2 \to \infty$ or if the scalar K is equal to zero, the action of GR is obtained. As shown in [17], the field equations in this theory are fourth order,

$$G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} = 0, \tag{3}$$

where

$$K_{\mu\nu} = 2\nabla_\rho\nabla^\rho R_{\mu\nu} - \tfrac{1}{2}\left(\nabla_\mu\nabla_\nu R + g_{\mu\nu}\nabla_\rho\nabla^\rho R\right) - 8R_{\mu\rho}R^\rho{}_\nu + \tfrac{9}{2}RR_{\mu\nu} + g_{\mu\nu}\left(3R^{\rho\lambda}R_{\rho\lambda} - \tfrac{13}{8}R^2\right), \tag{4}$$

and $K = g^{\mu\nu}K_{\mu\nu}$.

A plethora of solutions for these field equations have been found, some of them cited in Section 1. We focus on the asymptotically AdS stationary black hole family obtained in [17], given in the form presented in [18] including the case of extreme rotation. The metric is

$$ds^2 = -NFdt^2 + \frac{dr^2}{F} + r^2(d\phi + N^\phi dt)^2 \tag{5}$$

with

$$N = \left[1 + \frac{bl^2}{4\sigma}(1-\xi)\right]^2, \qquad N^\phi = -\frac{a}{2r^2}(\mu - b\sigma), \tag{6}$$

$$F = \frac{\sigma^2}{r^2}\left[\frac{\sigma^2}{l^2} + \frac{b}{2}(1+\xi)\sigma + \frac{b^2l^2}{16}(1-\xi)^2 - \mu\xi\right], \tag{7}$$

$$\sigma = \left[r^2 - \frac{\mu}{2} l^2 (1-\xi) - \frac{b^2 l^4}{16} (1-\xi)^2 \right]^{1/2}, \qquad \xi^2 = 1 - \frac{a^2}{l^2}. \tag{8}$$

Here, $\mu = 4GM$, M being the mass measured with respect to the zero mass black hole. The angular momentum is given by $J = Ma$ and b is the hair parameter. The rotational parameter a satisfies $-l \leq a \leq l$ and the extreme case is obtained when $|a| = l$. The parameter l is the AdS radius related to the cosmological constant in the usual way, i.e., $\Lambda = -l^{-2}$. The mass parameter μ is bounded from below, depending on the sign of the hair parameter.

$$\text{if } b \leq 0 \Rightarrow \mu \geq -\frac{b^2 l^2}{4}, \qquad \text{if } b > 0 \Rightarrow \mu \geq -\frac{b^2 l^2}{8}(1-\xi). \tag{9}$$

These solutions possess one or more event horizons, an ergosphere, and in general there is a curvature singularity always hidden by the event horizon. Also, for $b \leq 0$, the extreme limit $|a| = l$ corresponds to a cylindrical end, in everything similar to what happens for extreme Kerr. For details of this analysis, please refer to [24] and references therein.

3. Circular Geodesics

If we denote by a dot the derivative with respect to proper time (or the affine parameter for null geodesics) and by u^μ the tangent vector, the geodesics satisfy

$$-\kappa = g_{\mu\nu} u^\mu u^\nu = -NF\dot{t}^2 + \frac{\dot{r}^2}{F} + r^2(\dot{\phi} + N^\phi \dot{t})^2 \tag{10}$$

with

$$\kappa = \begin{cases} 1, & \text{for timelike geodesics,} \\ 0, & \text{for null geodesics.} \end{cases} \tag{11}$$

The metric possesses the Killing vectors ∂_t and ∂_ϕ, so the corresponding constants of motion for the geodesics are

$$E = -g_{\mu\nu} \partial_t^\mu u^\nu = NF\dot{t} - r^2 N^\phi (\dot{\phi} + N^\phi \dot{t}), \qquad L = g_{\mu\nu} \partial_\phi^\mu u^\nu = r^2 (\dot{\phi} + N^\phi \dot{t}). \tag{12}$$

Using these expressions in (10) and rearranging terms, we obtain the one dimensional radial equation of motion:

$$\dot{r}^2 = V_r, \tag{13}$$

where, we define

$$V_r = \frac{1}{N}(E + N^\phi L)^2 - F\left(\frac{L^2}{r^2} + \kappa\right). \tag{14}$$

The derivative of V_r with respect to r, denoted by a prime, is

$$V_r' = \frac{E + N^\phi L}{N^2}\left[2LNN^{\phi'} - (E + N^\phi L)N'\right] + \frac{2L^2 F}{r^3} - F'\left(\kappa + \frac{L^2}{r^2}\right). \tag{15}$$

If we restrict our attention to circular geodesics, these must to satisfy the conditions

$$V_r = 0, \qquad V_r' = 0. \tag{16}$$

To analyze the stability of circular geodesics, we follow the method based on Lyapunov exponents presented in [23]. We refer the reader to that work for additional details. In the case in question, the principal Lyapunov exponent is given by

$$\lambda = \sqrt{\frac{V_r''}{2\dot{t}^2}}, \tag{17}$$

and the unstable orbits are those that have $V_r'' > 0$. We can associate an instability timescale (or Lyapunov timescale) to the unstable geodesics by $T_\lambda = \frac{1}{\lambda}$, which is a measure of how fast instability would be noticed. This can be compared with the orbital timescale, $T_\Omega = \frac{2\pi}{\Omega}$, being Ω the angular velocity of the geodesic, ($\Omega = \frac{\dot{\phi}}{\dot{t}}$), to calculate the corresponding critical exponent $\gamma = \frac{T_\lambda}{T_\Omega}$.

4. Static Black Hole

If we consider the static case, $a = 0$, then

$$\xi = 1, \qquad \sigma = r, \qquad N = 1, \qquad N^\phi = 0, \qquad F = \frac{r^2}{l^2} + br - \mu, \tag{18}$$

and the metric takes the simple form

$$ds^2 = -Fdt^2 + \frac{dr^2}{F} + r^2 d\phi^2. \tag{19}$$

The horizon is located at

$$r_+ = \frac{l}{2}\left(-lb + \sqrt{l^2b^2 + 4\mu}\right), \tag{20}$$

and the mass parameter μ has the following ranges according to the sign of b:

$$b \le 0 \Rightarrow \mu \ge -\frac{l^2b^2}{4}, \qquad b > 0 \Rightarrow \mu \ge 0. \tag{21}$$

Considering the geodesics, the constants of motion are:

$$E = F\dot{t}, \qquad L = r^2\dot{\phi}. \tag{22}$$

In the null case, the equations of motion can be integrated analytically, which has been done in [21].

4.1. Timelike Circular Geodesics

Now $\kappa = 1$, the potential is

$$V_r = E^2 - \left(\frac{r^2}{l^2} + br - \mu\right)\left(\frac{L^2}{r^2} + 1\right), \tag{23}$$

and its first two derivatives are

$$V_r' = -\frac{2}{l^2}r - b + \frac{L^2}{r^3}(br - 2\mu), \qquad V_r'' = -\frac{2}{l^2} - \frac{2L^2}{r^4}(br - 3\mu). \tag{24}$$

From the conditions $V_r = 0$, $V_r' = 0$, we have

$$E^2 = \frac{2F^2}{br - 2\mu}, \qquad L^2 = \frac{r^3(2r + l^2b)}{l^2(br - 2\mu)}. \tag{25}$$

Given that E must be real, we need that

$$br - 2\mu > 0. \tag{26}$$

It is necessary to separate the analysis according to the sign of b. If $b = 0$, then there are no circular geodesics. If $b > 0$, the condition (26) gives

$$r > r_E = \frac{2\mu}{b}. \tag{27}$$

It can be checked that $r_E \geq r_+$. It is also necessary that $L^2 > 0$, which means

$$2r + l^2 b > 0, \tag{28}$$

that is

$$r > r_L = -\frac{l^2 b}{2}, \tag{29}$$

which is always satisfied as $r_L < 0$. If $b < 0$ then condition (29) remains the same, but condition (27) is replaced by

$$r < r_E = \frac{2\mu}{b}, \tag{30}$$

and it can be checked that

$$r_E < r_L, \tag{31}$$

which implies that there are no circular geodesics. To summarize, if $b \leq 0$, then there are no circular geodesics, if $b > 0$ there are circular geodesics for $r > r_E$ and there are no circular geodesics for $r_E > r > r_+$.

Having found the circular geodesics, we now turn our attention to the issue of stability. From (24) and (25) the stability condition $V_r'' < 0$ takes the form

$$3br^2 + (l^2 b^2 - 8\mu)r - 3l^2 b > 0. \tag{32}$$

The roots of this quadratic equation are

$$r_{s\pm} = \frac{1}{6b}\left(-l^2 b^2 + 8\mu \pm \sqrt{l^4 b^4 + 20 l^2 b^2 \mu + 64\mu^2}\right), \tag{33}$$

and considering that $b > 0$ and $\mu \geq 0$, we have

$$r_{s-} \leq 0, \qquad r_{s+} \geq r_E. \tag{34}$$

This means that for $b > 0$ there are three regions outside the horizon. For $r_+ < r < r_E$ there are no circular geodesics, for $r_E < r < r_{s+}$ there are circular geodesics but they are unstable, and for $r_{s+} < r$ there are circular geodesics and they are stable.

The angular velocity for a given geodesic is

$$\Omega = \frac{\dot{\phi}}{\dot{t}} = \frac{LF}{Er^2} = \pm\sqrt{\frac{1}{l^2} + \frac{b}{2r}}. \tag{35}$$

It is interesting to note that the angular velocity is a decreasing function of the distance to the horizon, as generally expected, but that it has a non-zero asymptotic value, which is clearly the inverse of the AdS radius l.

4.2. Null Circular Geodesics

Considering now the null case, $\kappa = 0$, the positions of the null geodesics can be found by the condition $V_r' = 0$, which is equivalent to

$$2F - rF' = 0, \tag{36}$$

and there is at most one null geodesic located at

$$r_n = \frac{2\mu}{b}. \tag{37}$$

If $b = 0$, then there is no circular geodesic, if $b < 0$ then $r_n \leq r_+$ and there is also no circular geodesic. For $b > 0$, the relation between E and L is obtained from $V_r = 0$,

$$\frac{E}{L} = \pm\frac{\sqrt{F}}{r} = \pm\sqrt{\frac{1}{l^2} + \frac{b^2}{4\mu}} = \Omega_n. \tag{38}$$

The second derivative of the potential is

$$V_r'' = \frac{L^2 b^4}{8\mu^3} > 0 \tag{39}$$

and therefore the null geodesic is unstable. Note that the null geodesic is the limiting case of the timelike geodesics, being the innermost circular geodesic. The other quantities of interest, regarding the instability of the geodesic, can be easily calculated,

$$\lambda_n = \Omega_n\sqrt{\mu}, \qquad \gamma_n = \frac{1}{2\pi\sqrt{\mu}}. \tag{40}$$

4.3. Isoperimetric Surfaces

In this brief section, we want to explore if it is possible to relate the ranges and behavior of circular geodesics with the isoperimetric structure of the spacetime studied in [22].

Regarding the isoperimetric structure, all circles of constant radius in a constant-t slice of the spacetime are isoperimetric surfaces. For $b \leq 0$, all of them are also stable. For $b > 0$, the isoperimetric surfaces are stable in the range $r_+ < r < r_c$ with

$$r_c = \frac{2}{b}(1 + \mu), \tag{41}$$

and are unstable for $r > r_c$.

It seems that there is no direct relationship between the ranges of existence and the stability of circular geodesics and the properties of isoperimetric surfaces.

5. Conclusions

We have analyzed the existence and stability of timelike and null circular geodesics in a family of asymptotically AdS static black hole solutions of NMG. We conclude that the sign of the hair parameter determines the existence of such geodesics. For $b \leq 0$, there are no circular geodesics. For $b > 0$, there is a region close to the horizon without circular geodesics followed by an unstable null geodesic, a region of unstable timelike geodesics, and a region of stable timelike geodesics. The $b = 0$ case is the well-known BTZ black hole, for which the geodesics can be obtained in analytic form [19]. Qualitatively, the case $b < 0$ is closely related with the BTZ black hole and differs from the $b > 0$ case. This was also observed regarding the structure of the horizon with respect to the extreme rotating case in [24] and regarding the isoperimetric structure of the spacetime in [22].

An interesting phenomenon, in the $b > 0$ case, was analyzed in [21]: namely, that due to the hair parameter, a black hole repulsive behavior was observed in null geodesics. This also appears in our analysis, although in a subtle form. For $b \leq 0$, there are no circular geodesics, which can be interpreted as that gravitational attraction is "too attractive". When the hair parameter is positive, it has a repulsive effect that counteracts the attraction to the black hole and allows for the existence of circular geodesics. A quantitative measure of this effect can be seen from the position of the null circular geodesic (37), which is closer to the horizon as b increases.

Note that from (35), the velocity profile can be obtained for test particles that orbit around the black hole. By inverting the argument, if the velocity profile is known then the parameters l and b of the spacetime can be obtained. Also, the innermost circular orbit allows us to obtain the value of μ, and therefore, from the velocity profile all the spacetime parameters of the black hole can be determined. Another way of obtaining the value of μ is through the critical exponent (40), which surprisingly depends only on μ.

Finally, we tried to relate geodesics to isoperimetric surfaces, without success. At this point, it is not clear whether or not there is a relationship between them.

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