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Review

Mathematical Treatment of Saturated Macroscopic Flow in Heterogeneous Porous Medium: Evaluating Darcy's Law

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Abstract: We present a rigorous mathematical treatment of water flow in saturated heterogeneous porous media based on the classical Navier-Stokes formulation that includes vorticity in a heterogeneous porous media. We used the mathematical approach proposed in 1855 by James Clark Maxwell. We show that flow in heterogeneous media results in a flow field described by a heterogeneous complex lamellar vector field with rotational flows, compared to the homogeneous lamellar flow field that results from Darcy's law. This analysis shows that Darcy's Law does not accurately describe flow in a heterogeneous porous medium and we encourage precise laboratory experiments to determine under what conditions these issues are important. We publish this work to encourage others to perform numerical and laboratory experiments to determine the circumstances in which this derivation is applicable, and in which the complications can be disregarded.

Keywords: porous media; heterogeneous; analytical solution; mathematical treatment; lamellar vector field; rotational flow

1. Introduction

We present a rigorous mathematical treatment of water flow in saturated heterogeneous porous media that shows that flow in heterogeneous media involves a complex flow field with rotational flows, a condition not addressed by Darcy's law. Our work parallels work presented by Gupta, et al. [1], though we used a different mathematical approach and assumptions. We assumed saturated flow, while Gupta, Sposito and Bhattacharya [1] assumed unsaturated flow and used the Buckingham–Darcy flux law with the equation of continuity. We started from a more basic position and used the math proposed by 24 year-old James Clark Maxwell and sent to William Thompson dated 13 September 1855, which included the theoretical basis to describe "*The Fluid Flow in a Resistive Media*". Maxwell's work is reported in a paper of Narasimhan [2], who located this earlier work. Henry Darcy published his experimental studies on the macroscopic flow through a homogeneous saturated porous medium and derived his empirical law about one year after Maxwell's work in 1856 [3].

The earlier work by Gupta, Sposito and Bhattacharya [1] has led to applications in stochastic modeling [4–7]. More recent work includes [8] who uses an entropy approach for mathematical analysis. We hope that our mathematical treatment inspires experiments, following in Darcy's footprints, to validate our work and to better determine practical impacts and in what flow regimes these issues are important.

Analytical studies of the mathematics of groundwater flow provide a characterization that is difficult to obtain from numerical studies [9–16]. The resulting equations clearly show relationships among the various parameters and help refine our understanding of behaviors [16,17].

1.1. Darcy' Law for Flow in a Homogeneous Saturated Porous Medium

Darcy's law was empirically derived from macroscopic flow through a homogeneous saturated porous medium. Which infers that Darcy's law and Darcy's corroborating experiments are only applicable to a uniform media, or more precisely, only describe macroscopic flow through a homogenous saturated porous media. We will start with a formal mathematical statement for flow in such a medium and use this as a basis for comparison with later equations involving saturated flow in heterogeneous porous media. As part of this treatment, we will clearly define physically and mathematically a heterogeneous porous medium.

Consider the movement of a single homogeneous fluid through a homogeneous porous media. It is appropriate to describe the macroscopic impelling force vector field as the gradient of the scalar energy potential defined on a unit fluid volume basis [18] as:

$$\phi = p + \rho g z \tag{1}$$

and for a homogeneous porous material, the saturated hydraulic conductivity K is constant, i.e.,

$$K = constant \tag{2}$$

which gives as the macroscopic dynamic flux discharge vector, d.

$$\overline{d} = -K\nabla\phi \tag{3}$$

where $\overline{d}$ is the Darcian macroscopic volumetric flux discharge vector (L^3/TL^2), ϕ is the scalar energy potential per unit volume of fluid and is defined here as always positive in space and time (FL/L^3), p is the fluid pressure, (F/L^2), ρ is the mass density of the homogeneous fluid (FT^2/L^4), g is the gravitational scalar (L/T^2), and z is the vertical Cartesian coordinate axis directed outward from the center of the earth (L), and ∇ is the vector mathematical gradient operator (1/L).

It is useful to further classify Equation (3) in terms of the accepted classical field theory types: Equation (3) is a classical irrotational lamellar vector field from continuum mechanics [19,20].

1.2. Heterogeneous Porous Material Defined

If a property of a porous media shows a different function dependence at different spatial points in the media, it is defined as "Heterogeneous" [1], where a heterogeneous porous medium has unlike quantities or differing characteristics at different spatial locations. If a porous medium is heterogeneous with respect to some property, then that property is functionally dependent upon spatial location. If we consider the hydraulic conductivity, K, as heterogeneous, then:

$$K = F_1(x, y, z) = kf_1(x, y, z)/\mu \tag{4}$$

where K is the scalar saturated hydraulic conductivity (L^4/FT), F_1 is a function of the variables in parenthesis, which is at least once differentiable (L^4/FT), and x, y, and z are the Cartesian spatial coordinates with z oriented vertically away from the earth's center (L). For later convenience, we included the second equality here, where k is the scalar intrinsic permeability of the saturated heterogeneous media (L^2), μ is the dynamic viscosity of the homogeneous fluid (FT/L^2), and f_1 is a function of the variables in parenthesis, which, again, is at least once differentiable (L^2). We can also state that $\overline{d}$, is continuous and differentiable everywhere in the porous media. This statement is both the necessary and sufficient condition on $\overline{d}$.

The macroscopic flux discharge, $\overline{d}$, and the fluid pore velocity vector, $\overline{u}$ in the heterogeneous porous media are related as:

$$\overline{u} = \frac{\overline{d}}{P_e} \tag{5}$$

where $\overline{u}$ is the macroscopic saturated fluid pore velocity vector in the heterogeneous porous media (L/T), $\overline{d}$, is the saturated macroscopic fluid flux vector in the heterogeneous porous media (L^3/TL^2), and P_e is the macroscopic scalar effective porosity (L^3/L^3). With $P_e = F_2(x, y, z)$, where F_2 is a function of space that is at least once differentiable (L^3/L^3).

Spatial variation of several other properties, including medium compressibility, thermal conductivity, exchange capacity, etc., are implied by heterogeneous media. In this paper, however, the phrase will be used only to apply to heterogeneity of hydraulic conductivity, intrinsic permeability, and the effective porosity.

1.3. A Commonly Used but Questionable Law for Flow in a Heterogeneous Saturated Porous Medium

One can postulate a direct extension of Darcy's Law for macroscopic flow in heterogeneous porous media and this extension has been widely used through the years. From that tradition comes the possible Dynamic Law for $\overline{d}$:

$$\overline{d} = -K\nabla\phi \tag{6}$$

where K is defined by Equation (4) and all of the other variables are as previously stated; noting, of course, that $\overline{d}$ now represents the macroscopic volumetric flux vector for flow in a saturated heterogeneous porous media.

Equation (3) for flow in a homogeneous medium is in the form of a simpler classical irrotational flow or lamellar vector field of continuum mechanics, while Equation (6) for flow in a heterogeneous media is in the form of a more involved complex lamellar vector field [19,20]. This complex lamellar vector field involves rotational flow. Extensive derivations and discussions of these two forms are provided in [21–23] which provide the necessary and sufficient conditions defining both homogeneous lamellar flow and the heterogeneous complex lamellar flow.

Specifically, we note the rotational flux vector, E, introduced by media heterogeneity where $(\ ) \times (\ )$ denotes the vector cross product and Equation (6) is entered into the last equality, then Equation (6) can be expanded by vector identities [24,25] to give:

$$\overline{E} = curl\left(\overline{d}\right) = (\nabla) \times \left(\overline{d}\right) = (\nabla) \times (-K\nabla\phi) = K(\nabla\phi) \times (\nabla\phi) - (\nabla \mathrm{K}) \times (\nabla\phi) \tag{7}$$

The first term on the right hand side vanishes, since the cross product of the gradients of the same scalar is always zero, and the remaining term is conveniently made positive by changing the order of the vector product, i.e.,

$$\overline{E} = (\nabla\phi) \times (\nabla \mathrm{K}) \tag{8}$$

Equation (8) demonstrates that the complex lamellar field involves rotational flow, when the flow occurs heterogeneous media. However, when the porous material is homogeneous then by Equation (2).

$$\nabla K = 0 \tag{9}$$

hence by Equation (8), $\overline{E}$ is identically zero for flow in a homogeneous media. Though not vital to the theme of this paper, there are a few cases in heterogeneous porous systems—always very special in nature—where $\overline{E} = 0$ that are individually enumerated in [23].

2. A Proposed Law for Flow in a Heterogeneous Saturated Porous Medium

From the rotational aspects found in the previous section, the question arises: Can we develop a more general description for describing macroscopic saturated flow in heterogeneous porous media?

We will explore this potential in this manuscript. We will base the proposed law upon a fluid mechanics conceptual approach using a Navier-Stokes formulation that incorporates the presence of a heterogeneous porous media on a macroscopic basis.

The earliest available technical references mathematically evaluate the transition from classical fluid mechanics analysis to consideration of flow in homogeneous porous media [26–28]. This work retains the rotational flow aspects of fluid mechanics. Lapwood [26], in fact, retains in his Equation (1.1) the "convective" or "total time derivative" operator present in the Navier-Stokes approach. Other related applications come from the literature on coupled thermal processes in porous media; this includes work studying coupled heat and flow analysis for analysis of geothermal reservoirs in New Zealand which was motivated by basic theoretical and excellent experimental work in [29–32]. This work includes many coupled thermal porous media flow papers summarized by [33] and continuing with work by [34–44]. In these papers, only rarely is material heterogeneity mentioned in porous media coupled process literature [1]. When heterogeneity is considered, only the effects of single discrete changes in material properties on the convective circulation patterns are examined or treated, with examples in Vincourt [45] and Strack [46].

Extended the Navier–Stokes Based Equation Describing Heterogeneous Systems

We propose a description of flow in heterogeneous porous media formulated starting with a simplified statement of the Navier–Stokes equation. Consider an incompressible homogeneous fluid flow combined with a macroscopic representation of flow in heterogeneous porous media. Accordingly, flow through heterogeneous, but isotropic porous media, of a homogeneous fluid of mass density, $\underline{p}$, moving at a velocity, μ, (the macroscopic fluid pore velocity) and related to the fluid flux vector, $\overline{d}$, (Equation (5)) may be written as:

$$\rho\frac{D\overline{u}}{Dt} = -\nabla p - \rho g\nabla z - \frac{\mu}{k}\overline{d} \tag{10}$$

where ρ is the constant mass density of fluid in units (M/L^3), $\frac{D}{Dt}$ is the classic convective or total derivative operator with time (1/T), t is the time variable (T), p is the fluid pressure (F/L^2), g is the gravitational scalar constant (L/T^2), μ is the dynamic viscosity of the fluid (FT/L^2), ∇ is the mathematical gradient operator (1/L), and the other variables are as already defined.

Writing the convective or total derivative in vector form in Equation (10) gives:

$$\rho\left[\frac{\partial\overline{u}}{\partial t} + (\overline{u}\cdot\nabla)\overline{u}\right] = -\nabla p - \rho g\nabla z - \frac{\mu}{k}\overline{d} \tag{11}$$

This equation has been reported in other papers, in particular [29] (Equation (6)), [33] (Equation (2)), [44] (Equation (2)), and [36] (Equation (1)). These papers presented equations that are identical or can be made identical by use of Equation (5). These publications span over 40 years of technical literature, suggesting the consistency and acceptance of the approach for introducing the rotational flow aspects caused by thermal coupled fluid processes.

Next, consider the vector operator term on the left side of the equals sign in Equation (11) and expanding that term to yield:

$$\rho(\overline{u}\cdot\nabla)\overline{u} = \rho\nabla\left[\frac{1}{2}|\overline{u}|^2\right] - \rho[\overline{u}]\times[(\nabla)\times(\overline{u})] \tag{12}$$

The expanded terms on the right of the equals sign in Equation (12) each have special physical significance or are designated in a special way as discussed below. The function inside the gradient operator of the first term denoted by *K. E.* or:

$$K.E. = \frac{1}{2}|\overline{u}|^2 \tag{13}$$

is the spatially varying kinetic energy per unit mass of fluid. The right-most term in Equation (12) denoted by the vector S is:

$$\overline{S} = (\nabla) \times (\overline{u}) \tag{14}$$

which represents the rotational flow vector field induced by the heterogeneity of the porous media. In other words, Equation (14) describes the specific rotational components of flow in heterogeneous porous media. The last term in Equation (12) is recognized as the classical Lamb Vector field, $\overline{L}$, [47,48] and is defined as:

$$\overline{L} = -\rho(\overline{u}) \times \left(\overline{S}\right) \tag{15}$$

which we can substitute into Equation (14), which gives:

$$\overline{L} = -\rho[(\overline{u})] \times [(\nabla) \times (\overline{u})] \tag{16}$$

we can write Equation (12) using Equation (15) as:

$$\rho(\overline{u}\cdot\nabla)\overline{u} = \underset{\textit{Kinetic Energy}}{\rho\nabla\left[\tfrac{1}{2}|\overline{u}|^2\right]} \quad \underset{\textit{Lamb Vector}}{-\,\rho(\overline{u}) \times \left(\overline{S}\right)} \tag{17}$$

and substitute this expanded Equation (17) into (11) and rearranging which gives:

$$\overline{d} = -\left(\frac{k}{\mu}\right)\left\{\nabla p + \rho g \nabla z + \left(\frac{1}{2}\right)\nabla[|\overline{u}|^2]\right\} - \left(\frac{k}{\mu}\right)\left\{\frac{\partial \overline{u}}{\partial t} - \rho(\overline{u}) \times \left(\overline{S}\right)\right\}. \tag{18}$$

Since the fluid is homogeneous, the sum of terms containing gradient expressions may be conveniently rewritten as:

$$\overline{d} = -\left(\frac{k}{\mu}\right)\nabla\left\{p + \rho g \nabla z + p\left(\frac{1}{2}\right)|\overline{u}|^2\right\} - \left(\frac{k}{\mu}\right)\left\{\frac{\partial \overline{u}}{\partial t} - \rho(\overline{u}) \times \left(\overline{S}\right)\right\} \tag{19}$$

which is the extended dynamic equation for describing flow in heterogeneous porous media. Since the fluid is homogeneous, it is useful to define the potential, Φ, on a unit volume basis as:

$$\Phi = p + \rho g z + \left(\frac{1}{2}\right)\rho|\overline{u}|^2 \tag{20}$$

and the saturated scalar hydraulic conductivity, K, for the heterogeneous porous media, as:

$$K = \left(\frac{k}{\mu}\right) \tag{21}$$

We can now substitute Equations (15), (20), and (21) into (19) which yields:

$$\overline{d} = -K\nabla\Phi - K\left\{\rho\frac{\partial \overline{u}}{\partial t}\right\} - K\left(\overline{L}\right) \tag{22}$$

Equation (22) is the extended dynamic equation for flow in heterogeneous porous materials in the more compact notational form.

The significance of the various terms in Equation (22) warrants consideration. The energy potential, Φ, in Equation (22) as defined in (20) is the fluid mechanics Bernoulli sum of pressure, elevation potential, and kinetic energy with each of the three terms expressed as energy per unit volume of fluid. The kinetic energy or third term in the Bernoulli energy summation is usually much smaller than the sum of the pressure and potential energy terms for flow in porous media. Traditionally the kinetic energy term has been assumed to be insignificant for flow in porous media [18,49,50]. Accordingly, the traditional reduced potential, ϕ, is represented as:

$$\phi = p + \rho g z \tag{23}$$

The middle term on the right side of Equation (22) is the local acceleration-induced inertial effects in the porous media flow. Such local acceleration (i.e., the partial derivative of the fluid pore velocity with time) is minuscule in porous media. Local acceleration is small because the fluid pore velocities are small. But more important, any flow changes in porous flow systems generally occur slowly and over long time periods; therefore, the local acceleration is small. Traditionally, the local acceleration effects are considered negligibly small and so have been dropped [18,49,50]. Under these assumptions then, approximately in transient cases or satisfied exactly under steady flow:

$$\frac{\partial \overline{u}}{\partial t} = 0 \tag{24}$$

we can derive a reduced form of Equation (22) by using Equations (21), (23), and (24) in (19) to give:

$$\overline{d} = -K\nabla\phi - K\left(\overline{L}\right) \tag{25}$$

The last term in Equation (25) incorporates the rotational flow effects into the extended dynamic equation for heterogeneous porous systems. This additional term is the product of the hydraulic conductivity, K, and the Lamb Vector, L.

3. An Extended Fluid Pore Velocity Expression in Heterogeneous Porous Materials

We now develop an extended equation for the fluid pore velocity in a heterogeneous porous medium. We substitute into Equation (11) Equations (21), and (24) to obtain, after multiplying by K,

$$K\rho(\overline{u}\cdot\nabla)\overline{u} = -K(\nabla \mathrm{p} - \rho \mathrm{g}\nabla \mathrm{z}) - \overline{d} \tag{26}$$

We now use Equation (23) which gives

$$\overline{d} = -K\nabla\phi - K\rho(\overline{u}\cdot\nabla)\overline{u} \tag{27}$$

The use Equation (5) solving and substitute into (27) which yields

$$P_e\overline{u} = -K\nabla\phi - K\rho(\overline{u}\cdot\nabla)\overline{u} \tag{28}$$

We can now rearrange Equation (28) as follows

$$[P_e + \; K\rho(\overline{u}\cdot\nabla)]\overline{u} = -K\nabla\phi \tag{29}$$

which after substituting Equation (5) and expanding gives

$$(\overline{u}\cdot\nabla) = \nabla\cdot\left[\frac{\overline{d}}{P_e}\right] = \frac{1}{P_e}\nabla\cdot\overline{d} + \frac{1}{(P_e)^2}\nabla P_e \tag{30}$$

In the right most part of this equation, $\nabla\cdot\overline{d} = 0$, by Conservation of Mass, so,

$$(\overline{u}\cdot\nabla) = \nabla\cdot\left[\frac{\overline{d}}{P_e}\right] = \frac{1}{P_e}\nabla \ln P_e \tag{31}$$

We can now substitute Equation (31) into Equation (29), which gives:

$$\left[P_e + \frac{K\rho}{P_e}\nabla \ln P_e\right]\overline{u} = -K\nabla\phi \tag{32}$$

We can then solve for the fluid pore velocity, $\overline{\mu}$, which yields

$$\overline{u} = \frac{P_e K}{(P_e)^2 + K\rho\nabla \ln P_e}\nabla\phi = \frac{\overline{d}}{P_e} \tag{33}$$

Equation (33) is the extended fluid pore velocity expression in heterogeneous porous materials with both hydraulic conductivity and effective porosity spatial variations included as derived using the Navier–Stokes equations.

For discussion purposes, we rewrite Equation (33), derived Navier–Stokes expression, as

$$\overline{d} = -K\nabla\phi - K\left(\overline{L}\right) \tag{34}$$

The last term in Equation (34) is the term proposed to incorporate any rotational flow effects into an extended dynamic equation for describing flow in heterogeneous porous materials. The significant rotational flow effects are contained in the first right hand term, $-K\nabla\phi$ [23].

We now consider if the term $-K\left(\overline{L}\right)$ in Equation (34) is required to describe the rotational effects of flow in heterogeneous porous systems.

Consider the rotation introduced by the material heterogeneity, which is the total torque component denoted by the vector $\overline{T}$, then, using Equation (34), we can write:

$$\overline{T} = (\nabla)\times\left(\overline{d}\right) = (\nabla)\times\left(\left\{[-K\nabla\phi] - \left[K\left(\overline{L}\right)\right]\right\}\right) = (\nabla)\times(-K\nabla\phi) - (\nabla)\times\left(K\left(\overline{L}\right)\right) \tag{35}$$

We can then substitute Equations (7) and (8) into Equation (35), which yields:

$$\overline{T} = \overline{E} - (\nabla)\times\left(K\left(\overline{L}\right)\right) \tag{36}$$

if we denote the Lamb rotational vector component as $\overline{N}$, we get:

$$\overline{N} = -(\nabla)\times\left(K\left(\overline{L}\right)\right) \tag{37}$$

which we can then substitute into Equation (36) which gives:

$$\overline{T} = \overline{E} - \overline{N}. \tag{38}$$

The previous question asked above may be now stated: Are both rotational components, $\overline{E}$, and, $\overline{N}$, required to appropriately describe the rotational effects of flow in heterogeneous porous materials? Closely related- what are the relative magnitudes of $\overline{E}$ and $\overline{N}$ in the total rotation, $\overline{T}$; that occur in heterogeneous porous media? Such must be answered to determine the applicability of this work and the situations in which this additional complexity is required.

Unfortunately, in our opinion, further theoretical answers to these questions are not presently possible; but anticipate advanced by capable theoretical researchers. We encourage experiments to explore this theoretical derivation to determine under what conditions such an extension is required and to validate this derivation.

4. Conclusions and Recommendations

We present a rigorous mathematical treatment of water flow in saturated heterogeneous porous media based on the classical Navier–Stokes formulation that includes vorticity in a heterogeneous porous media. We show that flow in heterogeneous media results in a flow field described by a heterogeneous complex lamellar vector field with rotational flows, compared to the homogeneous lamellar flow field that results from Darcy's law.

Through this theoretical analysis, we provided an extension to the often used Law of Flow (Equation (34)) for a homogeneous fluid in a heterogeneous porous material. Both this extension and the often used Law of Flow requires appropriate testing to determine under what conditions these issues are important and should be considered and under what conditions the more simplified Darcy's Law can be used.

Specifically, Equation (6) needs to be both appropriately tested experimentally. By so doing, demonstrating, which of the two, the postulated Equation (34) or the often used Equation (6), is appropriate for specific circumstances. We show, theoretically, that Darcy's Law does not accurately describe flow in a heterogeneous porous media and encourage precise laboratory experiments to determine under what conditions these issues are important.

While this work is similar to the Gupta, Sposito and Bhattacharya [1] manuscript "*Water Flow Through Inhomogeneous Porous Medium*", that paper is an extension that considers partially saturated flow which traditionally has been accepted as being described by the Buckingham-Darcy Flow Law. Here we consider saturated flow in a heterogeneous porous medium which has traditionally been accepted as being described by Darcy's Law. Thus, these two papers represent different flow systems with the complexity of the partially saturated inhomogeneous system much greater than the saturated heterogeneous flow system considered here. While the flow conditions and mathematical approaches are different, the results stated by Gupta, Sposito and Bhattacharya [1] reinforce and in most cases directly corroborate our work.

We publish this work to explicitly encourage others to test this derivation numerically and experimentally. Specifically, experimental work needs to be performed to determine under which circumstances Equation (6) is sufficient to describe flow conditions and under which conditions Equation (34) is required to characterize the effects of rotational flows. We look forward, with interest, to see numerical or experimental work inspired by our work that addresses these issues.

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