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Some Similarity Measures for Interval-Valued Picture Fuzzy Sets and Their Applications in Decision Making

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Abstract: Similarity measures, distance measures and entropy measures are some common tools considered to be applied to some interesting real-life phenomena including pattern recognition, decision making, medical diagnosis and clustering. Further, interval-valued picture fuzzy sets (IVPFSs) are effective and useful to describe the fuzzy information. Therefore, this manuscript aims to develop some similarity measures for IVPFSs due to the significance of describing the membership grades of picture fuzzy set in terms of intervals. Several types cosine similarity measures, cotangent similarity measures, set-theoretic and grey similarity measures, four types of dice similarity measures and generalized dice similarity measures are developed. All the developed similarity measures are validated, and their properties are demonstrated. Two well-known problems, including mineral field recognition problems and multi-attribute decision making problems, are solved using the newly developed similarity measures. The superiorities of developed similarity measures over the similarity measures of picture fuzzy sets, interval-valued intuitionistic fuzzy sets and intuitionistic fuzzy sets are demonstrated through a comparison and numerical examples.

Keywords: similarity measures; interval-valued picture fuzzy sets; pattern recognition; multi-attribute decision making

1. Introduction

The concept of fuzzy set (FS), a tool that deals with uncertainty, was proposed by Zadeh [1] in 1965. In FS, there is a function from a non-empty set to the closed unit interval [0, 1] which gives the degree of membership of an object to a non-empty set. FSs have many applications in decision making, pattern recognition, etc. A generalization of FSs called interval-valued fuzzy sets (IvFSs) were also proposed by Zadeh [2]. In IvFSs, the membership degree is expressed by an interval which is basically sub-interval of [0, 1]. Like FSs, IvFSs have many applications in the field of decision making, pattern recognition, etc.

Another generalization of FSs called intuitionistic FSs (IFSs), was proposed by Atanassov [3]. In IFSs, there are two functions named as the membership function and non-membership function with the condition that sum of the membership and non-membership must belong to a closed unit interval. Many authors used IFSs to develop applications in decision making, pattern recognition, etc. Atanassov and Gargov [4] proposed the concept of interval-valued IFSs (IvIFSs). In IvIFSs the degrees of membership and of non-membership are expressed by intervals which are sub-intervals of [0, 1] and they keep a condition that the sum of supremums of these intervals must belong to [0, 1].

Coung [5,6] proposed the concept of picture FSs (PFSs) which is generalization of IFSs. PFSs consist of three functions named as membership, abstinence and non-membership functions. Like in IFSs, PFSs also have the condition that the sum of all three values of a point must belong to a closed unit interval. The concept of interval-valued PFSs (IvPFSs) was also proposed in [7]. In IvPFSs, the degrees of membership, abstinence and non-membership are given in closed sub-intervals of [0, 1] and have a condition that the sum of the supremum of all three subintervals must belong to a closed unit interval. Obviously, IvPFSs can describe fuzzy information more easily than FSs, IFSs, IvFSs and PFSs.

Similarity measures (SMs) are an important measure about the similarity between two objects. Now, many researchers developed different SMs and used them in the fields of MADM, pattern recognition, mineral field recognition, building material recognition, strategy decision making, etc. Dhengfeng and Chuntian [8] proposed SMs for IFSs and then used them to solve a pattern recognition problem. Hung and Yang [9] used the Hausdorff distance to propose a new SM and used these SMs for pattern recognition problem. Ye [10] proposed some cosine SMs for IFSs. Xu [11] solved MADM problems using proposed SMs. Hwang et al. [12] proposed new SMs based on the Jaccard index for IFSs. Nguyen [13] proposed some SMs and dissimilarity measures for IFSs. Garg [14] proposed some improved cosine SMs for IFSs and solved decision-making problems using these SMs. Szmidt and Kecprzyk [15] proposed SMs for IFSs and used them to solve medical diagnostic reasoning problems. Meng and Chen [16] proposed SMs using entropy and then used these SMs to solve pattern recognition problems. Tang et al. [17] proposed dice SMs and generalized dice SMs and applied these SMs to group decision making. Xu and Chen [18] made a comparative overview of different SMs, which are proposed using Hamming distance, weighted Euclidean distance, etc. Moreover, this overview is also extended for interval-valued intuitionistic fuzzy sets (IvIFSs). Ye [19] proposed cosine SMs for IvIFSs and solved MADM problems using these SMs. Wei et al. [20] proposed SMs using entropy measures for IvIFSs and applied these SMs to MADM, pattern recognition and medical diagnosis problems. Lui et al. [21] proposed interval-valued intuitionistic fuzzy ordered weighted cosine similarity measure and used these SMs to solve investment decision making problems. Salvachandran et al. [22] solved a pattern recognition problem using SMs for complex vague soft set. Liao and Xu [23] proposed SMs for hesitant fuzzy linguistic information and solved qualitative decision-making problems. Chen and Chang [24] used transformation techniques to propose SMs and applied these SMs to solve pattern recognition problems. Rani and Garg [25] proposed distance measures between complex IFSs to solve decision making problems. Mishra et al. [26] proposed SMs with intuitionistic fuzzy WASPAS method and solved MADM problem with them. Garg and Kumar [27] used set pair analysis theory to propose new SMs and applied proposed SMs to decision making problems. Garg [28] proposed distance and SMs for intuitionistic multiplicative preference relation and solved pattern recognition and medical diagnoses. Obviously, there are a large number of research results about SMs for different FSs.

For the SMs based on PFSs, there are some achievements. Wei [29] proposed some SMs for PFSs and applied these SMs to mineral field recognition and building material recognition applications. Son [30] proposed generalized picture distance measure and applied it to picture fuzzy clustering. Wei [31] proposed some SMs for PFSs and applied these to strategy decision making problem. Wei and Gao [32] proposed generalized dice SMs for PFSs and applied these to building material recognition. Wei [33] solved MADM problem using picture fuzzy cross entropy. In decision making problems, the decision makers have to give their results in the form of different fuzzy framework but if there is a large amount of data then it is difficult to cover it in fuzzy framework. For example, in the case of "daily mean temperature of a city", there could be multiple readings are taken at different stations within that particular city, all of which are presented in the dataset. To convert this data into one IvPFN or any other fuzzy framework there are some methods discussed in [34–36]. Some decision-making problems for different tools for uncertainty are discussed in [37–44]. SMs defined above have their own significance but these SMs fail when information given as:

$\begin{pmatrix}[0.40, 0.50],\\ [0.12, 0.23],\\ [0.19, 0.23]\end{pmatrix}$	$\begin{pmatrix}[0.71, 0.79],\\ [0.07, 0.09],\\ [0.10, 0.12]\end{pmatrix}$	$\begin{pmatrix}[0.21, 0.30],\\ [0.12, 0.22],\\ [0.39, 0.44]\end{pmatrix}$
$\begin{pmatrix}[0.27, 0.43],\\ [0.08, 0.28],\\ [0.16, 0.23]\end{pmatrix}$	$\begin{pmatrix}[0.42, 0.51],\\ [0.13, 0.29],\\ [0.07, 0.27]\end{pmatrix}$	$\begin{pmatrix}[0.14, 0.23],\\ [0.21, 0.29],\\ [0.30, 0.40]\end{pmatrix}$

Thus, in this manuscript, some improved SMs are developed for interval-valued picture fuzzy sets (IvPFSs). The key features of this paper are:

1. To propose different SMs, such as cosine SMs, cosine SMs based on cosine function, SMs based on cotangent function, grey SMs, set-theoretic SMs, dice SMs and generalized dice SMs for IvPFSs.
2. To develop applications to strategy decision making and mineral recognition. In such applications, we will describe that how the opinion of decision makers can be brought into the picture fuzzy environment using a decision matrix and to be processed by using the proposed approach. Once the picture fuzzy data is processed, we then utilize the score and accuracy functions to analyse the obtained results.
3. To discuss the advantages of proposed new methods over the existing SMs of other fuzzy frameworks.
4. To make a comparative study with some existing SMs and to show the superiority and effectiveness of our proposed work.
5. To discuss some future aspects of our proposed study where the applicability could be improved.

In this manuscript, Section 2 gives some basic definitions that make an ease for reader to understand these basic concepts. In Section 3, we propose some SMs for interval-valued picture fuzzy information SMs, such as cosine SMs, cosine SMs based on cosine function, SMs based on cotangent function, grey SM, set-theoretic SM, dice SMs and generalized dice SMs. In Section 4, applications of strategy decision making and mineral recognition are established and results are studied. Section 5 is about advantages of proposed work over existing work. Section 6 offers the conclusion to the whole manuscript.

2. Preliminaries

In this section, we define some basic notions that will help readers to understand later sections. From now to onward, if stated, otherwise, X is used as universal set.

Definition 1. [1] *A FS on* X *is defined as*:

$$T = \{(x, m(x)) |\, x \in X\}, \tag{1}$$

where $m: X \to [0, 1]$ *is called membership function and it describes the degree of membership of an object to a non-empty set. Each* $m(x)$ *is called fuzzy number*.

Definition 2. *[2] An IvFS on* X *is defined as:*

$$T = \{(x, m_T(x)) |\, x \in X\}, \tag{2}$$

where $m_T = [m_{TL}, m_{TU}]$ *is subinterval of* [0, 1] *and it expresses the degree of membership by sub-interval and each* $m_T(x)$ *is called interval-valued fuzzy number (IvFN).*

Definition 3. *[3] An IFS on* X *is defined as*:

$$T = \{(x, m(x), n(x)) |\, x \in X\}, \tag{3}$$

where $m, n: X \rightarrow [0,1]$ *are called membership and non-membership functions. An IFS has a condition that the sum of both functions must lie in unit interval and the degree of refusal is defined as* $r(x) = 1 - \left(m(x) + n(x)\right)$. *A duplet* $\left(m(x), n(x)\right)$ *is called an intuitionistic fuzzy number (IFN).*

Definition 4. *[5] An IvIFS on a universal set* X *is defined as*:

$$T = \left\{\left(x, m_T(x), n_T(x)\right) | \ x \in X\right\}, \tag{4}$$

where $m_T = [m_{TL}, m_{TU}], n_T = [n_{TL}, n_{TU}]$ *and* $m_T, n_T: X \rightarrow [0,1]$. *An IvIFS has a condition that the sum of supremum of membership and non-membership functions must lie in unit interval. A duplet* $\left(m_T(x), n_T(x)\right)$ *is called interval-valued intuitionistic fuzzy number (IvIFN).*

Definition 5. *[7] A PFS on* X *is defined as*:

$$T = \left\{\left(x, m(x), i(x), n(x)\right) | \ x \in X\right\}, \tag{5}$$

where $m, i, n: X \rightarrow [0,1]$ *are called membership, abstinence and non-membership functions. A PFS has a condition that the sum of all three functions must lie in unit interval and the degree of refusal is defined as* $r(x) = 1 - \left(m(x) + i(x) + n(x)\right)$. *A triplet* $\left(m(x), i(x), n(x)\right)$ *is called a picture fuzzy number (PFN).*

Definition 6. *[7] An IvPFS on a universal set* X *is defined as*:

$$T = \left\{\left(x, m_T(x), i_T(x), n_T(x)\right) | \ x \in X\right\}, \tag{6}$$

where $m_T = [m_{TL}, m_{TU}], i_T = [i_{TL}, i_{TU}], n_T = [n_{TL}, n_{TU}]$ *and* $m_T, i_T, n_T: X \rightarrow [0,1]$. *An IvPFS has a condition that the sum of supremum of all three functions must lie in unit interval. A triplet* $\left(m_T(x), i_T(x), n_T(x)\right)$ *is called interval-valued picture fuzzy number (IvPFN).*

3. Similarity Measures

In this section, some SMs like cosine SMs, cosine SMs based on cosine and cotangent functions, grey SMs, set-theoretic SMs, dice SMs and generalized dice SMs are defined for IvPFSs. Some basic properties of these SMs are also defined.

3.1. Cosine Similarity Measures for IvPFSs

In this subsection, some cosine SMs and weighted cosine SMs for IvPFSs are defined and some basic properties of these SMs are also discussed.

Definition 7. *For any two IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$ *and* $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$, *an interval-valued picture fuzzy cosine similarity measure (IvPFCSM) between* T_1 *and* T_2 *is defined as*:

$$IvPFCSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{\begin{matrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ + i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) \end{matrix}}{\sqrt{\begin{matrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + m^2_{T_1U}(x_t) \\ + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t) \end{matrix}} \sqrt{\begin{matrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + m^2_{T_2U}(x_t) + \\ i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t) \end{matrix}}} \tag{7}$$

For $k = 1$ the above equation becomes correlation coefficient between IvPFSs.

Theorem 1. *For any two IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$ *and* $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$, *cosine similarity measure satisfies the following properties*:

i. $0 \leq IvPFCSM^1(T_1, T_2) \leq 1$.
ii. $IvPFCSM^1(T_1, T_2) = IvPFCSM^1(T_2, T_1)$
iii. $For\ T_1 = T_2,\ IvPFCSM^1(T_1, T_2) = 1$.

Proof.

(i). As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFCSM^1(T_1, T_2)$ belongs to [0, 1].
(ii). Holds trivially.
(iii). If $T_1 = T_2$ then $m_{T_1L} = m_{T_2L}$, $m_{T_1U} = m_{T_2U}$, $i_{T_1L} = i_{T_2L}$, $i_{T_1U} = i_{T_2U}$, $n_{T_1L} = n_{T_2L}$ and $n_{T_1U} = n_{T_2U}$.

and then

$$IvPFCSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{m^2_{T_1L}(x_t)+i^2_{T_1L}(x_t)+n^2_{T_1L}(x_t)+m^2_{T_1U}(x_t)+i^2_{T_1U}(x_t)+n^2_{T_1U}(x_t)}{m^2_{T_1L}(x_t)+i^2_{T_1L}(x_t)+n^2_{T_1L}(x_t)+m^2_{T_1U}(x_t)+i^2_{T_1U}(x_t)+n^2_{T_1U}(x_t)}$$

$$= \frac{1}{k}\sum_{t=1}^{k} 1 = \frac{1}{k}k = 1.$$

Definition 8. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *weighted cosine similarity measure between* T_1 *and* T_2 *is defined as:*

$$IvPFWCSM^1(T_1, T_2) = \sum_{t=1}^{k} \omega_t \frac{\begin{array}{c} m_{T_1L}(x_t)m_{T_2L}(x_t)+i_{T_1L}(x_t)i_{T_2L}(x_t)+n_{T_1L}(x_t)n_{T_2L}(x_t)+m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t)+n_{T_1U}(x_t)n_{T_2U}(x_t) \end{array}}{\sqrt{\begin{array}{c} m^2_{T_1L}(x_t)+i^2_{T_1L}(x_t)+n^2_{T_1L}(x_t)+m^2_{T_1U}(x_t)+ \\ i^2_{T_1U}(x_t)+n^2_{T_1U}(x_t) \end{array}}\sqrt{\begin{array}{c} m^2_{T_2L}(x_t)+i^2_{T_2L}(x_t)+n^2_{T_2L}(x_t)+m^2_{T_2U}(x_t)+ \\ i^2_{T_2U}(x_t)+n^2_{T_2U}(x_t) \end{array}}} \tag{8}$$

where $\omega = (\omega_1, \ldots\ldots\ldots, \omega_k)$ *is a weight vector satisfies* $\omega_t \in [0,1]$ *and* $\sum_{t=1}^{k} \omega_t = 1$.

Theorem 2. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *the weighted cosine similarity measure satisfies the following properties:*

i. $0 \le IvPFWCSM^1(T_1, T_2) \le 1$
ii. $IvPFWCSM^1(T_1, T_2) = IvPFWCSM^1(T_2, T_1)$
iii. $For\ T_1 = T_2,\ IvPFWCSM^1(T_1, T_2) = 1$

Proof.

(i). As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFWCSM^1(T_1, T_2)$ belongs to [0, 1].
(ii). Holds trivially.
(iii). If $T_1 = T_2$ then $m_{T_1L} = m_{T_2L}$, $m_{T_1U} = m_{T_2U}$, $i_{T_1L} = i_{T_2L}$, $i_{T_1U} = i_{T_2U}$, $n_{T_1L} = n_{T_2L}$ and $n_{T_1U} = n_{T_2U}$.

and then

$$IvPFWCSM^1(T_1, T_2) = \sum_{t=1}^{k} \omega_t \frac{m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + m^2_{T_1U}(x_t) + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t)}{m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + m^2_{T_1U}(x_t) + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t)} = \sum_{t=1}^{k} \omega_t = 1. \tag{9}$$

Definition 9. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *the cosine and weighted cosine similarity measure based on four functions membership, abstinence, non-membership and refusal between* T_1 *and* T_2 *are defined as:*

$$IvPFCSM^2(T_1,T_2) = \frac{1}{k}\sum_{t=1}^{k}\frac{\begin{array}{c}m_{T_1L}(x_t)m_{T_2L}(x_t)+i_{T_1L}(x_t)i_{T_2L}(x_t)+n_{T_1L}(x_t)n_{T_2L}(x_t)+r_{T_1L}(x_t)r_{T_2L}(x_t)+m_{T_1U}(x_t)m_{T_2U}(x_t)+\\ i_{T_1U}(x_t)i_{T_2U}(x_t)+n_{T_1U}(x_t)n_{T_2U}(x_t)+r_{T_1U}(x_t)r_{T_2U}(x_t)\end{array}}{\sqrt{\begin{array}{c}m^2_{T_1L}(x_t)+i^2_{T_1L}(x_t)+n^2_{T_1L}(x_t)+r^2_{T_1L}(x_t)+\\ m^2_{T_1U}(x_t)+i^2_{T_1U}(x_t)+n^2_{T_1U}(x_t)+r^2_{T_1U}(x_t)\end{array}}\sqrt{\begin{array}{c}m^2_{T_2L}(x_t)+i^2_{T_2L}(x_t)+n^2_{T_2L}(x_t)+r^2_{T_2L}(x_t)+\\ m^2_{T_2U}(x_t)+i^2_{T_2U}(x_t)+n^2_{T_2U}(x_t)+r^2_{T_2U}(x_t)\end{array}}} \tag{10}$$

And:

$$IvPFWCSM^2(T_1,T_2) = \sum_{t=1}^{k}\omega_j\frac{\begin{array}{c}m_{T_1L}(x_t)m_{T_2L}(x_t)+i_{T_1L}(x_t)i_{T_2L}(x_t)+n_{T_1L}(x_t)n_{T_2L}(x_t)+r_{T_1L}(x_t)r_{T_2L}(x_t)+m_{T_1U}(x_t)m_{T_2U}(x_t)+\\ i_{T_1U}(x_t)i_{T_2U}(x_t)+n_{T_1U}(x_t)n_{T_2U}(x_t)++r_{T_1U}(x_t)r_{T_2U}(x_t)\end{array}}{\sqrt{\begin{array}{c}m^2_{T_1L}(x_t)+i^2_{T_1L}(x_t)+n^2_{T_1L}(x_t)+r^2_{T_1L}(x_t)+\\ m^2_{T_1U}(x_t)+i^2_{T_1U}(x_t)+n^2_{T_1U}(x_t)+r^2_{T_1U}(x_t)\end{array}}\sqrt{\begin{array}{c}m^2_{T_2L}(x_t)+i^2_{T_2L}(x_t)+n^2_{T_2L}(x_t)+r^2_{T_2L}(x_t)+\\ m^2_{T_2U}(x_t)+i^2_{T_2U}(x_t)+n^2_{T_2U}(x_t)+r^2_{T_2U}(x_t)\end{array}}} \tag{11}$$

where weight vector $\omega = (\omega_1, \ldots\ldots\ldots, \omega_k)^T$ *is with a condition that for* $t = 1,2,\ldots\ldots\ldots,k$ $\omega_t \in [0,1]$ and $\sum_{t=1}^{k}\omega_t = 1$.

Theorem 3. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *cosine and weighted cosine similarity measures based on four functions satisfy the following properties*:

i. $0 \le IvPFCSM^2(T_1,T_2) \le 1.$
ii. $IvPFCSM^2(T_1,T_2) = IvPFCSM^2(T_2,T_1)$
iii. $For\ T_1 = T_2,\ IvPFCSM^2(T_1,T_2) = 1.$
iv. $0 \le IvPFWCSM^2(T_1,T_2) \le 1$
v. $IvPFWCSM^2(T_1,T_2) = IvPFWCSM^2(T_2,T_1)$
vi. $For\ T_1 = T_2,\ IvPFWCSM^2(T_1,T_2) = 1$

Proof. The proofs are similar to Theorems 1 and 2. □

3.2. Cosine Similarity Measures for IvPFSs Based on Cosine Function

In this subsection some cosine SMs based on cosine function and some weighted cosine SMs based on cosine function for IvPFSs are defined. Some basic properties of these SMs are also discussed.

Definition 10. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *cosine similarity measures based on cosine function between these two IvPFNs are defined as:*

$$IvPFCsSM^1(T_1,T_2) = \frac{1}{k}\sum_{t=1}^{k}\cos\left\{\frac{\pi}{2}\left[\begin{array}{c}|m_{T_1L}-m_{T_2L}|\vee|i_{T_1L}-i_{T_2L}|\vee|n_{T_1L}-n_{T_2L}|\vee|m_{T_1U}-m_{T_2U}|\\ \vee|i_{T_1U}-i_{T_2U}|\vee|n_{T_1U}-n_{T_2U}|\end{array}\right]\right\} \tag{12}$$

$$IvPFCsSM^2(T_1,T_2) = \frac{1}{k}\sum_{t=1}^{k}\cos\left\{\frac{\pi}{4}\left[\begin{array}{c}|m_{T_1L}-m_{T_2L}|+|i_{T_1L}-i_{T_2L}|+|n_{T_1L}-n_{T_2L}|+|m_{T_1U}-m_{T_2U}|\\ +|i_{T_1U}-i_{T_2U}|+|n_{T_1U}-n_{T_2U}|\end{array}\right]\right\} \tag{13}$$

Further, cosine similarity measures using four functions membership, abstinence, non-membership and refusal are defined as:

$$IvPFCsSM^3(T_1,T_2) = \frac{1}{k}\sum_{t=1}^{k}\cos\left\{\frac{\pi}{2}\left[\begin{array}{c}|m_{T_1L}-m_{T_2L}|\vee|i_{T_1L}-i_{T_2L}|\vee|n_{T_1L}-n_{T_2L}|\vee|r_{T_1L}-r_{T_2L}|\\ \vee|m_{T_1U}-m_{T_2U}|\vee|i_{T_1U}-i_{T_2U}|\vee|n_{T_1U}-n_{T_2U}|\vee|r_{T_1U}-r_{T_2U}|\end{array}\right]\right\} \tag{14}$$

$$IvPFCsSM^4(T_1,T_2) = \frac{1}{k}\sum_{t=1}^{k}\cos\left\{\frac{\pi}{4}\left[\begin{array}{c}|m_{T_1L}-m_{T_2L}|+|i_{T_1L}-i_{T_2L}|+|n_{T_1L}-n_{T_2L}|+|r_{T_1L}-r_{T_2L}|\\ +|m_{T_1U}-m_{T_2U}|+|i_{T_1U}-i_{T_2U}|+|n_{T_1U}-n_{T_2U}|+|r_{T_1U}-r_{T_2U}|\end{array}\right]\right\} \tag{15}$$

Theorem 4. *For any three IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$, $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$ *and* $T_3 = (m_{T_3}, i_{T_3}, n_{T_3})$, *all IvPFCsSMs satisfy the following properties for p = 1, 2, 3, 4.*

i. $0 \leq IvPFCsSM^p(T_1, T_2) \leq 1$.
ii. $IvPFCsSM^p(T_1, T_2) = IvPFCsSM^p(T_2, T_1)$
iii. For $T_1 = T_2$, $IvPFCsSM^p(T_1, T_2) = 1$.
iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $IvPFCsSM^p(T_1, T_3) \leq IvPFCsSM^p(T_1, T_2)$
v. and $IvPFCsSM^p(T_1, T_3) \leq IvPFCsSM^p(T_2, T_3)$

Proof.

(i). Since value of cosine function lies in [0, 1], so it is obvious that value of $IvPFCsSM^p(T_1, T_2)$ lies in [0, 1] for all $p = 1,2,3,4$.

(ii). Trivially hold.

(iii). For $T_1 = T_2$, $m_{T_1L} = m_{T_2L}$, $m_{T_1U} = m_{T_2U}$, $i_{T_1L} = i_{T_2L}$, $i_{T_1U} = i_{T_2U}$, $n_{T_1L} = n_{T_2L}$, $n_{T_1U} = n_{T_2U}$, $r_{T_1L} = r_{T_2L}$ and $r_{T_1U} = r_{T_2U}$. This shows that:

$|m_{T_1L} - m_{T_2L}| = 0$, $|i_{T_1L} - i_{T_2L}| = 0$, $|n_{T_1L} - n_{T_2L}| = 0$, $|m_{T_1U} - m_{T_2U}| = 0$, $|i_{T_1U} - i_{T_2U}| = 0$, $|n_{T_1U} - n_{T_2U}| = 0$.

Thus, $IvPFCsSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \cos\{0\} = \frac{1}{k}\sum_{t=1}^{k} 1 = 1$.

Similarly, for $p = 2,3,4$, the others can also be proved.

(iv). For $T_1 \subseteq T_2 \subseteq T_3$, $m_{T_1L} \leq m_{T_2L} \leq m_{T_3L}$ also $m_{T_1U} \leq m_{T_2U} \leq m_{T_3U}$.

Similarly, $i_{T_1L} \leq i_{T_2L} \leq i_{T_3L}$, $i_{T_1U} \leq i_{T_2U} \leq i_{T_3U}$, $n_{T_1L} \geq n_{T_2L} \geq n_{T_3L}$ and $n_{T_1U} \geq n_{T_2U} \geq n_{T_3U}$.
For $t = 1,2, \ldots\ldots, k$ we have:

$$|m_{T_1L} - m_{T_2L}| \leq |m_{T_1L} - m_{T_3L}|$$
$$|i_{T_1L} - i_{T_2L}| \leq |i_{T_1L} - i_{T_3L}|$$
$$|n_{T_1L} - n_{T_2L}| \geq |n_{T_1L} - n_{T_3L}|$$
$$|m_{T_1U} - m_{T_2U}| \leq |m_{T_1U} - m_{T_3U}|$$
$$|i_{T_1U} - i_{T_2U}| \leq |i_{T_1U} - i_{T_3U}|$$
$$|n_{T_1U} - n_{T_2U}| \geq |n_{T_1U} - n_{T_3U}|$$

As cosine function is decreasing in $\left[0, \frac{\pi}{2}\right]$, so $IvPFCsSM^1(T_1, T_3) \leq IvPFCsSM^1(T_1, T_2)$ and also by following same method it can be proved that $IvPFCsSM^1(T_1, T_3) \leq IvPFCsSM^1(T_2, T_3)$.

Similarly, for $p = 2,3,4$, the others can also be proved. □

Definition 11. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *weighted cosine similarity measures based on cosine function between these two IvPFNs are defined as:*

$$IvPFWCsSM^1(T_1, T_2) = \sum_{t=1}^{k} \omega_t \cos\left\{\frac{\pi}{2}\left[\begin{array}{c}|m_{T_1L} - m_{T_2L}| \vee |i_{T_1L} - i_{T_2L}| \vee |n_{T_1L} - n_{T_2L}| \vee |m_{T_1U} - m_{T_2U}| \\ \vee |i_{T_1U} - i_{T_2U}| \vee |n_{T_1U} - n_{T_2U}|\end{array}\right]\right\} \quad (16)$$

$$IvPFWCsSM^2(T_1, T_2) = \sum_{t=1}^{k} \omega_t \cos\left\{\frac{\pi}{4}\left[\begin{array}{c}|m_{T_1L} - m_{T_2L}| + |i_{T_1L} - i_{T_2L}| + |n_{T_1L} - n_{T_2L}| + |m_{T_1U} - m_{T_2U}| \\ + |i_{T_1U} - i_{T_2U}| + |n_{T_1U} - n_{T_2U}|\end{array}\right]\right\} \quad (17)$$

Further, the weighted cosine similarity measures using four functions membership, abstinence, non-membership and refusal are defined as:

$$IvPFWCsSM^3(T_1,T_2) = \sum_{t=1}^{k}\omega_t \cos\left\{\frac{\pi}{2}\left[\begin{array}{c}|m_{T_1L}-m_{T_2L}|\vee|i_{T_1L}-i_{T_2L}|\vee|n_{T_1L}-n_{T_2L}|\vee|r_{T_1L}-r_{T_2L}|\\ \vee|m_{T_1U}-m_{T_2U}|\vee|i_{T_1U}-i_{T_2U}|\vee|n_{T_1U}-n_{T_2U}|\vee|r_{T_1U}-r_{T_2U}|\end{array}\right]\right\} \tag{18}$$

$$IvPFWCsSM^4(T_1,T_2) = \sum_{t=1}^{k}\omega_t \cos\left\{\frac{\pi}{4}\left[\begin{array}{c}|m_{T_1L}-m_{T_2L}|+|i_{T_1L}-i_{T_2L}|+|n_{T_1L}-n_{T_2L}|+|r_{T_1L}-r_{T_2L}|\\ +|m_{T_1U}-m_{T_2U}|+|i_{T_1U}-i_{T_2U}|+|n_{T_1U}-n_{T_2U}|+|r_{T_1U}-r_{T_2U}|\end{array}\right]\right\} \tag{19}$$

where weight vector $\omega = (\omega_1, \ldots\ldots\ldots, \omega_k)^T$ *is with a condition that for* $t = 1,2,\ldots\ldots\ldots,k$ $\omega_t \in [0,1]$ *and* $\sum_{t=1}^{k}\omega_t = 1$.

Theorem 5. *For any three IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$, $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$ *and* $T_3 = (m_{T_3}, i_{T_3}, n_{T_3})$, *all IvPFWCsSMs satisfy the following properties for t=1, 2, 3, 4.*

i. $0 \le IvPFWCsSM^t(T_1,T_2) \le 1$.
ii. $IvPFWCsSM^t(T_1,T_2) = IvPFWCsSM^t(T_2,T_1)$
iii. For $T_1 = T_2$, $IvPFWCsSM^t(T_1,T_2) = 1$.
Iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $IvPFWCsSM^t(T_1,T_3) \le IvPFWCsSM^t(T_1,T_2)$ and $IvPFWCsSM^t(T_1,T_3) \le IvPFWCsSM^t(T_2,T_3)$

Proof.

(i). Since value of cosine function lies in [0, 1], so it is obvious that value of $IvPFWCsSM^t(T_1,T_2)$ lies in [0, 1] for all $t = 1,2,3,4$.
(ii). Trivially hold.
(iii). For $T_1 = T_2$, $m_{T_1L} = m_{T_2L}$, $m_{T_1U} = m_{T_2U}$, $i_{T_1L} = i_{T_2L}$, $i_{T_1U} = i_{T_2U}$, $n_{T_1L} = n_{T_2L}$, $n_{T_1U} = n_{T_2U}$, $r_{T_1L} = r_{T_2L}$ and $r_{T_1U} = r_{T_2U}$. This shows that:

$|m_{T_1L} - m_{T_2L}| = 0$, $|i_{T_1L} - i_{T_2L}| = 0$, $|n_{T_1L} - n_{T_2L}| = 0$, $|m_{T_1U} - m_{T_2U}| = 0$, $|i_{T_1U} - i_{T_2U}| = 0$, $|n_{T_1U} - n_{T_2U}| = 0$.

Thus:

$$IvPFWCsSM^1(T_1,T_2) = \sum_{t=1}^{k}\omega_t \cos\{0\} = \sum_{t=1}^{k}\omega_t = 1$$

Similarly, for $t = 2,3,4$, they can also be proved.

(iv). For $T_1 \subseteq T_2 \subseteq T_3$, $m_{T_1L} \le m_{T_2L} \le m_{T_3L}$ also $m_{T_1U} \le m_{T_2U} \le m_{T_3U}$.

Similarly, $i_{T_1L} \le i_{T_2L} \le i_{T_3L}$, $i_{T_1U} \le i_{T_2U} \le i_{T_3U}$, $n_{T_1L} \ge n_{T_2L} \ge n_{T_3L}$ and $n_{T_1U} \ge n_{T_2U} \ge n_{T_3U}$.
For $t = 1,2,\ldots\ldots.,k$ we have:

$$|m_{T_1L} - m_{T_2L}| \le |m_{T_1L} - m_{T_3L}|$$

$$|i_{T_1L} - i_{T_2L}| \le |i_{T_1L} - i_{T_3L}|$$

$$|n_{T_1L} - n_{T_2L}| \ge |n_{T_1L} - n_{T_3L}|$$

$$|m_{T_1U} - m_{T_2U}| \le |m_{T_1U} - m_{T_3U}|$$

$$|i_{T_1U} - i_{T_2U}| \le |i_{T_1U} - i_{T_3U}|$$

$$|n_{T_1U} - n_{T_2U}| \ge |n_{T_1U} - n_{T_3U}|$$

As cosine function is decreasing in $\left[0, \frac{\pi}{2}\right]$, *so* $IvPFWCsSM^1(T_1,T_3) \le IvPFWCsSM^1(T_1,T_2)$ and also by following same method it can be proved that $IvPFWCsSM^1(T_1,T_3) \le IvPFWCsSM^1(T_2,T_3)$.

Similarly, for $t = 2,3,4$, the others can also be proved. □

3.3. Similarity Measures for IvPSs Based on Cotangent Function

In this subsection, we proposed some cotangent SMs based on cotangent function and some weighted cotangent SMs based on cotangent function for IvPFSs, and some basic properties of these SMs are also discussed.

Definition 12. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *a cotangent similarity measure based on cotangent function between these two IvPFNs is defined as:*

$$IvPFCtSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \cot\left\{\frac{\pi}{4} + \frac{\pi}{4}\begin{bmatrix}|m_{T_1L} - m_{T_2L}| \vee |i_{T_1L} - i_{T_2L}| \vee |n_{T_1L} - n_{T_2L}| \vee |m_{T_1U} - m_{T_2U}| \\ \vee |i_{T_1U} - i_{T_2U}| \vee |n_{T_1U} - n_{T_2U}|\end{bmatrix}\right\} \quad (20)$$

Further, then cotangent similarity measures using four functions membership, abstinence, non-membership and refusal is defined as:

$$IvPFCtSM^2(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \cot\left\{\frac{\pi}{4} + \frac{\pi}{4}\begin{bmatrix}|m_{T_1L} - m_{T_2L}| \vee |i_{T_1L} - i_{T_2L}| \vee |n_{T_1L} - n_{T_2L}| \vee |r_{T_1L} - r_{T_2L}| \vee |m_{T_1U} - m_{T_2U}| \\ \vee |i_{T_1U} - i_{T_2U}| \vee |n_{T_1U} - n_{T_2U}| \vee |r_{T_1U} - r_{T_2U}|\end{bmatrix}\right\} \quad (20)$$

Theorem 6. *For any three IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$, $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$ *and* $T_3 = (m_{T_3}, i_{T_3}, n_{T_3})$, *all IvPFCtSMs satisfy the following properties for t = 1, 2:*

i. $0 \leq \mathrm{IvPFCtSM}^t(T_1, T_2) \leq 1$.
ii. $\mathrm{IvPFCtSM}^t(T_1, T_2) = \mathrm{IvPFCtSM}^t(T_2, T_1)$.
iii. For $T_1 = T_2$, $\mathrm{IvPFCtSM}^t(T_1, T_2) = 1$.
Iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $\mathrm{IvPFCtSM}^t(T_1, T_3) \leq \mathrm{IvPFCtSM}^t(T_1, T_2)$ and $\mathrm{IvPFCtSM}^t(T_1, T_3) \leq \mathrm{IvPFCtSM}^t(T_2, T_3)$.

Proof. The proofs are similar as in Theorem 4. □

Definition 13. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *a weighted cotangent similarity measure based on cotangent function between these two IvPFNs is defined as:*

$$IvPFWCtSM^1(T_1, T_2) = \sum_{t=1}^{k} \omega_t \cot\left\{\frac{\pi}{4} + \frac{\pi}{4}\begin{bmatrix}|m_{T_1L} - m_{T_2L}| \vee |i_{T_1L} - i_{T_2L}| \vee |n_{T_1L} - n_{T_2L}| \vee |m_{T_1U} - m_{T_2U}| \\ \vee |i_{T_1U} - i_{T_2U}| \vee |n_{T_1U} - n_{T_2U}|\end{bmatrix}\right\} \quad (21)$$

Further, then weighted cotangent similarity measure using four functions membership, abstinence, non-membership and refusal is defined as:

$$IvPFWCtSM^2(T_1, T_2) = \sum_{t=1}^{k} \omega_t \cot\left\{\frac{\pi}{4} + \frac{\pi}{4}\begin{bmatrix}|m_{T_1L} - m_{T_2L}| \vee |i_{T_1L} - i_{T_2L}| \vee |n_{T_1L} - n_{T_2L}| \vee |r_{T_1L} - r_{T_2L}| \\ \vee |m_{T_1U} - m_{T_2U}| \vee |i_{T_1U} - i_{T_2U}| \vee |n_{T_1U} - n_{T_2U}| \vee |r_{T_1U} - r_{T_2U}|\end{bmatrix}\right\} \quad (22)$$

where weight vector $\omega = (\omega_1, \ldots\ldots\ldots, \omega_k)^T$ *is with a condition that for* $t = 1,2, \ldots\ldots\ldots, k$ $\omega_t \in [0,1]$ *and* $\sum_{t=1}^{k} \omega_t = 1$.

Theorem 7. *For any three IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$, $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$ *and* $T_3 = \left(m_{T_3}, i_{T_3}, n_{T_3}\right)$*, all IvPFWCtSMs satisfy the following properties for t = 1, 2:*

i. $0 \leq IvPFWCtSM^t(T_1, T_2) \leq 1$.
ii. $IvPFWCtSM^t(T_1, T_2) = IvPFWCtSM^t(T_2, T_1)$
iii. For $T_1 = T_2$, $IvPFWCtSM^t(T_1, T_2) = 1$.
iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $IvPFWCtSM^t(T_1, T_3) \leq IvPFWCtSM^t(T_1, T_2)$ and $IvPFWCtSM^t(T_1, T_3) \leq IvPFWCtSM^t(T_2, T_3)$.

Proof. The proofs are similar as in Theorem 5. □

3.4. Set-Theoretic Similarity Measures and Grey Similarity Measures for IvPFSs

In this subsection, set-theoretic SM, Grey SM and weighted set-theoretic SM, weighted Grey SM for IvPFSs are defined, and some basic properties of these SMs are also discussed.

Definition 14. *For any two IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$ *and* $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$*, an interval-valued picture fuzzy set-theoretic similarity measure (IvPFStSM) between these IvPFNs is defined as:*

$$IvPFStSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{\begin{matrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ + i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) \end{matrix}}{max\left(\begin{matrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) \\ + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{matrix}, \begin{matrix} m_{T_2L}^2(x_t) + i_{T_2L}^2(x_t) + n_{T_2L}^2(x_t) + m_{T_2U}^2(x_t) + \\ i_{T_2U}^2(x_t) + n_{T_2U}^2(x_t) \end{matrix}\right)} \quad (23)$$

Theorem 8. *For any three IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$, $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$ *and* $T_3 = \left(m_{T_3}, i_{T_3}, n_{T_3}\right)$*, the IvPFStSM satisfies the following properties:*

i. $0 \leq IvPFStSM^1(T_1, T_2) \leq 1$.
ii. $IvPFStSM^1(T_1, T_2) = IvPFStSM^1(T_2, T_1)$
iii. For $T_1 = T_2$, $IvPFStSM^1(T_1, T_2) = 1$.
iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $IvPFStSM^1(T_1, T_3) \leq IvPFStSM^1(T_1, T_2)$ and $IvPFStSM^1(T_1, T_3) \leq IvPFStSM^1(T_2, T_3)$

Proof.

(i). As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFStSM^1(T_1, T_2)$ belongs to [0, 1].
(ii). Holds trivially.
(iii). If $T_1 = T_2$ then $m_{T_1L} = m_{T_2L}$, $m_{T_1U} = m_{T_2U}$, $i_{T_1L} = i_{T_2L}$, $i_{T_1U} = i_{T_2U}$, $n_{T_1L} = n_{T_2L}$ and $n_{T_1U} = n_{T_2U}$.

and then

$$IvPFStSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t)}{m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t)}$$

$$= \frac{1}{k}\sum_{t=1}^{k} 1$$

$$= \frac{1}{k}k = 1$$

Definition 15. *For any two IvPFNs* $T_1=\left(m_{T_1},i_{T_1},n_{T_1}\right)$ *and* $T_2=\left(m_{T_2},i_{T_2},n_{T_2}\right)$*, an interval-valued picture fuzzy weighted set-theoretic similarity measure (IvPFWStSM) between these IvPFNs is defined as:*

$$IvPFWStSM^1(T_1,T_2)=\sum_{t=1}^{k}\omega_t\frac{\begin{array}{c}m_{T_1L}(x_t)m_{T_2L}(x_t)+i_{T_1L}(x_t)i_{T_2L}(x_t)+n_{T_1L}(x_t)n_{T_2L}(x_t)+m_{T_1U}(x_t)m_{T_2U}(x_t)\\+i_{T_1U}(x_t)i_{T_2U}(x_t)+n_{T_1U}(x_t)n_{T_2U}(x_t)\end{array}}{max\begin{pmatrix}m_{T_1L}^2(x_t)+i_{T_1L}^2(x_t)+n_{T_1L}^2(x_t)+m_{T_1U}^2(x_t)+ & m_{T_2L}^2(x_t)+i_{T_2L}^2(x_t)+n_{T_2L}^2(x_t)+m_{T_2U}^2(x_t)+\\ i_{T_1U}^2(x_t)+n_{T_1U}^2(x_t) & , \quad i_{T_2U}^2(x_t)+n_{T_2U}^2(x_t)\end{pmatrix}} \tag{24}$$

where weight vector $\omega=(\omega_1,\ldots\ldots\ldots,\omega_k)^T$ *is with the condition that for* $t=1,2,\ldots\ldots\ldots,k$ $\omega_t\in[0,1]$ *and* $\sum_{t=1}^{k}\omega_t=1$.

Theorem 9. *For any three IvPFNs* $T_1=\left(m_{T_1},i_{T_1},n_{T_1}\right)$*,* $T_2=\left(m_{T_2},i_{T_2},n_{T_2}\right)$ *and* $T_3=\left(m_{T_3},i_{T_3},n_{T_3}\right)$*, the IvPFWStSM satisfies the following properties:*

i. $0\le IvPFWStSM^1(T_1,T_2)\le 1$.
ii. $IvPFWStSM^1(T_1,T_2)=IvPFWStSM^1(T_2,T_1)$.
iii. For $T_1=T_2$, $IvPFWStSM^1(T_1,T_2)=1$.
iv. Consider $T_1\subseteq T_2\subseteq T_3$, then $IvPFWStSM^1(T_1,T_3)\le IvPFWStSM^1(T_1,T_2)$ and $IvPFWStSM^2(T_1,T_3)\le IvPFWStSM^1(T_2,T_3)$.

Proof. The proof is similar as in Theorem 2. □

Definition 16. *For any two IvPFNs* $T_1=\left(m_{T_1},i_{T_1},n_{T_1}\right)$ *and* $T_2=\left(m_{T_2},i_{T_2},n_{T_2}\right)$*, an interval-valued picture fuzzy grey similarity measure (IvPFGSM) between these IvPFNs is defined as:*

$$\begin{aligned}IvPFGSM^1(T_1,T_2)\\=\frac{1}{3k}\sum_{t=1}^{k}&\frac{\Delta m_{L(min)}+\Delta m_{U(min)}+\Delta m_{L(max)}+\Delta m_{U(max)}}{\Delta m_L+\Delta m_U+\Delta m_{L(max)}+\Delta m_{U(max)}}\\&+\frac{\Delta i_{L(min)}+\Delta i_{U(min)}+\Delta i_{L(max)}+\Delta i_{U(max)}}{\Delta i_L+\Delta i_U+\Delta i_{L(max)}+\Delta i_{U(max)}}\\&+\frac{\Delta n_{L(min)}+\Delta n_{U(min)}+\Delta n_{L(max)}+\Delta n_{U(max)}}{\Delta n_L+\Delta n_U+\Delta n_{L(max)}+\Delta n_{U(max)}}\end{aligned} \tag{25}$$

where $\Delta m_{L(min)}=\min\{|m_{T_1L}-m_{T_2L}|\}$, $\Delta m_{U(min)}=\min\{|m_{T_1U}-m_{T_2U}|\}$, $\Delta m_L=|m_{T_1L}-m_{T_2L}|$, $\Delta m_U=|m_{T_1U}-m_{T_2U}|$, $\Delta m_{L(max)}=\max\{|m_{T_1L}-m_{T_2L}|\}$, $\Delta m_{U(max)}=\max\{|m_{T_1U}-m_{T_2U}|\}$, $\Delta i_{L(min)}=\min\{|i_{T_1L}-i_{T_2L}|\}$, $\Delta i_{U(min)}=\min\{|i_{T_1U}-i_{T_2U}|\}$, $\Delta i_L=|i_{T_1L}-i_{T_2L}|$, $\Delta i_U=|i_{T_1U}-i_{T_2U}|$, $\Delta i_{L(max)}=\max\{|i_{T_1L}-i_{T_2L}|\}$, $\Delta i_{U(max)}=\max\{|i_{T_1U}-i_{T_2U}|\}$, $\Delta n_{L(min)}=\min\{|n_{T_1L}-n_{T_2L}|\}$, $\Delta n_{U(min)}=\min\{|n_{T_1U}-n_{T_2U}|\}$, $\Delta n_L=|n_{T_1L}-n_{T_2L}|$, $\Delta n_U=|n_{T_1U}-n_{T_2U}|$, $\Delta n_{L(max)}=\max\{|n_{T_1L}-n_{T_2L}|\}$, $\Delta n_{U(max)}=\max\{|n_{T_1U}-n_{T_2U}|\}$.

Theorem 10. *For any three IvPFNs* $T_1=\left(m_{T_1},i_{T_1},n_{T_1}\right)$*,* $T_2=\left(m_{T_2},i_{T_2},n_{T_2}\right)$ *and* $T_3=\left(m_{T_3},i_{T_3},n_{T_3}\right)$*, the IvPFGSM satisfies the following properties:*

i. $0\le IvPFGSM^1(T_1,T_2)\le 1$.
ii. $IvPFGSM^1(T_1,T_2)=IvPFGSM^1(T_2,T_1)$
iii. For $T_1=T_2$, $IvPFGSM^1(T_1,T_2)=1$.
iv. Consider $T_1\subseteq T_2\subseteq T_3$, then $IvPFGSM^1(T_1,T_3)\le IvPFGSM^1(T_1,T_2)$ and $IvPFGSM^1(T_1,T_3)\le IvPFGSM^1(T_2,T_3)$.

Definition 17. *For any two IvPFNs* $T_1=\left(m_{T_1},i_{T_1},n_{T_1}\right)$ *and* $T_2=\left(m_{T_2},i_{T_2},n_{T_2}\right)$*, an interval-valued picture fuzzy weighted grey similarity measure (IvPFWGSM) between these IvPFNs is defined as:*

$$IvPFWGSM^1(T_1,T_2) = \frac{1}{3}\sum_{t=1}^{k}\omega_t \frac{\Delta m_{L(min)} + \Delta m_{U(min)} + \Delta m_{L(max)} + \Delta m_{U(max)}}{\Delta m_L + \Delta m_U + \Delta m_{L(max)} + \Delta m_{U(max)}} + \frac{\Delta i_{L(min)} + \Delta i_{U(min)} + \Delta i_{L(max)} + \Delta i_{U(max)}}{\Delta i_L + \Delta i_U + \Delta i_{L(max)} + \Delta i_{U(max)}} + \frac{\Delta n_{L(min)} + \Delta n_{U(min)} + \Delta n_{L(max)} + \Delta n_{U(max)}}{\Delta n_L + \Delta n_U + \Delta n_{L(max)} + \Delta n_{U(max)}} \tag{21}$$

where weight vector $\omega = (\omega_1, \ldots\ldots\ldots, \omega_k)^T$ *is with a condition that for* $t = 1,2,\ldots\ldots\ldots,k$ $\omega_t \in [0,1]$ *and* $\sum_{t=1}^{k}\omega_t = 1$ *and* $\Delta m_{L(min)} = \min\{|m_{T_1L} - m_{T_2L}|\}$, $\Delta m_{U(min)} = \min\{|m_{T_1U} - m_{T_2U}|\}$, $\Delta m_L = |m_{T_1L} - m_{T_2L}|$, $\Delta m_U = |m_{T_1U} - m_{T_2U}|$, $\Delta m_{L(max)} = \max\{|m_{T_1L} - m_{T_2L}|\}$, $\Delta m_{U(max)} = \max\{|m_{T_1U} - m_{T_2U}|\}$, $\Delta i_{L(min)} = \min\{|i_{T_1L} - i_{T_2L}|\}$, $\Delta i_{U(min)} = \min\{|i_{T_1U} - i_{T_2U}|\}$, $\Delta i_L = |i_{T_1L} - i_{T_2L}|$, $\Delta i_U = |i_{T_1U} - i_{T_2U}|$, $\Delta i_{L(max)} = \max\{|i_{T_1L} - i_{T_2L}|\}$, $\Delta i_{U(max)} = \max\{|i_{T_1U} - i_{T_2U}|\}$, $\Delta n_{L(min)} = \min\{|n_{T_1L} - n_{T_2L}|\}$, $\Delta n_{U(min)} = \min\{|n_{T_1U} - n_{T_2U}|\}$, $\Delta n_L = |n_{T_1L} - n_{T_2L}|$, $\Delta n_U = |n_{T_1U} - n_{T_2U}|$, $\Delta n_{L(max)} = \max\{|n_{T_1L} - n_{T_2L}|\}$, $\Delta n_{U(max)} = \max\{|n_{T_1U} - n_{T_2U}|\}$.

Theorem 11. *For any three IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$, $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$ *and* $T_3 = (m_{T_3}, i_{T_3}, n_{T_3})$, *the IvPFWGSM satisfies the following properties*:

i. $0 \le IvPFWGSM^1(T_1,T_2) \le 1$.
ii. $IvPFWGSM^1(T_1,T_2) = IvPFWGSM^1(T_2,T_1)$.
iii. For $T_1 = T_2$, $IvPFWGSM^1(T_1,T_2) = 1$.
iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $IvPFWGSM^1(T_1,T_3) \le IvPFWGSM^1(T_1,T_2)$ and $IvPFWGSM^1(T_1,T_3) \le IvPFWGSM^1(T_2,T_3)$.

3.5. Some Dice Similarity Measures for IvPFSs

In this subsection, some dice SMs and weighted dice SMs for IvPFSs are defined. Some basic properties of these SMs are discussed.

Definition 18. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *some dice similarity measures for these IvPFNs is defined as:*

$$IvPFDSM^1(T_1,T_2) = \frac{1}{k}\sum_{t=1}^{k}\frac{2\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) + i_{T_1U}(x_t)i_{T_2U}(x_t) \\ +n_{T_1U}(x_t)n_{T_2U}(x_t)\end{pmatrix}}{\begin{pmatrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + m^2_{T_1U}(x_t) + \\ i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t)\end{pmatrix} + \begin{pmatrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + m^2_{T_2U}(x_t) + \\ i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t)\end{pmatrix}} \tag{22}$$

$$IvPFDSM^2(T_1,T_2) = \frac{1}{k}\sum_{t=1}^{k}\frac{2\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + r_{T_1L}(x_t)r_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) + r_{T_1U}(x_t)r_{T_2U}(x_t)\end{pmatrix}}{\begin{pmatrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + r^2_{T_1L}(x_t) \\ +m^2_{T_1U}(x_t) + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t) + r^2_{T_1U}(x_t)\end{pmatrix} + \begin{pmatrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + r^2_{T_2L}(x_t) + m^2_{T_2U}(x_t) + \\ i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t) + r^2_{T_2U}(x_t)\end{pmatrix}} \tag{23}$$

$$IvPFDSM^3(T_1,T_2) = \frac{\sum_{t=1}^{k} 2\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) + i_{T_1U}(x_t)i_{T_2U}(x_t) \\ +n_{T_1U}(x_t)n_{T_2U}(x_t)\end{pmatrix}}{\sum_{t=1}^{k}\begin{pmatrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + m^2_{T_1U}(x_t) + \\ i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t)\end{pmatrix} + \sum_{t=1}^{k}\begin{pmatrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + m^2_{T_2U}(x_t) + \\ i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t)\end{pmatrix}} \tag{24}$$

$$IvPFDSM^4(T_1,T_2) = \frac{2\sum_{t=1}^{k}\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + r_{T_1L}(x_t)r_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) + r_{T_1U}(x_t)r_{T_2U}(x_t)\end{pmatrix}}{\sum_{t=1}^{k}\begin{pmatrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + r^2_{T_1L}(x_t) + \\ m^2_{T_1U}(x_t) + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t) + r^2_{T_1U}(x_t)\end{pmatrix} + \sum_{t=1}^{k}\begin{pmatrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + r^2_{T_2L}(x_t) + \\ m^2_{T_2U}(x_t) + i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t) + +r^2_{T_2U}(x_t)\end{pmatrix}} \tag{25}$$

Theorem 12. *For any three IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$, $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$ *and* $T_3 = \left(m_{T_3}, i_{T_3}, n_{T_3}\right)$, *all IvPFDSMs satisfy the following properties for* $p = 1,2,3,4$:

i. $0 \le IvPFDSM^p(T_1, T_2) \le 1$.
ii. $IvPFDSM^p(T_1, T_2) = IvPFDSM^p(T_2, T_1)$
iii. For $T_1 = T_2$, $IvPFDSM^p(T_1, T_2) = 1$.
iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $IvPFDSM^p(T_1, T_3) \le IvPFDSM^p(T_1, T_2)$ and $IvPFDSM^p(T_1, T_3) \le IvPFDSM^p(T_2, T_3)$

Proof.

(i). As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFDSM^1(T_1, T_2)$ belongs to [0, 1].
(ii). Holds trivially.
(iii). If $T_1 = T_2$ then $m_{T_1L} = m_{T_2L}$, $m_{T_1U} = m_{T_2U}$, $i_{T_1L} = i_{T_2L}$, $i_{T_1U} = i_{T_2U}$, $n_{T_1L} = n_{T_2L}$ and $n_{T_1U} = n_{T_2U}$

and then

$$\begin{aligned} &IvPFDSM^1(T_1, T_2) \\ &= \frac{1}{k}\sum_{t=1}^{k} \frac{2\left(m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t)\right)}{\begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + \\ m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix} + \begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + \\ m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix}} \\ &= \frac{1}{k}\sum_{t=1}^{k} \frac{2\left(m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t)\right)}{2\begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + \\ m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix}} \\ &= \frac{1}{k}\sum_{t=1}^{k} 1 \\ &= \frac{1}{k} k = 1. \end{aligned}$$

Definition 19. *For any two IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$ *and* $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$, *some weighted dice similarity measures between these IvPFNs are defined as:*

$$\begin{aligned} &IvPFWDSM^1(T_1, T_2) \\ &= \sum_{t=1}^{k} \omega_t \frac{2\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) + i_{T_1U}(x_t)i_{T_2U}(x_t) \\ +n_{T_1U}(x_t)n_{T_2U}(x_t) \end{pmatrix}}{\begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) + \\ i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix} + \begin{pmatrix} m_{T_2L}^2(x_t) + i_{T_2L}^2(x_t) + n_{T_2L}^2(x_t) + m_{T_2U}^2(x_t) + \\ i_{T_2U}^2(x_t) + n_{T_2U}^2(x_t) \end{pmatrix}} \end{aligned} \tag{26}$$

$$\begin{aligned} &IvPFWDSM^2(T_1, T_2) \\ &= \sum_{t=1}^{k} \omega_t \frac{2\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + r_{T_1L}(x_t)r_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) + r_{T_1U}(x_t)r_{T_2U}(x_t) \end{pmatrix}}{\begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + r_{T_1L}^2(x_t) \\ +m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) + r_{T_1U}^2(x_t) \end{pmatrix} + \begin{pmatrix} m_{T_2L}^2(x_t) + i_{T_2L}^2(x_t) + n_{T_2L}^2(x_t) + r_{T_2L}^2(x_t) + m_{T_2U}^2(x_t) + \\ i_{T_2U}^2(x_t) + n_{T_2U}^2(x_t) + r_{T_2U}^2(x_t) \end{pmatrix}} \end{aligned} \tag{27}$$

$$\begin{aligned} &IvPFWDSM^3(T_1, T_2) \\ &= \frac{2\sum_{t=1}^{k} \omega_t^2 \begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) + i_{T_1U}(x_t)i_{T_2U}(x_t) \\ +n_{T_1U}(x_t)n_{T_2U}(x_t) \end{pmatrix}}{\sum_{t=1}^{k} \omega_t^2 \begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) + \\ i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix} + \sum_{t=1}^{k} \omega_t^2 \begin{pmatrix} m_{T_2L}^2(x_t) + i_{T_2L}^2(x_t) + n_{T_2L}^2(x_t) + m_{T_2U}^2(x_t) + \\ i_{T_2U}^2(x_t) + n_{T_2U}^2(x_t) \end{pmatrix}} \end{aligned} \tag{28}$$

$$\begin{aligned} &IvPFWDSM^4(T_1, T_2) \\ &= \frac{2\sum_{t=1}^{k} \omega_t^2 \begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + r_{T_1L}(x_t)r_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) + r_{T_1U}(x_t)r_{T_2U}(x_t) \end{pmatrix}}{\sum_{t=1}^{k} \omega_t^2 \begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + r_{T_1L}^2(x_t) + \\ m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) + r_{T_1U}^2(x_t) \end{pmatrix} + \sum_{t=1}^{k} \omega_t^2 \begin{pmatrix} m_{T_2L}^2(x_t) + i_{T_2L}^2(x_t) + n_{T_2L}^2(x_t) + r_{T_2L}^2(x_t) + \\ m_{T_2U}^2(x_t) + i_{T_2U}^2(x_t) + n_{T_2U}^2(x_t) + +r_{T_2U}^2(x_t) \end{pmatrix}} \end{aligned} \tag{29}$$

where weight vector $\omega = (\omega_1, \dots \dots \dots, \omega_k)^T$ *is with a condition that for* $t = 1,2, \dots \dots \dots, k$ $\omega_t \in [0,1]$ *and* $\sum_{t=1}^{k} \omega_t = 1$.

Theorem 13. *For any three IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$, $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$ *and* $T_3 = (m_{T_3}, i_{T_3}, n_{T_3})$, *all IvPFWDSMs satisfy the following properties for* $p = 1,2,3,4$:

i. $0 \le IvPFWDSM^p(T_1, T_2) \le 1$.
ii. $IvPFWDSM^p(T_1, T_2) = IvPFWDSM^p(T_2, T_1)$
iii. For $T_1 = T_2$, $IvPFWDSM^p(T_1, T_2) = 1$.
iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $IvPFWDSM^p(T_1, T_3) \le IvPFWDSM^p(T_1, T_2)$ and $IvPFWDSM^p(T_1, T_3) \le IvPFWDSM^p(T_2, T_3)$

Proof.

(i). As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFGDSM^1(T_1, T_2)$ belongs to [0, 1],.
(ii). Holds trivially.
(iii). If $T_1 = T_2$ then $m_{T_1L} = m_{T_2L}$, $m_{T_1U} = m_{T_2U}$, $i_{T_1L} = i_{T_2L}$, $i_{T_1U} = i_{T_2U}$, $n_{T_1L} = n_{T_2L}$ and $n_{T_1U} = n_{T_2U}$.

Then:

$$\begin{aligned}
&IvPFGDSM^1(T_1, T_2) \\
&= \sum_{t=1}^{k} \omega_t \frac{2\left(m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t)\right)}{\begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + \\ m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix} + \begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + \\ m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix}} \\
&= \sum_{t=1}^{k} \omega_t \frac{2\left(m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t)\right)}{2\begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + \\ m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix}} \\
&= \sum_{t=1}^{k} \omega_t = 1.
\end{aligned}$$

Similarly we can prove the others for $p = 2,3,4$. □

Definition 20. *For any two IvPFNs* $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ *and* $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$, *some generalized dice similarity measures between these IvPFNs are defined as:*

$$\begin{aligned}
&IvPFGDSM^1(T_1, T_2) \\
&= \frac{1}{k}\sum_{t=1}^{k} \frac{\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) + i_{T_1U}(x_t)i_{T_2U}(x_t) \\ + n_{T_1U}(x_t)n_{T_2U}(x_t) \end{pmatrix}}{\lambda\begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + m_{T_1U}^2(x_t) + \\ i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix} + (1-\lambda)\begin{pmatrix} m_{T_2L}^2(x_t) + i_{T_2L}^2(x_t) + n_{T_2L}^2(x_t) + \\ m_{T_2U}^2(x_t) + i_{T_2U}^2(x_t) + n_{T_2U}^2(x_t) \end{pmatrix}}
\end{aligned} \tag{30}$$

$$\begin{aligned}
&IvPFGDSM^2(T_1, T_2) \\
&= \frac{1}{k}\sum_{t=1}^{k} \frac{\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + r_{T_1L}(x_t)r_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ + i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) + r_{T_1U}(x_t)r_{T_2U}(x_t) \end{pmatrix}}{\lambda\begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + r_{T_1L}^2(x_t) \\ + m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) + r_{T_1U}^2(x_t) \end{pmatrix} + (1-\lambda)\begin{pmatrix} m_{T_2L}^2(x_t) + i_{T_2L}^2(x_t) + n_{T_2L}^2(x_t) + r_{T_2L}^2(x_t) \\ + m_{T_2U}^2(x_t) + i_{T_2U}^2(x_t) + n_{T_2U}^2(x_t) + r_{T_2U}^2(x_t) \end{pmatrix}}
\end{aligned} \tag{31}$$

$$\begin{aligned}
&IvPFGDSM^3(T_1, T_2) \\
&= \frac{\sum_{t=1}^{k}\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) + i_{T_1U}(x_t)i_{T_2U}(x_t) \\ + n_{T_1U}(x_t)n_{T_2U}(x_t) \end{pmatrix}}{\lambda\sum_{t=1}^{k}\begin{pmatrix} m_{T_1L}^2(x_t) + i_{T_1L}^2(x_t) + n_{T_1L}^2(x_t) + \\ m_{T_1U}^2(x_t) + i_{T_1U}^2(x_t) + n_{T_1U}^2(x_t) \end{pmatrix} + (1-\lambda)\sum_{t=1}^{k}\begin{pmatrix} m_{T_2L}^2(x_t) + i_{T_2L}^2(x_t) + n_{T_2L}^2(x_t) + \\ m_{T_2U}^2(x_t) + i_{T_2U}^2(x_t) + n_{T_2U}^2(x_t) \end{pmatrix}}
\end{aligned} \tag{32}$$

$$IvPFGDSM^4(T_1,T_2) = \frac{\sum_{t=1}^{k}\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + r_{T_1L}(x_t)r_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) + r_{T_1U}(x_t)r_{T_2U}(x_t) \end{pmatrix}}{\lambda\sum_{t=1}^{k}\begin{pmatrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + r^2_{T_1L}(x_t) + \\ m^2_{T_1U}(x_t) + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t) + r^2_{T_1U}(x_t) \end{pmatrix} + (1-\lambda)\sum_{t=1}^{k}\begin{pmatrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + r^2_{T_2L}(x_t) + \\ m^2_{T_2U}(x_t) + i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t) + +r^2_{T_2U}(x_t) \end{pmatrix}} \tag{33}$$

where $0 \le \lambda \le 1$.

Theorem 14. *For any three IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$, $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$ *and* $T_3 = \left(m_{T_3}, i_{T_3}, n_{T_3}\right)$ *then all IvPFDSMs satisfy the following properties for* $t = 1,2,3,4$:

i. $0 \le IvPFGDSM^t(T_1,T_2) \le 1$.
ii. $IvPFGDSM^t(T_1,T_2) = IvPFGDSM^t(T_2,T_1)$.
iii. For $T_1 = T_2$, $IvPFGDSM^t(T_1,T_2) = 1$.
iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $IvPFGDSM^t(T_1,T_3) \le IvPFGDSM^t(T_1,T_2)$ and $IvPFGDSM^t(T_1,T_3) \le IvPFGDSM^t(T_2,T_3)$.

Definition 21. *For any two IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$ *and* $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$, *some weighted dice similarity measures between IvPFNs are defined as:*

$$IvPFWGDSM^1(T_1,T_2) = \sum_{t=1}^{k}\omega_t\frac{\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) \end{pmatrix}}{\lambda\begin{pmatrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + \\ m^2_{T_1U}(x_t) + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t) \end{pmatrix} + (1-\lambda)\begin{pmatrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + \\ m^2_{T_2U}(x_t) + i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t) \end{pmatrix}} \tag{34}$$

$$IvPFWGDSM^2(T_1,T_2) = \sum_{t=1}^{k}\omega_t\frac{\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + r_{T_1L}(x_t)r_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) + r_{T_1U}(x_t)r_{T_2U}(x_t) \end{pmatrix}}{\lambda\begin{pmatrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + r^2_{T_1L}(x_t) \\ +m^2_{T_1U}(x_t) + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t) + r^2_{T_1U}(x_t) \end{pmatrix} + (1-\lambda)\begin{pmatrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + r^2_{T_2L}(x_t) \\ +m^2_{T_2U}(x_t) + i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t) + r^2_{T_2U}(x_t) \end{pmatrix}} \tag{35}$$

$$IvPFWGDSM^3(T_1,T_2) = \frac{\sum_{t=1}^{k}\omega_t^2\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) \end{pmatrix}}{\lambda\sum_{t=1}^{k}\omega_t^2\begin{pmatrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) \\ +m^2_{T_1U}(x_t) + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t) \end{pmatrix} + (1-\lambda)\sum_{t=1}^{k}\omega_t^2\begin{pmatrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) \\ +m^2_{T_2U}(x_t) + i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t) \end{pmatrix}} \tag{36}$$

$$IvPFWGDSM^4(T_1,T_2) = \frac{\sum_{t=1}^{k}\omega_t^2\begin{pmatrix} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + r_{T_1L}(x_t)r_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) + r_{T_1U}(x_t)r_{T_2U}(x_t) \end{pmatrix}}{\lambda\sum_{t=1}^{k}\omega_t^2\begin{pmatrix} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + r^2_{T_1L}(x_t) + \\ m^2_{T_1U}(x_t) + i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t) + r^2_{T_1U}(x_t) \end{pmatrix} + (1-\lambda)\sum_{t=1}^{k}\omega_t^2\begin{pmatrix} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + r^2_{T_2L}(x_t) + \\ m^2_{T_2U}(x_t) + i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t) + +r^2_{T_2U}(x_t) \end{pmatrix}} \tag{37}$$

where weight vector $\omega = (\omega_1, \ldots\ldots\ldots, \omega_k)^T$ *have a condition that for* $t = 1,2,\ldots\ldots\ldots,k$ $\omega_t \in [0,1]$ *and* $\sum_{t=1}^{k}\omega_t = 1$.

Theorem 15. *For any three IvPFNs* $T_1 = \left(m_{T_1}, i_{T_1}, n_{T_1}\right)$, $T_2 = \left(m_{T_2}, i_{T_2}, n_{T_2}\right)$ *and* $T_3 = \left(m_{T_3}, i_{T_3}, n_{T_3}\right)$, *all IvPFWDSMs satisfy the following properties for* $t = 1,2,3,4$:

i. $0 \le IvPFWGDSM^t(T_1,T_2) \le 1$.
ii. $IvPFWGDSM^t(T_1,T_2) = IvPFWGDSM^t(T_2,T_1)$
iii. For $T_1 = T_2$, $IvPFWGDSM^t(T_1,T_2) = 1$.
iv. Consider $T_1 \subseteq T_2 \subseteq T_3$, then $IvPFWGDSM^t(T_1,T_3) \le IvPFWGDSM^t(T_1,T_2)$ and $IvPFWGDSM^t(T_1,T_3) \le IvPFWGDSM^t(T_2,T_3)$

4. Applications for Strategy Decision Making and Mineral Fields Recognition

In this section, applications for strategy decision making and mineral field recognition are developed with the help of numerical examples that show the reliability of proposed SMs.

4.1. Numerical Example for Strategy Decision Making

A company wants to launch a new product and board of governors have to decide one strategy. For this purpose, there are three strategies to be selected shown as follows:

1. g_1: Make a product for rich persons
2. g_2: Make a product for every persons
3. g_3: Make a product for poor persons

In order to do the best selection, it is necessary to compare these three strategies with popular product in the existing market, so we give a best strategy g: a popular product in the existing market.

In addition, in order to evaluate these strategies, there are the following five attributes (which weight vector $= (0.25,0.2,0.15,0.18,0.22)^T$) to be used:

1. S_1: Risk of loss
2. S_2: Barriers in the development of business
3. S_3: Impact on society
4. S_4: Impact on environment
5. S_6: Growth analysis

The decision maker gives the evaluation values for strategies according to attributes which are shown in Table 1.

Table 1. Decision values for strategy decision making.

	g_1	g_2	g_3	g
S_1	$\begin{pmatrix}[0.26,0.31],\\ [0.12,0.24],\\ [0.21,0.39]\end{pmatrix}$	$\begin{pmatrix}[0.32,0.37],\\ [0.15,0.28],\\ [0.05,0.12]\end{pmatrix}$	$\begin{pmatrix}[0.23,0.46],\\ [0.1,0.15],\\ [0.31,0.36]\end{pmatrix}$	$\begin{pmatrix}[0.05,0.1],\\ [0.18,0.29],\\ [0.43,0.57]\end{pmatrix}$
S_2	$\begin{pmatrix}[0.25,0.46],\\ [0.03,0.13],\\ [0.17,0.23]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.35],\\ [0.09,0.17],\\ [0.37,0.47]\end{pmatrix}$	$\begin{pmatrix}[0.41,0.56],\\ [0.03,0.09],\\ [0.14,0.27]\end{pmatrix}$	$\begin{pmatrix}[0.45,0.53],\\ [0.1,0.17],\\ [0.01,0.13]\end{pmatrix}$
S_3	$\begin{pmatrix}[0.08,0.26],\\ [0.16,0.37],\\ [0.02,0.29]\end{pmatrix}$	$\begin{pmatrix}[0.25,0.31],\\ [0.21,0.29],\\ [0.3,0.39]\end{pmatrix}$	$\begin{pmatrix}[0.07,0.16],\\ [0.24,0.32],\\ [0.47,0.51]\end{pmatrix}$	$\begin{pmatrix}[0.23,0.41],\\ [0.07,0.17],\\ [0.11,0.26]\end{pmatrix}$
S_4	$\begin{pmatrix}[0.2,0.4],\\ [0.1,0.3],\\ [0.1,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.14,0.25],\\ [0.13,0.19],\\ [0.41,0.53]\end{pmatrix}$	$\begin{pmatrix}[0.17,0.21],\\ [0.07,0.14],\\ [0.51,0.61]\end{pmatrix}$	$\begin{pmatrix}[0.14,0.28],\\ [0.12,0.24],\\ [0.06,0.36]\end{pmatrix}$
S_5	$\begin{pmatrix}[0.48,0.57],\\ [0.22,0.3],\\ [0.0,0.07]\end{pmatrix}$	$\begin{pmatrix}[0.31,0.41],\\ [0.02,0.09],\\ [0.39,0.47]\end{pmatrix}$	$\begin{pmatrix}[0.35,0.39],\\ [0.11,0.23],\\ [0.06,0.21]\end{pmatrix}$	$\begin{pmatrix}[0.19,0.31],\\ [0.04,0.08],\\ [0.49,0.59]\end{pmatrix}$

The similarity measure of three alternatives$g_1, g_2$and$g_3$with g with respect to weight vector $w = (0.25,0.2,0.15,0.18,0.22)^T$ are calculated by using the formulas of similarity measures, which are shown in Table 2.

Table 2. Similarity measures for strategy decision making.

$SM's$	(g_1, g)	(g_2, g)	(g_3, g)
$IvPFWCSM^1$	**0.7961**	0.7794	0.7898
$IvPFWCSM^2$	**0.8341**	0.7811	0.7758
$IvPFWCsSM^1$	**0.8931**	0.8720	0.8416
$IvPFWCsSM^2$	0.6643	**0.7347**	0.6754
$IvPFWCsSM^3$	**0.8931**	0.8551	0.8404
$IvPFWCsSM^4$	**0.6101**	0.5558	0.4932
$IvPFWCtSM^1$	**0.6574**	0.6195	0.5695
$IvPFWCtSM^2$	**0.6254**	0.5909	0.5679
$IvPFWStSM^1$	**0.7111**	0.6576	0.6405
$IvPFWGSM^1$	0.7843	0.7937	**0.8168**
$IvPFWDSM^1$	**0.7886**	0.7621	0.7625
$IvPFWDSM^2$	**0.8317**	0.7791	0.7744
$IvPFWDSM^3$	0.7396	0.7592	**0.7677**
$IvPFWDSM^4$	**0.2772**	0.2166	0.2738
$IvPFWGDSM^1$	0.7537	0.7948	**0.8021**
$IvPFWGDSM^2$	**0.8187**	0.7477	0.7578
$IvPFWGDSM^3$	0.6997	0.7457	**0.7665**
$IvPFWGDSM^4$	**0.2716**	0.2078	0.2668

From Table 2, we can know the different similarity definitions can get the different similarity measures, however, in 18 similarity measures, there are 13 similarity measures in which (g_1, g) is the biggest, there are one similarity measure in which (g_2, g) is the biggest, and there are 4 similarity measures in which (g_3, g) is the largest.

So we can get g_1 is best option for company is to launch product for rich persons.

4.2. Numerical Example for Mineral Fields Recognition

Let us consider three kinds of mineral fields g_1, g_2 and g_3. Each of them is featured by five minerals $\{s_1, s_2, s_3, s_4, s_5\}$ and the weight vector of minerals is $= (0.25, 0.2, 0.15, 0.18, 0.22)^T$. The evaluation values for three kinds of mineral fields under the five minerals are shown in Table 3.

Now consider an existing best mineral field g and we have to check that which field is most similar to g. Experts evaluate each field under the consideration of five minerals.

Table 3. Decision values for mineral fields recognition.

	g_1	g_2	g_3	g
s_1	$\begin{pmatrix}[0.37,0.49],\\ [0.03,0.11],\\ [0.34,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.23,0.33],\\ [0.13,0.20],\\ [0.11,0.19]\end{pmatrix}$	$\begin{pmatrix}[0.12,0.35],\\ [0.07,0.18],\\ [0.22,0.32]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.28],\\ [0.07,0.15],\\ [0.31,0.50]\end{pmatrix}$
s_2	$\begin{pmatrix}[0.07,0.23],\\ [0.11,0.29],\\ [0.21,0.33]\end{pmatrix}$	$\begin{pmatrix}[0.13,0.31],\\ [0.02,0.13],\\ [0.22,0.44]\end{pmatrix}$	$\begin{pmatrix}[0.26,0.44],\\ [0.02,0.08],\\ [0.16,0.27]\end{pmatrix}$	$\begin{pmatrix}[0.33,0.51],\\ [0.02,0.17],\\ [0.20,0.21]\end{pmatrix}$
s_3	$\begin{pmatrix}[0.27,0.36],\\ [0.09,0.19],\\ [0.13,0.18]\end{pmatrix}$	$\begin{pmatrix}[0.09,0.19],\\ [0.17,0.31],\\ [0.22,0.36]\end{pmatrix}$	$\begin{pmatrix}[0.14,0.19],\\ [0.21,0.32],\\ [0.36,0.41]\end{pmatrix}$	$\begin{pmatrix}[0.17,0.37],\\ [0.04,0.14],\\ [0.22,0.36]\end{pmatrix}$
s_4	$\begin{pmatrix}[0.09,0.43],\\ [0.12,0.21],\\ [0.14,0.35]\end{pmatrix}$	$\begin{pmatrix}[0.12,0.21],\\ [0.08,0.13],\\ [0.24,0.49]\end{pmatrix}$	$\begin{pmatrix}[0.13,0.19],\\ [0.08,0.22],\\ [0.48,0.58]\end{pmatrix}$	$\begin{pmatrix}[0.12,0.24],\\ [0.11,0.21],\\ [0.36,0.49]\end{pmatrix}$
s_5	$\begin{pmatrix}[0.16,0.48],\\ [0.14,0.30],\\ [0.01,0.11]\end{pmatrix}$	$\begin{pmatrix}[0.13,0.34],\\ [0.01,0.23],\\ [0.31,0.42]\end{pmatrix}$	$\begin{pmatrix}[0.28,0.38],\\ [0.10,0.20],\\ [0.14,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.15,0.26],\\ [0.09,0.17],\\ [0.43,0.56]\end{pmatrix}$

The similarity measures of three alternatives with g with respect to weight vector $w = (0.25,0.2,0.15,0.18,0.22)^T$ are calculated by using the formulas of similarity measures, which are shown in Table 4.

Table 4. Similarity measures for mineral field recognition.

$SM's$	(g_1, g)	(g_2, g)	(g_3, g)
$IvPFWCSM^1$	0.8154	0.9028	**0.9382**
$IvPFWCSM^2$	0.8413	0.9100	**0.9255**
$IvPFWCsSM^1$	0.8983	0.9436	**0.9565**
$IvPFWCsSM^2$	0.7963	0.8998	**0.9112**
$IvPFWCsSM^3$	0.8963	0.9342	**0.9472**
$IvPFWCsSM^4$	0.6449	0.8118	**0.8305**
$IvPFWCtSM^1$	0.6468	0.7273	**0.7607**
$IvPFWCtSM^2$	0.6093	0.6986	**0.7379**
$IvPFWStSM^1$	0.6911	0.7701	**0.7952**
$IvPFWGSM^1$	0.7426	0.7702	**0.8132**
$IvPFWDSM^1$	0.8030	0.8877	**0.9252**
$IvPFWDSM^2$	0.8396	0.9092	**0.9244**
$IvPFWDSM^3$	0.8010	0.8902	**0.9252**
$IvPFWDSM^4$	0.3039	0.2822	**0.3195**
$IvPFWGDSM^1$	0.7736	0.8122	**0.8990**
$IvPFWGDSM^2$	0.8584	0.9129	**0.9260**
$IvPFWGDSM^3$	0.7702	0.8067	**0.8851**
$IvPFWGDSM^4$	0.3122	0.2832	**0.3200**

From Table 4, we can obtain that the g_3 is most similar to g, so we can select the g_3.

5. Advantages

In this section, we explain the advantages of the proposed SMs.

5.1. Some Special Cases

We prove the generalization of proposed works. For this, we consider two IvPFNs $T_1 = (m_{T_1}, i_{T_1}, n_{T_1})$ and $T_2 = (m_{T_2}, i_{T_2}, n_{T_2})$

$$IvPFCSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{\begin{array}{c} m_{T_1L}(x_t)m_{T_2L}(x_t) + i_{T_1L}(x_t)i_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +i_{T_1U}(x_t)i_{T_2U}(x_t) + n_{T_1U}(x_t)n_{T_2U}(x_t) \end{array}}{\sqrt{\begin{array}{c} m^2_{T_1L}(x_t) + i^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + m^2_{T_1U}(x_t) \\ +i^2_{T_1U}(x_t) + n^2_{T_1U}(x_t) \end{array}} \sqrt{\begin{array}{c} m^2_{T_2L}(x_t) + i^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + m^2_{T_2U}(x_t) + \\ i^2_{T_2U}(x_t) + n^2_{T_2U}(x_t) \end{array}}}$$

1. When lower and upper value of intervals becomes equal, then the above equation becomes SM for PFSs:

$$PFCSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{m_{T_1}(x_t)m_{T_2}(x_t) + i_{T_1}(x_t)i_{T_2}(x_t) + n_{T_1}(x_t)n_{T_2}(x_t)}{\sqrt{m^2_{T_1}(x_t) + i^2_{T_1}(x_t) + n^2_{T_1}(x_t)}\sqrt{m^2_{T_2}(x_t) + i^2_{T_2}(x_t) + n^2_{T_2}(x_t)}}$$

2. For $i_{T_1} = [0,0]$ the above equation becomes SM for interval valued intuitionistic fuzzy number:

$$IvIFCSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{\begin{array}{c} m_{T_1L}(x_t)m_{T_2L}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t) \\ +n_{T_1U}(x_t)n_{T_2U}(x_t) \end{array}}{\sqrt{\begin{array}{c} m^2_{T_1L}(x_t) + n^2_{T_1L}(x_t) + m^2_{T_1U}(x_t) \\ +n^2_{T_1U}(x_t) \end{array}} \sqrt{\begin{array}{c} m^2_{T_2L}(x_t) + n^2_{T_2L}(x_t) + m^2_{T_2U}(x_t) \\ +n^2_{T_2U}(x_t) \end{array}}}$$

3. For $i_{T_1} = [0,0]$ and the upper and lower values of membership and non-membership intervals become equal, then the above equation becomes SM for intuitionistic fuzzy number:

$$IFCSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{m_{T_1}(x_t)m_{T_2}(x_t) + n_{T_1L}(x_t)n_{T_2L}(x_t)}{\sqrt{m_{T_1}^2(x_t) + n_{T_1}^2(x_t)}\sqrt{m_{T_2}^2(x_t) + n_{T_2}^2(x_t)}}$$

4. For $i_{T_1} = [0,0]$, $n_{T_1} = [0,0]$ and $i_{T_2} = [0,0]$, $n_{T_2} = [0,0]$, the above equation becomes SM for IvFN:

$$IvFCSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{m_{T_1L}(x_t)m_{T_2L}(x_t) + m_{T_1U}(x_t)m_{T_2U}(x_t)}{\sqrt{m_{T_1L}^2(x_t) + m_{T_1U}^2(x_t)}\sqrt{m_{T_2L}^2(x_t) + m_{T_2U}^2(x_t)}}$$

5. For $i_{T_1} = [0,0]$, $n_{T_1} = [0,0]$ and $i_{T_2} = [0,0]$, $n_{T_2} = [0,0]$ and the upper and lower values of membership intervals become equal, then the above equation becomes SM for FN:

$$FCSM^1(T_1, T_2) = \frac{1}{k}\sum_{t=1}^{k} \frac{m_{T_1}(x_t)m_{T_2}(x_t)}{\sqrt{m_{T_1}^2(x_t) + m_{T_2}^2(x_t)}}$$

Similarly, we can reduce all other similarities in interval-valued intuitionistic, intuitionistic and picture fuzzy environment.

Thus, we can know the proposed SMs are more general than some existing SMs.

5.2. Comparative Study

The main advantage of proposed works is that the existing SMs cannot handle the information given in IvPFNs, but they can handle the information given in intuitionistic, interval-valued intuitionistic and picture fuzzy environment. Hence, the proposed SMs are more generalized than those of existing SMs.

Example 1. *Here, an example for interval-valued intuitionistic fuzzy information has been taken from* [15] *and solved by the proposed SMs. An investment company wants to invest its money in some business and they have four alternatives* $\{g_1, g_2, g_3, g_4\}$ *and must select one from these alternatives. Thus, they evaluate these alternatives on the base of three attributes* $\{s_1, s_2, s_3\}$ *with a weight vector* $(0.35, 0.25, 0.40)$, *the evaluation values are shown in Table 5.*

Now we can use the similarity measure of each alternative with the ideal alternative to select the best one.

Table 5. Decision makers for comparative study.

	s_1	s_2	s_3
g_1	$\begin{pmatrix}[0.4,0.5],\\ [0.0,0.0]\\ [0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\ [0.0,0.0]\\ [0.2,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.3],\\ [0.0,0.0]\\ [0.5,0.6]\end{pmatrix}$
g_2	$\begin{pmatrix}[0.6,0.7],\\ [0.0,0.0]\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.7],\\ [0.0,0.0]\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.7],\\ [0.0,0.0]\\ [0.1,0.2]\end{pmatrix}$
g_3	$\begin{pmatrix}[0.3,0.6],\\ [0.0,0.0]\\ [0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.6],\\ [0.0,0.0]\\ [0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.6],\\ [0.0,0.0]\\ [0.1,0.3]\end{pmatrix}$
g_4	$\begin{pmatrix}[0.7,0.8],\\ [0.0,0.0]\\ [0.1,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.7],\\ [0.0,0.0]\\ [0.1,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.3,0.4],\\ [0.0,0.0]\\ [0.1,0.2]\end{pmatrix}$

By using the above information, the interval-valued intuitionistic fuzzy cosine similarity measure (IvIFCSM) can be found as given:

$$IvIFCSM^1(g_1, g) = 0.5645,$$
$$IvIFCSM^1(g_2, g) = 0.8637,$$
$$IvIFCSM^1(g_3, g) = 0.7768,$$
$$IvIFCSM^1(g_4, g) = 0.7801.$$

These results are similar as in [15]. Thus, this proves the effectiveness of the proposed works.

6. Conclusions

In this paper, the existing SMs in picture fuzzy environment are discussed and their limitations are discussed that the existing SMs cannot handle the information given in IvPFNs. To overcome this problem, the SMs for interval-valued picture fuzzy information are proposed, include cosine SMs, SMs using cosine function, SMs using cotangent function, set-theoretic SM, grey SM, dice and generalized dice SMs for IvPFSs, and some basic properties of all these SMs are also discussed, then the proposed SMs are applied to decision making problems with the help of numerical examples. In addition, advantages of proposed works are also discussed. In future, some other tools of correlation coefficient and distance measure could also be developed. Further, to improve this work, such SMs can be defined in T-spherical fuzzy environments [45–46] and interval valued T-spherical fuzzy environment [47] where one has a variety of choices for the selection of membership, abstinence and non-membership grades.

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