Journal of
Risk and Financial
Management

MDPI

Article

The Solvency II Standard Formula, Linear Geometry, and Diversification

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Academic Editor: Michael McAleer
Received: 28 February 2017; Accepted: 10 May 2017; Published: 18 May 2017

Abstract: The core of risk aggregation in the Solvency II Standard Formula is the so-called square root formula. We argue that it should be seen as a means for the aggregation of different risks to an overall risk rather than being associated with variance-covariance based risk analysis. Considering the Solvency II Standard Formula from the viewpoint of linear geometry, we immediately find that it defines a norm and therefore provides a homogeneous and sub-additive tool for risk aggregation. Hence, Euler's Principle for the reallocation of risk capital applies and yields explicit formulas for capital allocation in the framework given by the Solvency II Standard Formula. This gives rise to the definition of *diversification functions*, which we define as monotone, subadditive, and homogeneous functions on a convex cone. Diversification functions constitute a class of models for the study of the aggregation of risk and diversification. The aggregation of risk measures using a diversification function preserves the respective properties of these risk measures. Examples of diversification functions are given by seminorms, which are monotone on the convex cone of non-negative vectors. Each L^p norm has this property, and any scalar product given by a non-negative positive semidefinite matrix does as well. In particular, the Standard Formula is a diversification function and hence a risk measure that preserves homogeneity, subadditivity and convexity.

Keywords: Solvency II; standard formula; risk measure; diversification; aggregation; monotony; homogeneity; subadditivity; Euler's Principle; capital allocation

MSC: 91B30

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1. Introduction

The Solvency II standard formula is a means to assign the so-called solvency capital requirement to an insurance or reinsurance company. The undertaking has to have enough own funds to cover its capital requirement and the ratio of both is called the solvency ratio, which thereby should be greater or at least be equal to 1.

The solvency capital requirement is the sum of the basic solvency capital requirement – which aggregates the market, life, non-life, health, and counterparty risk – and the adjustments for operational risk, deferred taxes and others.

Let us denote the solvency capital requirements for the modules market, life, non-life, health, and counterparty risk with $S_1, \ldots, S_5$ respectively and combine them as a vector $S = (S_1, \ldots, S_5)^{\mathrm{T}}$. Then, the basic solvency capital requirement is derived by the formula:

$$\text{basic solvency capital requirement} = \sqrt{S^{\mathrm{T}} A S}, \tag{1}$$

where A is a positive definite matrix of correlation parameters (European Parliament and Council 2009, Appendix IV).

The risk modules themselves consist of sub-modules that are aggregated in the same manner. The market risk module, for example, consists of the interest rate, equity, spread, property, currency, and concentration risk sub-modules, which are aggregated by a similar formula, but with a different matrix A.

In the following, we will focus on the square-root Formula (1). All the other features of the Solvency II standard formula within the modules and sub-modules are not in our scope.

There has been some debate whether or not Formula (1) is a suitable way for the aggregation of risk, and how to allocate risk capital in this framework (cf. e.g., De Angelis and Granito 2015; Dittrich et al. 2003; Filipovic 2009). However, all these investigations assume, to the best of our knowledge, that there is an additive decomposition of some portfolio (or aggregated outcome) X in terms of sub-portfolios (or contributions) $X_1, \ldots, X_N$:

$$X = \sum_{k=1}^{N} X_k. \tag{2}$$

This is also assumed in general and in insurance-related literature on risk aggregation and allocation (cf. e.g., Bauer and Zanjani 2015; Boonen et al. 2017; Denault 2001; Dhaene et al. 2012; Kalkbrener 2005; Rueschendorf 2013; Tasche 2008).

This means that the information on the aggregation of several risks and the diversification effects are encoded in the *joint* distribution of the marginals $X_1, \ldots, X_N$, which then determines the distribution of X. This amounts to knowing the copula of the joint distribution of the marginals.

In this paper, we take a different position and do *not* assume that we have a model with an additive structure (2) and do *not* assume that we have a copula, or joint distribution. Instead, we explore the case in which the joint distribution is unknown—and ask whether Formula (1) may be a sound way for the aggregation of risk in case one does not know everything about the joint distribution, i.e., the copula.

Our point of view can be interpreted in the way that we try to find a good model for diversification, which is as easy as possible—however as feasible as needed—and which does not need, in particular, all the information on the model and its parameters that are unavoidable, when one wants to build up a joint distribution of risks of different nature.

Our main findings are:

- The Standard Formula (1) allows for the Euler principle of capital allocation, *independent of any assumption on distributions*. It even allows for diversification (i.e., is sub-additive) independently of whether the underlying risk measure is sub-additive or not.
- The Standard Formula is an example of a general principle of how to aggregate risk, namely, an example of a diversification function as defined below. In particular, the Standard Formula (1) can be interpreted as a risk measure.
- It is important that the correlation parameters of the Standard Formula (i.e., the elements of A) are not negative. This is crucial in order to obtain the properties of a risk measure.

We will specify these statements in the following. The article is organized as follows:

In Section 2, we introduce diversification functions—which can be applied to the risks of a given portfolio, or to the risks of sub-portfolios of a portfolio, or to the risks of business lines within a company, or generally speaking to any "portfolio of risks".

In Section 3, we establish Euler's principle of capital allocation for diversification functions.

In Section 4, we apply the theory to the aggregation of risk in the Solvency II standard formula.

Section 5 is the conclusion with a summary of the results.

2. Diversification Functions

A risk functional is defined on a domain $\mathcal{X} \subseteq L^0$ of random variables (risks) with values in $\mathbb{R} \cup \{\infty\}$. We assume that $\mathcal{X}$ is a convex cone, i.e., $\alpha X + \beta Y \in \mathcal{X}$ for all $X, Y \in \mathcal{X}$ and $\alpha, \beta > 0$. A risk functional R may have one or more of these properties[1]:

- R is *monotone*, if $R(X) \geq R(Y)$ whenever $X \leq Y$ a.s. In this case, the risk functional is called a *risk measure*.
- R is *subadditive*, if $R(X+Y) \leq R(X) + R(Y)$.
- R is *homogeneous*, if $R(tX) = tR(X)$ for all $t > 0$.
- R is *cash invariant*, if $R(X+a) = R(X) - a$ for all $a \in \mathbb{R}$. (For this to make sense, we have to assume that $\mathcal{X}$ contains all constants $a \in \mathbb{R}$.)
- R is *convex*, if $R(\alpha X + (1-\alpha)Y) \leq \alpha R(X) + (1-\alpha)R(Y)$ for all $\alpha \in (0,1)$.
- R is *version independent*, if $R(X) = R(Y)$ for all $X \overset{d}{=} Y$.
- R is *comonotone additive*, if $R(X+Y) = R(X) + R(Y)$ for X, Y comonotone.

Note that a homogeneous risk functional is subadditive if and only if it is convex.

Definition 1. *Let $B \subseteq (\mathbb{R} \cup \{\infty\})^n$ be a convex cone and $f : B \to \mathbb{R} \cup \{\infty\}$ a function. Then,*

- *f is* monotone (non-decreasing), *if $f(r) \leq f(s)$ for all $r, s \in B$ with $r_k \leq s_k$ for all $1 \leq k \leq n$.*
- *f is* subadditive, *if $f(r+s) \leq f(r) + f(s)$ for all $r, s \in B$.*
- *f is* homogeneous, *if $f(tr) = tf(r)$ for all $r \in B$ and $t > 0$.*
- *f is* convex, *if $f(\alpha r + (1-\alpha)s) \leq \alpha f(r) + (1-\alpha) f(s)$ for all $r, s \in B$ and $\alpha \in (0,1)$.*
- *f is* additive, *if $f(r+s) = f(r) + f(s)$ for all $r, s \in B$.*

As is the case for risk functionals, a homogeneous function is subadditive if and only if it is convex. Due to the convention that lower outcomes correspond to a higher risk, the signs in the notion of monotony of a risk functional and a function are the opposite. If the unit vectors $e_1, \ldots, e_n$ of the standard basis in $\mathbb{R}^n$ are elements of B, monotony is equivalent to monotony in every argument, i.e.,

$$f(x) \leq f(x + ce_k) \text{ for all } x \in B, c > 0, 1 \leq k \leq n. \tag{3}$$

Note that we do not propose a property so as to maintain cash invariance. Cash invariance might not be a helpful concept for the study of diversification effects within a company or a portfolio.

Lemma 1. *Let*

$$R_k : \mathcal{X} \to B_k \subseteq \mathbb{R} \cup \{\infty\} \quad (1 \leq k \leq n) \tag{4}$$

be risk functionals and

$$B = B_1 \times \ldots \times B_n \tag{5}$$

be a convex cone. Let $f : B \to \mathbb{R} \cup \{\infty\}$ be a function. Then,

$$R = f(R_1, \ldots, R_n) \tag{6}$$

is a risk functional and the following holds:

- *If $R_1, \ldots, R_n$ and f are monotone, then R is monotone, i.e., a risk measure.*
- *If $R_1, \ldots, R_n$ are subadditive and f is monotone and subadditive, then R is subadditive.*
- *If $R_1, \ldots, R_n$ and f are homogeneous, then R is homogeneous.*
- *If $R_1, \ldots, R_n$ are convex and f is monotone and convex, then R is convex (albeit not cash invariant in general[2]).*

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[1] There is a lot of literature on risk measures. For our purposes, it is more than enough to rely on the textbook (Rueschendorf 2013, p. 142 ff.).

[2] A risk measure is called a *convex risk measure* in literature when it is convex, and cash invariant.

- *If $R_1, \ldots, R_n$ are version independent, then R is version independent.*
- *If $R_1, \ldots, R_n$ are comonotone additive and f is additive, then R is comonotone additive.*

The proof is straightforward and several facts of the lemma may be known to the experts. The point here is that the lemma paves the way for the following definition that establishes a class of models, which are useful for the study of the aggregation of risk, and diversification.

Definition 2. *Let $B \subseteq (\mathbb{R} \cup \{\infty\})^n$ be a convex cone and $f : B \to \mathbb{R} \cup \{\infty\}$ a function. We call f a* diversification function, *if f is monotone, homogeneous and subadditive.*

Note that there is a related notion in information theory, namely, of an *aggregation function* $f : [0,1]^n \to [0,1]$, which has a slightly different aim (Beliakov et al. 2007).

Lemma 1 implies:

Theorem 1. *Let $R_1, \ldots, R_n : \mathcal{X} \to [0, \infty)$ be non-negative, finite risk measures and $R = (R_1, \ldots, R_n)^{\mathrm{T}}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a seminorm and the restriction of f to $[0, \infty)^n$ be monotone. Then, the following holds:*

- *The restriction of f to $[0, \infty)^n$ is a diversification function.*
- *$f(R)$ is a risk measure.*
- *If $R_1, \ldots, R_n$ are homogeneous, then $f(R)$ is a homogeneous risk measure.*
- *If $R_1, \ldots, R_n$ are subadditive, then $f(R)$ is a subadditive risk measure.*
- *If $R_1, \ldots, R_n$ are convex, then $f(R)$ is a risk measure, which is convex (albeit not cash invariant in general).*

It is not unusual for a seminorm that its restriction to non-negative elements is monotone: All L^p-norms have this property.

The following Lemma shows that one may consider non-negative risk functionals, or risk measures. The proof is straightforward:

Lemma 2. *Let R be a risk functional and*

$$S = \max\{R; 0\}. \tag{7}$$

If R is monotone, subadditive, homogeneous, convex, or version independent, then S has the respective property as well.

With respect to symmetric bilinear forms, we have the following theorem:

Theorem 2. *Let $A \in \mathbb{R}^{n \times n}$ be positive semidefinite. The restriction to $[0, \infty)^n$ of the seminorm*

$$x \mapsto \|x\|_A = \sqrt{x^{\mathrm{T}} A x} \tag{8}$$

is monotone if and only if A is non-negative, i.e., has non-negative entries only.

Proof. Let A be non-negative. We show that the restriction of $\|\cdot\|_A$ to $[0, \infty)^n$ is monotone. Let $R, S \in \mathbb{R}^n$ with $R \geq S \geq 0$, i.e., $R_k \geq S_k \geq 0$ for all $1 \leq k \leq n$. Then,

$$\|R\|_A^2 - \|S\|_A^2 = (R - S)^{\mathrm{T}} A R + S^{\mathrm{T}} A (R - S) \geq 0 \tag{9}$$

because all contributions in the sum are non-negative.

Now assume that $a_{jk} = a_{kj} < 0$ for some $1 \leq j, k \leq n$. Consider $R = c\, e_j + e_k$ and $S = e_k$, where e_j and e_k are the respective unit vectors of the standard basis and $0 < c < 2|a_{jk}| / |a_{jj}|$ in case $a_{jj} \neq 0$ and 1 otherwise. Then, $0 \leq S \leq R$ and

$$\|R\|_A^2 - \|S\|_A^2 = c\, e_j^{\mathrm{T}} A (c\, e_j + e_k) + e_k^{\mathrm{T}} A c\, e_j = c^2 a_{jj} + 2c\, a_{jk} < 0, \tag{10}$$

which shows that monotony is violated. □

In fact, we can characterize diversification functions $f : [0,\infty)^n \to \mathbb{R}$ as follows:

Lemma 3. *Let $f : [0,\infty)^n \to \mathbb{R}$ be a diversification function and $f(0) = 0$. Then, there exists a seminorm $g : \mathbb{R}^n \to \mathbb{R}$ with $g(x) = f(x)$ for all $x \in [0,\infty)^n$, namely,*

$$g(x_1, \ldots, x_n) = f(|x_1|, \ldots, |x_n|) \quad \textit{for all } x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n. \tag{11}$$

If $f(x) > 0$ for all $x \in [0,\infty)^n \setminus \{0\}$, g defines a norm.

Proof. Because of $f(0) = 0$ and the monotony of f, we have $g(0) = 0$ and g is non-negative. By monotony and subadditivity of f,

$$g(x_1 + y_1, \ldots, x_n + y_n) \le f(|x_1| + |y_1|, \ldots, |x_n| + |y_n|) \le g(x_1, \ldots, x_n) + g(y_1, \ldots, y_n) \tag{12}$$

for all $(x_1, \ldots, x_n)^{\mathrm{T}}, (y_1, \ldots, y_n)^{\mathrm{T}} \in \mathbb{R}^n$. The homogeneity of f implies $g(tx) = |t|\, g(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. □

Theorem 3.

1. *Let $f : [0,\infty)^n \to \mathbb{R}$ be a function and $f(0) = 0$. Then, f is a diversification function if and only if there exists a seminorm $g : \mathbb{R}^n \to \mathbb{R}$, which is monotone on $[0,\infty)^n$ and $g(x) = f(x)$ for all $x \in [0,\infty)^n$.*
2. *Let $f : [0,\infty)^n \to \mathbb{R}$ be a function with $f(0) = 0$ and $f(x) \neq 0$ for $x \neq 0$. Then, f is a diversification function if and only if there exists a norm $g : \mathbb{R}^n \to \mathbb{R}$, which is monotone on $[0,\infty)^n$ and $g(x) = f(x)$ for all $x \in [0,\infty)^n$.*

3. Diversification Functions and the Euler Principle

Let $B \subseteq \mathbb{R}^n$ be a convex cone. A function $f : B \subseteq \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous of degree $\alpha \in \mathbb{R}$*, if

$$f(tx) = t^{\alpha} f(x) \text{ for all } t > 0 \text{ and } x \in B, x \neq 0. \tag{13}$$

A theorem of Euler states that a differentiable function is homogeneous of degree $\alpha > 0$ if and only if

$$\langle \nabla f(x), x \rangle = \alpha f(x) \text{ for all } x \in B, x \neq 0, \tag{14}$$

see (Tasche 1999). A function that is homogeneous and of degree 1 is briefly called homogeneous as well. Capital allocation by Equation (14) is called *Euler's principle* (Tasche 2008), i.e.,

Remark 1. *Let $f : B \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable diversification function. We call*

$$\omega_k(x) = \frac{\partial f}{\partial x_k}(x) \tag{15}$$

the sensitivity *of f with respect to x_k at x. Our interpretation is that $x_1, \ldots, x_n$ represent risk measurements, and $f(x)$ the aggregated risk measurement. The sensitivities serve to reallocate the overall risk $f(x)$ to the contributions $x_1, \ldots, x_n$. Namely, by Equation (14), we have*

$$f(x) = \omega_1(x)x_1 + \ldots + \omega_k(x)x_k. \tag{16}$$

The sensitivities are the marginal contributions of the individual risks to the overall risk. They depend on the allocation x.

The sensitivities are often not negative and not greater than one.

Definition 3. *Let $B \subseteq (\mathbb{R} \cup \{\infty\})^n$ be a convex cone, and $f : B \to \mathbb{R} \cup \{\infty\}$ a diversification function. We call f a* normalized diversification function, *if $e_k \in B$ for all unit vectors $e_1, \ldots, e_n$ of the standard basis in $\mathbb{R}^n$, and*

$$f(e_k) = 1 \text{ for all } 1 \leq k \leq n. \tag{17}$$

Theorem 4. *The sensitivities of a differentiable, normalized diversification function are not negative and not greater than 1, i.e.,*

$$\frac{\partial f}{\partial x_k}(x) \in [0, 1] \text{ for all } x \in B \text{ and } 1 \leq k \leq n. \tag{18}$$

Proof. We have by monotony

$$f(x + h\, e_k) \geq f(x) \text{ for all } x \in B, h > 0 \text{ and } 1 \leq k \leq n, \tag{19}$$

and we obtain by subadditivity

$$f(x + h\, e_k) \leq f(x) + f(h\, e_k) \text{ for all } x \in B, h > 0 \text{ and } 1 \leq k \leq n. \tag{20}$$

So

$$0 \leq \lim_{h \searrow 0} \frac{f(x + h\, e_k) - f(x)}{h} = \frac{\partial f}{\partial x_k}(x) \leq \lim_{h \searrow 0} \frac{f(h\, e_k)}{h} = 1, \tag{21}$$

as was to be shown. □

The following Lemmas work out the economic meaning of homogeneity. They both could be rephrased in saying "Diversification does not depend on the size of the portfolio, but only on its composition."—Assertion and Proof of Lemma 5 may be known to the reader.

Lemma 4. *Let $f : B \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable diversification function. The risk in direction of a risk allocation $R \in B$, $R \neq 0$ is proportional to the directional derivative of the risk in this direction, i.e.,*

$$f(y) = D_{\nu_R} f(R) \cdot |y| \quad \text{for all } y = |y|\, \nu_R, \text{ where } \nu_R = \frac{R}{|R|}. \tag{22}$$

Proof. By Euler's theorem, i.e., Equation (14), the directional derivative of the risk is

$$D_{\nu_R} f(R) = \langle \nabla f(R), \nu_R \rangle = \left\langle \nabla f(R), \frac{R}{|R|} \right\rangle = \frac{f(R)}{|R|} = f(\nu_R). \tag{23}$$

This implies

$$D_{\nu_R} f(R) \cdot |y| = f(|y|\, \nu_R) = f(y). \tag{24}$$

□

Lemma 5. *Let $f : B \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable diversification function. Let $R \in B, R \neq 0$ be some risk allocation and $c > 0$. Then, the sensitivities of cR and R coincide:*

$$\nabla f(cR) = \nabla f(R). \tag{25}$$

Proof. By homogeneity, we have for all $v \in \mathbb{R}^n$

$$\begin{aligned} \langle \nabla f(cR), v \rangle &= \lim_{h \to 0} \frac{1}{h} \Big(f(cR + hv) - f(cR) \Big) = \lim_{h \to 0} \frac{c}{h} \left(f\left(R + \frac{h}{c}\, v \right) - f(R) \right) \\ &= \lim_{h' \to 0} \frac{1}{h'} \Big(f(R + h'v) - f(R) \Big) = \langle \nabla f(R), v \rangle . \end{aligned}$$

This implies the assertion. □

More generally, the same proof shows that the gradient of a homogeneous function of degree $\alpha \in \mathbb{R}$ is homogeneous of degree $\alpha - 1$. (The case $\alpha = 0$ is somewhat degenerate, yet true.)

We conclude the section with a remark on the composition of diversification functions:

Lemma 6. *Let $B_1, \ldots, B_m$ be convex cones. Then, $B = B_1 \times \ldots \times B_m$ is a convex cone and vice versa.*

Lemma 7. *Let $C \subseteq (\mathbb{R} \cup \{\infty\})^m$, $B_k \subseteq \mathbb{R} \cup \{\infty\}$ be convex cones, $g_k : C \to B_k$ be diversification functions $(1 \leq k \leq n, m \in \mathbb{N})$, and $g = (g_1, \ldots, g_n)^{\mathrm{T}}$. Let*

$$f : B_1 \times \ldots \times B_n \longrightarrow \mathbb{R} \cup \{\infty\} \tag{26}$$

be a diversification function. Then,

$$f \circ g : C \longrightarrow \mathbb{R} \cup \{\infty\} \tag{27}$$

is a diversification function. The sensitivity ω_ℓ $(1 \leq \ell \leq m)$ of $f \circ g$ with respect to an allocation $S \in C$ is given by

$$\omega_\ell = \frac{\partial (f \circ g)}{\partial S_\ell}(S) = \langle \nabla f(g(S)), \partial_\ell g(S) \rangle. \tag{28}$$

Remark 2. *One often encounters the special case, in which there is only one function g_k, which depends on a coordinate S_ℓ. In this case, Equation (28) means*

$$\omega_\ell = \partial_k f(g(S))\, \partial_\ell g_k(S). \tag{29}$$

4. Application: The Solvency II Standard Formula

Theorems 1 and 2 apply to the Standard Formula (1) and show that $\|R\|_A$ is a diversification function on $[0, \infty)^n$:

Corollary 1. *Let $A \in [0, \infty)^{n \times n}$ be positive semidefinite and non-negative (i.e., A has non-negative entries). Then,*

$$R \in [0, \infty)^n \mapsto \|R\|_A = \sqrt{R^{\mathrm{T}} A R} \tag{30}$$

is a diversification function. If $R_1, \ldots, R_n$ are non-negative, finite risk measures and $R = (R_1, \ldots, R_n)^{\mathrm{T}}$, the following holds:

- *$\|R\|_A$ is a risk measure.*
- *If $R_1, \ldots, R_n$ are homogeneous, then $\|R\|_A$ is a homogeneous risk measure.*
- *If $R_1, \ldots, R_n$ are subadditive, then $\|R\|_A$ is a subadditive risk measure.*
- *If $R_1, \ldots, R_n$ are convex, then $\|R\|_A$ is a risk measure, which is convex (albeit not cash invariant in general).*

Thus, the standard formula can be interpreted as a subadditive risk measure and homogeneous risk measure, whenever the individual risk measures R_k, which are assumed to be non-negative and finite, share the property.[3]

From an economic point of view, it is reasonable to use a positive semidefinite and non-negative matrix A, as required by Corollary 1. In this regard, matrix A may well contain tail-correlations—defined in one or the other manner. Compare (Campbell et al. 2002; Mittnik 2014) with respect to the interconnection between the standard formula and tail correlations, and calibration issues.

[3] There is an exemption within the life risk module: The correlation coefficient longevity risk–mortality risk is -0.25 (European Commission 2015, p. 88).

Alternatively, matrix A is positive semidefinite if the correlation matrix is chosen:

Remark 3. *Let S be the covariance matrix and K the correlation matrix of a random vector Z. Then, S and K are positive semidefinite (Jacod and Protter 2000, p. 88). If there is no linear combination of the entries of Z that is constant, S and K are positive definite (Feller 1966, p. 83).*

However, choosing the correlation matrix goes along with two drawbacks:

- Pearson correlation coefficients do not refer to tail correlations.
- In general, one encounters negative correlation coefficients and the risk aggregation is not monotone (cf. Theorem 2), i.e., the standard formula is not a diversification function. (Unless, for example, the underlying risks are elliptically distributed, have a mean of zero, and the risk measures are a value at risk at the same level of confidence.)

The risk measure of the standard formula is value at risk. Value at risk is a homogeneous, cash invariant, version independent, comonotone additive risk measure, yet not subadditive or convex in general (Rueschendorf 2013, p. 147). Hence, the standard formula with value at risk is a homogeneous, version independent risk measure (to be precise, value at risk has to be maximized with zero to fit.). Of course, if the joint distribution of the overall risk was known, then it was by no means guaranteed that the standard formula resembled the value at risk of the joint distribution; however, given only the marginal risk measures, the standard formula is indeed a risk measure of the overall risk in spite of the fact of there being no further information on the joint distribution.

In particular, Euler's principle applies to the standard formula. Under the simplifying assumption that the underlying random variables have a normal distribution, this was already observed by (De Angelis and Granito 2015). We have by Theorem 4:

Corollary 2. *Let $A \in [0,\infty)^{n\times n}$ be positive semidefinite, non-negative, and the diagonal elements of A be 1. Then, $\|\cdot\|_A$ is a normalized diversification function on $[0,\infty)^n$ and*

$$\omega_k = \frac{\partial \, \|R\|_A}{\partial R_k} \in [0,1] \tag{31}$$

for all $R \in [0,\infty)^n \setminus \{0\}$ and $1 \leq k \leq n$.

The sensitivities can be computed as follows:

Lemma 8. *Let $R = (R_1,\ldots,R_N)^{\mathrm{T}} \in \mathbb{R}^N$ a vector of risk measurements and A be a symmetric matrix. Let the overall risk be given by*

$$\|R\|_A = \sqrt{R^{\mathrm{T}}AR}. \tag{32}$$

Then, the gradient of $R \mapsto \|R\|_A$ is given by

$$\nabla \, \|R\|_A = \frac{AR}{\|R\|_A} \text{ for all } \|R\|_A > 0. \tag{33}$$

The equation

$$\langle \nabla \, \|R\|_A \, , R \rangle = \|R\|_A \tag{34}$$

holds.

Note that, in this Lemma, A does not have to be positive definite or positive semi-definite. It suffices that A is symmetric. In any case, any positive definite or positive semi-definite matrix is symmetric by definition.

A weaker, yet more general assertion as compared to Corollary 2—without monotony—is given by the following:

Lemma 9. *Let A be positive semidefinite and the diagonal elements of A be 1. Then, the absolute value of the sensitivities*

$$\omega_k = \frac{\partial \|R\|_A}{\partial R_k} = \frac{1}{\|R\|_A} \sum_{\ell=1}^{N} R_\ell A_{\ell k}, \quad 1 \le k \le N \tag{35}$$

is not greater than 1 for all $1 \le k \le N$.

Proof. The assumption that the diagonal elements of A are 1 implies

$$\|e_k\|_A = 1 \tag{36}$$

for all unit vectors e_k of the standard basis $(1 \le k \le N)$. Hence, we obtain by the triangle inequality

$$|\omega_k| = \left|\frac{\partial \|R\|_A}{\partial R_k}\right| = \left|\lim_{t\to 0} \frac{\|R + t\, e_k\|_A - \|R\|_A}{t}\right| \le \lim_{t\to 0} \frac{|t|\, \|e_k\|_A}{|t|} = 1 \tag{37}$$

for all $1 \le k \le N$. □

We make a side remark on a special case, which is quite often encountered in practice: the concentration risk within the market risk module may vanish. In this case, its marginal contribution vanishes as well, i.e., if there was an increase of concentration risk, the market risk would be affected only slightly.

Remark 4. *In case the concentration risk within the market risk module vanishes, its sensitivity vanishes as well (provided the market risk is positive). This is because the correlation parameters of the concentration risk to the other sub-risks of the market risk are all zero. Therefore, market risk can be expressed in terms of concentration risk x as follows:*

$$f(x) = \sqrt{c + x^2}, \quad c > 0, \tag{38}$$

where $\sqrt{c}$ *denotes market risk without concentration risk. The sensitivity of the concentration risk for* $x = 0$ *is*

$$\omega = \frac{df}{dx}(0) = \left.\frac{x}{f(x)}\right|_{x=0} = 0. \tag{39}$$

From the point of view of a manager who is in charge of some sub-portfolio, it is a complication that sensitivities can change over time, for this means that the contribution of their sub-portfolio is not at their command alone. Therefore it is important that the sensitivities change only slightly, when the composition of risks changes. Of course, if there was turmoil in the portfolio and risks changed considerably, the sensitivities may change considerably as well—but this is reasonable and should be expected.

Theorem 5. *Let the matrix A be positive semidefinite, have diagonal elements of 1, and have non-negative entries. Let the risk vector R be non-negative, i.e., every entry of R be non-negative. Let D be a change of the risk vector and*

$$D \ge 0 \quad \text{or} \quad -R \le D \le 0. \tag{40}$$

Let $\|R + D\|_A > 0$. *Then, the following estimate on the change of the sensitivities holds:*

$$\left|\nabla \|R + D\|_A - \nabla \|R\|_A\right| \le \frac{\|D\|_A}{\|R + D\|_A}, \tag{41}$$

where the absolute value $|\cdot|$ *denotes the maximum norm.*

Proof. The case $\|D\|_A = 0$ is obvious. The case $\|R\|_A = 0$ follows from Lemma 9. So let $\| R \|_A > 0$, $\|D\|_A > 0$. We derive

$$\nabla \|R+D\|_A - \nabla \|R\|_A = \frac{A(R+D)}{\|R+D\|_A} - \frac{AR}{\|R\|_A} \\ = \frac{AR}{\|R\|_A}\left(\frac{\|R\|_A}{\|R+D\|_A} - 1\right) + \frac{AD}{\|D\|_A}\frac{\|D\|_A}{\|R+D\|_A}. \tag{42}$$

At first, we consider the case $D \geq 0$. Then,

$$\|R+D\|_A^2 - \|R\|_A^2 = D^{\mathrm{T}}A(R+D) + R^{\mathrm{T}}AD \tag{43}$$

is non-negative because there are solely non-negative numbers within the products. Hence, the bracket in Equation (42) is negative or zero. Therefore, the two summands on the right hand side of Equation (42) have a different sign in every component. We obtain with Lemma 9:

$$\left|\nabla \|R+D\|_A - \nabla \|R\|_A\right| \leq \max\left\{\left|\frac{\|R\|_A}{\|R+D\|_A} - 1\right|, \frac{\|D\|_A}{\|R+D\|_A}\right\}. \tag{44}$$

Now, the triangle inequality shows:

$$\left|\frac{\|R\|_A}{\|R+D\|_A} - 1\right| = \frac{\|R+D\|_A - \|R\|_A}{\|R+D\|_A} \leq \frac{\|D\|_A}{\|R+D\|_A}, \tag{45}$$

which implies Estimate (41).

The other case is $-R \leq D \leq 0$. In this case, the entries of D on the right-hand side of Equation (43) are non-positive, while all other entries are non-negative. Hence, both sides of Equation (43) are non-positive. This means that the bracket in Equation (42) is non-negative while the entries in the last term are non-positive. Hence, Estimate (44) holds in this case as well. We use the triangle inequality in the form:

$$\|R\|_A = \|R+D-D\|_A \leq \|R+D\|_A + \|-D\|_A \tag{46}$$

to obtain

$$\left|\frac{\|R\|_A}{\|R+D\|_A} - 1\right| = \frac{\|R\|_A - \|R+D\|_A}{\|R+D\|_A} \leq \frac{\|D\|_A}{\|R+D\|_A}. \tag{47}$$

This completes the proof. □

Note that the assumptions of Theorem 5 are fulfilled in particular if only one risk changes. In general, we have the following Theorem 6 with a weaker estimate:

Theorem 6. *Let the matrix A be positive semidefinite and have diagonal elements of 1. Let $R, D \in \mathbb{R}^n$ and $\|R+D\|_A > 0$. Then, the following estimate on the change of the sensitivities holds:*

$$\left|\nabla \|R+D\|_A - \nabla \|R\|_A\right| \leq 2\frac{\|D\|_A}{\|R+D\|_A}, \tag{48}$$

where the absolute value $|\cdot|$ denotes the maximum norm.

Proof. We apply the triangle inequality to Equation (42) and use Estimates (45) and (47), respectively. □

5. Conclusions

We define diversification functions to be monotone, subadditive and homogeneous functions on a convex cone. They maintain these properties, i.e., the diversification function of a vector of risk measures, which all have one ore more of these properties, is a risk measure itself with the respective properties.

A class of examples of diversification functions is given by seminorms, which are monotone on the convex cone of vectors with non-negative entries. This applies to seminorms given by symmetric bilinear forms, if and only if the positive semidefinite matrix defining the symmetric bilinear form has non-negative elements.

Due to homogeneity, Euler's principle of capital allocation works for diversification functions independently of any assumption on distributions. The sensitivities (i.e., marginal contributions of the several risks) of so-called normalized diversification functions are not negative and not greater than 1.

All this applies to the Solvency II standard formula, to be precise: to the aggregation of risk within the Solvency II standard formula. The Solvency II standard formula is a diversification function on the convex cone $[0, \infty)^n$ and hence maintains subadditivity and homogeneity, respectively, of the underlying risk measures.

We use the fact that the matrix A defining the risk aggregation is positive semidefinite and has non-negative elements. Consequently, A should be interpreted as a matrix of correlation parameters representing tail correlations rather than linear correlations. Moreover, the diagonal elements of A are 1, so the standard formula is a normalized diversification function with sensitivities (marginal risk contributions) between zero and one.

In particular, Euler's principle applies to the standard formula. We provide estimates for the change of the sensitivities when risks change, which are important for the feasibility of the theory.

Acknowledgments: I thank Mohammad Assadsolimani, Bruce Auchinleck, Daniel Berger, Katrin Credner, Andreea Hannich and Jonas Kaiser for many worthwhile discussions.

Conflicts of Interest: The author declares no conflict of interest.

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