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An Universal, Simple, Circular Statistics-Based Estimator of α for Symmetric Stable Family

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Abstract: The aim of this article is to obtain a simple and efficient estimator of the index parameter of symmetric stable distribution that holds universally, i.e., over the entire range of the parameter. We appeal to directional statistics on the classical result on wrapping of a distribution in obtaining the wrapped stable family of distributions. The performance of the estimator obtained is better than the existing estimators in the literature in terms of both consistency and efficiency. The estimator is applied to model some real life financial datasets. A mixture of normal and Cauchy distributions is compared with the stable family of distributions when the estimate of the parameter α lies between 1 and 2. A similar approach can be adopted when α (or its estimate) belongs to (0.5,1). In this case, one may compare with a mixture of Laplace and Cauchy distributions. A new measure of goodness of fit is proposed for the above family of distributions.

Keywords: Index parameter; estimation; wrapped stable; Hill estimator; characteristic function-based estimator; asymptotic; efficiency

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1. Introduction

Our motivation in this paper is to obtain a universal and efficient estimator of the tail index parameter α of symmetric stable distribution (explained in Section 2) Nolan (2003). This is achieved by appealing to methods available in circular statistics. We recall that there exist two popular estimators of α in the literature. The Hill estimator proposed by Hill (1975), which uses the linear function of the order statistics, however, can be used to estimate $\alpha \in [1, 2]$ only. Furthermore, it is also "extremely sensitive" to the choice of k (explained in Section 6) even for other values of α. Hill (1975) and Dufour and Kurz-Kim (2010) pointed out other drawbacks of the Hill estimator. The other estimator proposed by Anderson and Arnold (1993) is based on characteristic function approach. However, this estimator cannot be obtained in a closed form and is to be solved numerically. Furthermore, neither its asymptotic distribution nor its variance and bias are available in the literature.

Our approach in this paper appeals to circular statistics and is based on the method of trigonometric moments as in SenGupta (1996) and later also discussed in Jammalamadaka and SenGupta (2001). This stems from the very useful result which presents a closed analytical form of the density of a wrapped (circular) stable distribution obtained by wrapping the corresponding stable distribution which need not have any closed form analytic representation for arbitrary α. This result shows that α is preserved as the same parameter even after the wrapping. Furthermore, this paper presents a goodness of fit test based on the wrapped probability density function, which may be used as a necessary condition to ascertain the fit of the stable distribution. We exploit this approach with the real life examples. This estimator has a simple and elegant closed form expression. It is asymptotically normally distributed with mean α and variance available in a closed analytical form. Furthermore, from extensive simulations under parameter configurations encountered in financial data, it is exhibited that this new estimator outperforms both the estimators mentioned above almost uniformly in the entire comparable support of α. In Section 2, the probability density function of the wrapped stable distribution and some associated notations are introduced. The moment estimator of the index parameter is also defined in this section. Section 3 presents the derivation of the asymptotic distribution of the moment estimator defined in Section 2. In Section 4, an improved estimator of the index parameter is obtained. Section 5 shows the derivation of the asymptotic distribution of the improved estimator using the multivariate delta method. In addition, the asymptotic variance is computed for various values of the parameters through simulation. In Section 6, comparison of the performance of the improved estimator is made with those of the Hill estimator and the characteristic function-based estimator based on their root mean square errors through simulation. In Section 7, the procedure of the various computations is presented. In Section 8, applications of the proposed estimator is made on some real life data. We also conclude with remarks on the performance of the various estimators and some comments on future scope in Section 8. Finally, the tables showing the various computations and the figures on the applications of data are given in Appendices A, B and C.

2. The Trigonometric Moment Estimator

The regular symmetric stable distribution is defined through its characteristic function given by

$$\varphi(t) = \exp(it\mu - |\sigma t|^{\alpha})$$

where μ is the location parameter; σ is the scale parameter, which we take as 1; and α is the index or shape parameter of the distribution. Here, without loss of generality, we take $\mu = 0$.

From the stable distribution, we can obtain the wrapped stable distribution (the process of wrapping explained in Jammalamadaka and SenGupta (2001)). Suppose $\theta_1, \theta_2, ..., \theta_m$ is a random sample of size m drawn from the wrapped stable (given in Jammalamadaka and SenGupta (2001)) distribution whose probability density function is given by

$$f(\theta, \rho, \alpha, \mu) = \frac{1}{2\pi}[1 + 2\sum_{p=1}^{\infty} \rho^{p^{\alpha}} \cos p(\theta - \mu)] \quad 0 < \rho \leq 1, 0 < \alpha \leq 2, 0 < \mu \leq 2\pi \tag{1}$$

It is known in general from Jammalamadaka and SenGupta (2001) that the characteristic function of θ at the integer p is defined as,

$$\psi_{\theta}(p) = E[\exp(ip(\theta - \mu))] = \alpha_p + i\beta_p$$

$$\text{where} \quad \alpha_p = E\cos p(\theta - \mu) \quad \text{and} \quad \beta_p = E\sin p(\theta - \mu)$$

Furthermore, from Jammalamadaka and SenGupta (2001), it is known that for, the p.d.f given by Equation (1),

$$\psi_{\theta}(p) = \rho^{p^{\alpha}}$$

$$\text{Hence,} \quad E\cos p(\theta - \mu) = \rho^{p^{\alpha}} \quad \text{and} \quad E\sin p(\theta - \mu) = 0 \tag{2}$$

We define

$$\bar{C}_1 = \frac{1}{m}\sum_{i=1}^{m} \cos\theta_i, \quad \bar{C}_2 = \frac{1}{m}\sum_{i=1}^{m} \cos 2\theta_i, \quad \bar{S}_1 = \frac{1}{m}\sum_{i=1}^{m} \sin\theta_i$$

$$\text{and} \quad \bar{S}_2 = \frac{1}{m}\sum_{i=1}^{m} \sin 2\theta_i$$

Then, we note that $\bar{R}_1 = \sqrt{\bar{C}_1^{\,2} + \bar{S}_1^{\,2}}$ and $\bar{R}_2 = \sqrt{\bar{C}_2^{\,2} + \bar{S}_2^{\,2}}$

By the method of trigonometric moments estimation, equating $\bar{R}_1$ and $\bar{R}_2$ to the corresponding functions of the theoretical trigonometric moments, we get the estimator of index parameter α as (see SenGupta (1996)):

$$\hat{\alpha} = \frac{1}{\ln 2} \ln \frac{\ln \bar{R}_2}{\ln \bar{R}_1}$$

Then, we define $\bar{R}_j = \frac{1}{m} \sum_{i=1}^{m} \cos j(\theta_i - \bar{\theta})$, $j = 1, 2$ and $\bar{\theta}$ is the mean direction given by $\bar{\theta} = \arctan\left(\frac{\bar{S}_1}{\bar{C}_1}\right)$. Note that $\bar{R}_1 \equiv \bar{R}$.

We consider two special cases.

2.1. Special Case 1 : $\mu = 0, \sigma = 1$

We now consider the case as treated by Anderson and Arnold (1993), specifically $\mu = 0$ and $\sigma = 1$, and hence the concentration parameter $\rho = \exp(-1)$ as both parameters are known. This case may arise when one has historical data or prior information on the scale parameter. In such a case, the probability density function reduces to

$$f(\theta, \alpha) = \frac{1}{2\pi}[1 + 2 \sum_{p=1}^{\infty} \{\exp(-1)\}^{p^\alpha} \cos p\theta)], \quad 0 < \alpha \leq 2$$

In addition, by the method of trigonometric moments estimation, the estimator of index parameter α is given by

$$\hat{\alpha_1} = -\frac{\ln \bar{C}_2}{\ln 2}$$

2.2. Special Case 2 : $\mu = 0, \sigma$ Unknown

Next, we consider a general case when $\mu = 0$ and σ, and hence the estimator of the concentration parameter is $\rho = \bar{R}_1$. This case is especially useful in many real life applications, for example, for price changes in financial data, $\mu = 0$ is a standard assumption. In such a case, the probability density function reduces to

$$f(\theta, \alpha) = \frac{1}{2\pi}[1 + 2 \sum_{p=1}^{\infty} \rho^{p^\alpha} \cos p\theta)], \quad 0 < \alpha \leq 2$$

In addition, by the method of trigonometric moments estimation, the estimator of index parameter α is given by

$$\hat{\alpha_2} = \frac{1}{\ln 2} \ln \frac{\ln \bar{C}_2}{\ln \bar{C}_1}$$

As is also seen in Anderson and Arnold (1993), for financial data after using log-ratio transformation, the location parameter of the transformed variable becomes zero. Hence, the case of $\mu \neq 0$ was not considered by Anderson and Arnold (1993) and accordingly by us also for the comparison made in this paper.

3. Derivation of the Asymptotic Distribution of the Moment Estimator

Lemma 1.

$$\sqrt{m}(T_m - \mu) \xrightarrow{L} N_4(0, \Sigma)$$

$$where \quad T_m = (\bar{C}_1, \bar{C}_2, \bar{S}_1, \bar{S}_2)',$$

μ is the mean vector given by

$$\mu = (\rho \cos \mu_0, \rho^{2^\alpha} \cos 2\mu_0, \rho \sin \mu_0, \rho^{2^\alpha} \sin 2\mu_0)'$$

and Σ is the dispersion matrix given by

$$\Sigma = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}$$

where

$$A = \frac{\rho^{2^\alpha}\cos 2\mu_0 + 1 - 2\rho^2\cos^2\mu_0}{2}$$
$$B = \frac{\rho\cos\mu_0 + \rho^{3^\alpha}\cos 3\mu_0 - 2\rho^{2^\alpha+1}\cos\mu_0\cos 2\mu_0}{2}$$
$$C = \frac{\rho^{2^\alpha}\sin 2\mu_0 - 2\rho^2\cos\mu_0\sin\mu_0}{2}$$
$$D = \frac{\rho^{3^\alpha}\sin 3\mu_0 + \rho\sin\mu_0 - 2\rho^{2^\alpha+1}\cos\mu_0\sin 2\mu_0}{2}$$
$$E = \frac{\rho^{4^\alpha}\cos 4\mu_0 + 1 - 2(\rho^{2^\alpha})^2\cos^2 2\mu_0}{2}$$
$$F = \frac{\rho^{3^\alpha}\sin 3\mu_0 - \rho\sin\mu_0 - 2\rho^{2^\alpha+1}\cos 2\mu_0\sin\mu_0}{2}$$
$$G = \frac{\rho^{4^\alpha}\sin 4\mu_0 - 2(\rho^{2^\alpha})^2\cos 2\mu_0\sin 2\mu_0}{2}$$
$$H = \frac{1 - \rho^{2^\alpha}\cos 2\mu_0 - 2\rho^2\sin^2\mu_0}{2}$$
$$I = \frac{\rho\cos\mu_0 - \rho^{3^\alpha}\cos 3\mu_0 - 2\rho^{2^\alpha+1}\sin\mu_0\sin 2\mu_0}{2}$$
$$J = \frac{1 - \rho^{4^\alpha}\cos 4\mu_0 - 2(\rho^{2^\alpha})^2\sin^2 2\mu_0}{2}$$

Proof. The derivations for the proof are given in Appendix A.

Hence, assuming large sample size, central limit theorem Feller (1971) gives $(\bar{C}_1, \bar{C}_2, \bar{S}_1, \bar{S}_2)' \xrightarrow{L} N_4(\mu, \frac{\Sigma}{m})$ where μ is the mean vector given by

$$\mu = (\rho\cos\mu_0, \rho^{2^\alpha}\cos 2\mu_0, \rho\sin\mu_0, \rho^{2^\alpha}\sin 2\mu_0)'$$

and Σ is the dispersion matrix given by

$$\Sigma = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}$$

where $A, B, C, D, E, F, G, H, I$ and J are as defined above.
□

Theorem 1.

$$\sqrt{m}(\hat{\alpha} - \alpha) \xrightarrow{L} N(0, \gamma'\Sigma\gamma)$$

where

$$\gamma = \frac{1}{\ln 2}\left(\frac{-\cos\mu_0}{\rho\ln\rho}, \frac{\cos 2\mu_0}{\rho^{2^\alpha}\ln\rho^{2^\alpha}}, \frac{-\sin\mu_0}{\rho\ln\rho}, \frac{\sin 2\mu_0}{\rho^{2^\alpha}\ln\rho^{2^\alpha}}\right)'$$

and

$$\underline{\gamma}'\Sigma\underline{\gamma} = \frac{1}{(\ln 2)^2}\left[\frac{1+\rho^{2^\alpha} - 2\rho^2}{2(\rho\ln\rho)^2} + \frac{1+\rho^{4^\alpha} - 2(\rho^{2^\alpha})^2}{2(\rho^{2^\alpha}\ln\rho^{2^\alpha})^2} + \frac{2\rho^{2^\alpha+1} - \rho - \rho^{3^\alpha}}{\rho\ln\rho\rho^{2^\alpha}\ln\rho^{2^\alpha}}\right]$$

Proof. We know from Lemma 1 that $\sqrt{m}(T_m - \mu) \xrightarrow{L} N_4(0, \Sigma)$

Therefore, by delta method (given in Casella and Berger (2002)), we get

$\sqrt{m}(\hat{\alpha}-\alpha) \xrightarrow{L} N(0, \gamma'\Sigma\gamma)$ where

$$
\begin{aligned}
g(T_m) = \hat{\alpha} &= \frac{1}{\ln 2} \ln \frac{\ln \bar{R_2}}{\ln \bar{R_1}} \\
&= \frac{1}{\ln 2} \ln \frac{\ln \sqrt{\bar{C_2}^2 + \bar{S_2}^2}}{\ln \sqrt{\bar{C_1}^2 + \bar{S_1}^2}}
\end{aligned}
$$

$$
\gamma = \begin{pmatrix} \frac{\partial g}{\partial \bar{C_1}} \\ \frac{\partial g}{\partial \bar{C_2}} \\ \frac{\partial g}{\partial \bar{S_1}} \\ \frac{\partial g}{\partial \bar{S_2}} \end{pmatrix} \text{ at } \mu
$$

$$
= \frac{1}{\ln 2} \begin{pmatrix} \frac{\bar{C_1}}{-(\bar{C_1}^2+\bar{S_1}^2)\ln\sqrt{\bar{C_1}^2+\bar{S_1}^2}} \\ \frac{\bar{C_2}}{(\bar{C_2}^2+\bar{S_2}^2)\ln\sqrt{\bar{C_2}^2+\bar{S_2}^2}} \\ \frac{\bar{S_1}}{-(\bar{C_1}^2+\bar{S_1}^2)\ln\sqrt{\bar{C_1}^2+\bar{S_1}^2}} \\ \frac{\bar{S_2}}{(\bar{C_2}^2+\bar{S_2}^2)\ln\sqrt{\bar{C_2}^2+\bar{S_2}^2}} \end{pmatrix} \text{ at } \mu
$$

$$
= \frac{1}{\ln 2} \begin{pmatrix} \frac{-\cos\mu_0}{\rho\ln\rho} \\ \frac{\cos 2\mu_0}{\rho^{2^\alpha}\ln\rho^{2^\alpha}} \\ \frac{-\sin\mu_0}{\rho\ln\rho} \\ \frac{\sin 2\mu_0}{\rho^{2^\alpha}\ln\rho^{2^\alpha}} \end{pmatrix}
$$

$$
\begin{aligned}
\underline{\gamma}'\Sigma\underline{\gamma} = \frac{1}{(\ln 2)^2} \Bigg[& \frac{1+\rho^{2^\alpha}\cos^2 2\mu_0 - 2\rho^2(\cos^4\mu_0 + \sin^4\mu_0)}{2(\rho\ln\rho)^2} \\
&+ \frac{-\rho\cos^2\mu_0\cos 2\mu_0 - \rho^{3^\alpha}\cos 3\mu_0\cos\mu_0\cos 2\mu_0 + 2\rho^{2^\alpha+1}\cos^2\mu_0\cos^2 2\mu_0}{\rho\ln\rho\rho^{2^\alpha}\ln\rho^{2^\alpha}} \\
&+ \frac{\rho^{2^\alpha}\sin 2\mu_0\sin\mu_0\cos\mu_0 - 2\rho^2\cos^2\mu_0\sin^2\mu_0}{(\rho\ln\rho)^2} \\
&+ \frac{-\rho^{3^\alpha}\cos\mu_0\sin 2\mu_0\sin 3\mu_0 - \rho\sin\mu_0\cos\mu_0\sin 2\mu_0 + 2\rho^{2^\alpha+1}\cos^2\mu_0\sin^2 2\mu_0}{\rho\ln\rho\rho^{2^\alpha}\ln\rho^{2^\alpha}} \\
&+ \frac{\rho^{4^\alpha}\cos^2 4\mu_0 + 1 - 2(\rho^{2^\alpha})^2(\cos^4 2\mu_0 + \sin^4 2\mu_0)}{2(\rho^{2^\alpha}\ln\rho^{2^\alpha})^2} \\
&+ \frac{\rho^{4^\alpha}\sin 4\mu_0\sin 2\mu_0\cos 2\mu_0 - 2(\rho^{2^\alpha})^2\cos^2 2\mu_0\sin^2 2\mu_0}{(\rho^{2^\alpha}\ln\rho^{2^\alpha})^2} \\
&+ \frac{-\rho^{3^\alpha}\sin 3\mu_0\cos 2\mu_0\sin\mu_0 + \rho\sin^2\mu_0\cos 2\mu_0 + 2\rho^{2^\alpha+1}\cos^2 2\mu_0\sin^2\mu_0}{\rho\ln\rho\rho^{2^\alpha}\ln\rho^{2^\alpha}} \\
&+ \frac{-\rho\cos\mu_0\sin\mu_0\sin 2\mu_0 + \rho^{3^\alpha}\cos 3\mu_0\sin\mu_0\sin 2\mu_0 + 2\rho^{2^\alpha+1}\sin^2\mu_0\sin^2 2\mu_0}{\rho\ln\rho\rho^{2^\alpha}\ln\rho^{2^\alpha}} \Bigg] \\
= \frac{1}{(\ln 2)^2} \Bigg[& \frac{1+\rho^{2^\alpha} - 2\rho^2}{2(\rho\ln\rho)^2} + \frac{1+\rho^{4^\alpha} - 2(\rho^{2^\alpha})^2}{2(\rho^{2^\alpha}\ln\rho^{2^\alpha})^2} + \frac{2\rho^{2^\alpha+1} - \rho - \rho^{3^\alpha}}{\rho\ln\rho\rho^{2^\alpha}\ln\rho^{2^\alpha}} \Bigg]
\end{aligned}
$$

(using usual trigonometric identities and formulae)

□

Lemma 2.

$$\sqrt{m}(T'_m - \mu') \xrightarrow{L} N(0, \sigma^2)$$

where

$$T'_m = \bar{C}_2,$$

μ' is the mean given by

$$\mu' = \rho^{2^\alpha}$$

and σ^2 is the dispersion given by

$$\sigma^2 = \frac{\rho^{4^\alpha} + 1 - 2(\rho^{2^\alpha})^2}{2}$$

Proof. The derivations for the proof are given in Appendix B.

Hence, assuming large sample size, central limit theorem Feller (1971) gives $\bar{C}_2 \xrightarrow{L} N(\mu', \frac{\sigma^2}{m})$ where μ' is the mean given by $\mu' = \rho^{2^\alpha}$ and σ^2 is the dispersion given by $\sigma^2 = mV(\bar{C}_2)$, that is $\sigma^2 = \frac{\rho^{4^\alpha}+1-2(\rho^{2^\alpha})^2}{2}$. □

Theorem 2.

$$\sqrt{m}(\hat{\alpha}_1 - \alpha) \xrightarrow{L} N\left(0, \sigma^2 \left(\frac{\partial g}{\partial \mu'}\right)^2\right)$$

where

$$\frac{\partial g}{\partial \mu'} = -\frac{1}{(\ln 2)\rho^{2^\alpha}} \quad \textit{and}$$

$$\sigma^2 \left(\frac{\partial g}{\partial \mu'}\right)^2 = \frac{1}{(\ln 2)^2 \rho^{2^\alpha}}$$

Proof. We know from Lemma 2 that $\sqrt{m}(T'_m - \mu') \to N(0, \sigma^2)$ in distribution

Therefore, by delta method (given in Casella and Berger (2002)), we get $\sqrt{m}(\hat{\alpha}_1 - \alpha) \xrightarrow{L} N\left(0, \sigma^2 \left(\frac{\partial g(T'_m)}{\partial \mu'}\right)^2\right)$ where

$$g(T'_m) = -\frac{\ln \bar{C}_2}{\ln 2}$$

$$\frac{\partial g(T'_m)}{\partial \mu'} = -\frac{1}{\ln 2} \frac{1}{\bar{C}_2}\bigg|_{\mu'} = -\frac{1}{(\ln 2)\rho^{2^\alpha}}$$

$$\sigma^2 \left(\frac{\partial g(T'_m)}{\partial \mu'}\right)^2 = \frac{1}{(\ln 2)^2 \rho^{2^\alpha}}$$

□

Lemma 3.

$$\sqrt{m}(T''_m - \mu'') \xrightarrow{L} N_2(0, \Sigma')$$

where

$$T''_m = (\bar{C}_1, \bar{C}_2)'$$

μ'' is the mean vector given by

$$\mu'' = (\rho, \rho^{2^\alpha})'$$

and Σ' is the dispersion matrix given by:-

$$\Sigma' = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

where

$$A = \frac{\rho^{2^\alpha}+1-2\rho^2}{2}, \quad B = \frac{\rho+\rho^{3^\alpha}-2\rho^{2^\alpha+1}}{2}, \quad C = \frac{\rho^{4^\alpha}+1-2(\rho^{2^\alpha})^2}{2}$$

Proof. The derivations for the proof are given in Appendix C.

Hence, assuming large sample size, central limit theorem Feller (1971) gives

$$(\bar{C_1}, \bar{C_2})' \xrightarrow{L} N_2(\mu'', \frac{\Sigma'}{m})$$

where μ'' is the mean vector given by $\mu'' = (\rho, \rho^{2^\alpha})'$ and Σ' is the dispersion matrix given by $\Sigma' = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ where $A = \frac{\rho^{2^\alpha}+1-2\rho^2}{2}$, $B = \frac{\rho+\rho^{3^\alpha}-2\rho^{2^\alpha+1}}{2}$ and $C = \frac{\rho^{4^\alpha}+1-2(\rho^{2^\alpha})^2}{2}$. □

Theorem 3.

$$\sqrt{m}(\hat{\alpha_2} - \alpha) \xrightarrow{L} N(0, \gamma_1' \Sigma' \gamma_1)$$

where

$$\gamma_1 = \frac{1}{\ln 2}\left(\frac{-1}{\rho \ln \rho}, \frac{1}{\rho^{2^\alpha} \ln \rho^{2^\alpha}}\right)'$$

and

$$\underline{\gamma_1}'\Sigma'\underline{\gamma_1} = \frac{1}{(\ln 2)^2}\left[\frac{1+\rho^{2^\alpha}-2\rho^2}{2(\rho \ln \rho)^2} + \frac{1+\rho^{4^\alpha}-2(\rho^{2^\alpha})^2}{2(\rho^{2^\alpha}\ln \rho^{2^\alpha})^2} + \frac{2\rho^{2^\alpha+1}-\rho-\rho^{3^\alpha}}{\rho \ln \rho \rho^{2^\alpha} \ln \rho^{2^\alpha}}\right]$$

Proof. We know from Lemma 3 that $\sqrt{m}(T''_m - \mu'') \xrightarrow{L} N_2(0, \Sigma')$

Therefore, by delta method (given in Casella and Berger (2002)), we get $\sqrt{m}(\hat{\alpha_2} - \alpha) \xrightarrow{L} N(0, \gamma_1' \Sigma' \gamma_1)$ where $g(T''_m) = \frac{1}{\ln 2} \ln \frac{\ln \bar{C_2}}{\ln \bar{C_1}}$

$$\gamma_1 = \begin{pmatrix} \frac{\partial g}{\partial \bar{C_1}} \\ \frac{\partial g}{\partial \bar{C_2}} \end{pmatrix} \text{ at } \mu'' = \frac{1}{\ln 2}\begin{pmatrix} \frac{-1}{\bar{C_1}(\ln \bar{C_1})} \\ \frac{1}{\bar{C_2}(\ln \bar{C_2})} \end{pmatrix} \text{ at } \mu'' = \frac{1}{\ln 2}\begin{pmatrix} \frac{-1}{\rho \ln \rho} \\ \frac{1}{\rho^{2^\alpha} \ln \rho^{2^\alpha}} \end{pmatrix}$$

$$\underline{\gamma_1}'\Sigma'\underline{\gamma_1} = \frac{1}{(\ln 2)^2}\left[\frac{1+\rho^{2^\alpha}-2\rho^2}{2(\rho \ln \rho)^2} + \frac{1+\rho^{4^\alpha}-2(\rho^{2^\alpha})^2}{2(\rho^{2^\alpha}\ln \rho^{2^\alpha})^2} + \frac{2\rho^{2^\alpha+1}-\rho-\rho^{3^\alpha}}{\rho \ln \rho \rho^{2^\alpha} \ln \rho^{2^\alpha}}\right]$$

□

The above theorems imply the estimator to be consistent. Hence, in large samples, the performance of the estimator is reasonably good. Now, assuming the sample size is large, say 100, we calculate the asymptotic variance $\underline{\gamma}'\Sigma\underline{\gamma}/100$ of $g(T_m) = \hat{\alpha}$ for different values of α ranging from 0 to 2 and different values of ρ ranging from 0 to 1 in Table 1.

Table 1. Asymptotic Variances of the moment estimator $\hat{\alpha}$ and modified truncated estimator $\widehat{\alpha^*}$.

α	ρ	$V(\hat{\alpha})$	$V(\widehat{\alpha^*})$
0.2	0.2	0.179	0.097
0.2	0.4	0.093	0.058
0.2	0.6	0.084	0.053
0.2	0.8	0.118	0.070
0.4	0.2	0.211	0.148
0.4	0.4	0.094	0.079
0.4	0.6	0.079	0.069
0.4	0.8	0.107	0.088
0.6	0.2	0.270	0.209

Table 1. *Cont.*

α	ρ	$V(\hat{\alpha})$	$V(\widehat{\alpha^*})$
0.6	0.4	0.098	0.093
0.6	0.6	0.074	0.073
0.6	0.8	0.096	0.092
0.8	0.2	0.384	0.284
0.8	0.4	0.105	0.103
0.8	0.6	0.071	0.070
0.8	0.8	0.086	0.085
1.0	0.2	0.626	0.377
1.0	0.4	0.118	0.117
1.0	0.6	0.067	0.067
1.0	0.8	0.075	0.075

4. Improvement Over the Moment Estimator

The moment estimator need not always remain in the support of the true parameter α (that is (0,2]). Hence, the moment estimators proposed above do not need to be proper estimators of α. A modified estimator free from this defect is given by

$$\begin{aligned} \hat{\alpha^*} &= \hat{\alpha} \quad if\ 0 < \hat{\alpha} < 2 \\ &= 2 \quad if\ \hat{\alpha} \geq 2 \end{aligned}$$

(since support of α excludes non-positive values).

Thus, the density function of $\hat{\alpha^*}$ is given by

$$\begin{aligned} g(\hat{\alpha^*}) &= \frac{P[\hat{\alpha} < 2]}{P[\hat{\alpha} \geq 0]} \quad ; 0 < \hat{\alpha^*} < 2 \equiv -\infty < \hat{\alpha} < 2 \\ &= P[\hat{\alpha^*} = 2] \quad ; \hat{\alpha^*} = 2 \equiv \hat{\alpha} \geq\ 2 \\ &= \frac{P[\hat{\alpha} \geq 2]}{P[\hat{\alpha} \geq 0]} \quad ; \hat{\alpha^*} = 2 \equiv \hat{\alpha} \geq 2 \end{aligned}$$

where $f(\hat{\alpha})$ is the density function of $\hat{\alpha} \sim N(\alpha, \underline{\gamma}\prime\Sigma\underline{\gamma}/m)$. Therefore,

$$\begin{aligned} g(\hat{\alpha^*}) &= \frac{\Phi\left(\frac{2-\alpha}{\sqrt{\underline{\gamma}\prime\Sigma\underline{\gamma}/m}}\right)}{1 - \Phi\left(\frac{-\alpha}{\sqrt{\underline{\gamma}\prime\Sigma\underline{\gamma}/m}}\right)} \quad ; 0 < \hat{\alpha^*} < 2 \equiv -\infty < \hat{\alpha} < 2 \\ &= 1 - \frac{\Phi\left(\frac{2-\alpha}{\sqrt{\underline{\gamma}\prime\Sigma\underline{\gamma}/m}}\right)}{\Phi\left(\frac{\alpha}{\sqrt{\underline{\gamma}\prime\Sigma\underline{\gamma}/m}}\right)} \quad ; \hat{\alpha^*} = 2 \equiv \hat{\alpha} \geq 2 \end{aligned}$$

Thus, we get $g(\hat{\alpha^*})$ as a mixed distribution of one atomic mass function and a continuous function.

4.1. Special Case 1 : $\mu = 0, \sigma = 1$

Similar modifications can be made for the estimator $\hat{\alpha_1}$. Let it be denoted by $\hat{\alpha_1^*}$.

4.2. Special Case 2 : $\mu = 0, \sigma$ Unknown

Similar modifications can be made for the estimator $\hat{\alpha_2}$. Let it be denoted by $\hat{\alpha_2^*}$.

5. Derivation of the Asymptotic Distribution of the Modified Truncated Estimators

Now, using the asymptotic normal distribution of $\hat{\alpha}$, we can derive the same results for the modified truncated estimator of the index parameter α (given as below) as we have done for the method of moment estimator of α.

The mean of $\hat{\alpha^*}$ is given by

$$E(\hat{\alpha^*}) = 0.P(\hat{\alpha} < 0) + \int_0^2 \hat{\alpha} f(\hat{\alpha}) d\hat{\alpha} + 2.P(\hat{\alpha} > 2)$$

where $\sqrt{m}(\hat{\alpha} - \alpha) \to$N(0,$\underline{\gamma}\prime\Sigma\underline{\gamma}$) asymptotically (as noted above) and f($\hat{\alpha}$) =probability density function of $\hat{\alpha}$.

The above is equivalent to $\tau = \frac{\hat{\alpha}-\alpha}{\sqrt{\underline{\gamma}\prime\Sigma\underline{\gamma}/m}}$ $\to$N(0,1) asymptotically.

Let $\phi(\tau)$ and $\Phi(\tau)$ denote the p.d.f. and c.d.f. of τ, respectively.

Let $\sigma = \sqrt{\frac{\underline{\gamma}\prime\Sigma\underline{\gamma}}{m}}$. Then, we get,

$$E(\hat{\alpha^*}) = aP(\tau < a^*) + \int_{a^*}^{b^*} (\tau\sigma + \alpha)\phi(\tau) d\tau + bP(\tau > b^*)$$

$$\Rightarrow E(\hat{\alpha^*}) = \sigma\Big[\{\phi(a^*) - \phi(b^*)\}\Big] + \alpha\Big[\Phi(b^*) - \Phi(a^*)\Big]$$

$$= \alpha$$

since $[\Phi(b^*) - \Phi(a^*)] \to 1, b[1 - \Phi(b^*)] \to 0$ and $\sigma \to 0$ as $m \to$ infinity where a^*=$\frac{-\alpha}{\sqrt{\frac{\underline{\gamma}\prime\Sigma\underline{\gamma}}{m}}}$ and b^*=$\frac{2-\alpha}{\sqrt{\frac{\underline{\gamma}\prime\Sigma\underline{\gamma}}{m}}}$

$E(\hat{\alpha^*}^2) = 0^2.\, P(\hat{\alpha} < 0) + \int_0^2 \hat{\alpha}^2 f(\hat{\alpha}) \mathrm{d}\hat{\alpha} + 4.P(\hat{\alpha} > 2)$

$\Rightarrow E(\hat{\alpha^*}^2) = \sigma^2\Big[\{a^*\phi(a^*) - b^*\phi(b^*) + \Phi(b^*) - \Phi(a^*)\}\Big] + \alpha^2\{\Phi(b^*) - \Phi(a^*)\} + 2\alpha\sigma\{\phi(a^*) - \phi(b^*)\}$ since $b^2.\, [1 - \Phi(b')] \to 0$ as $m \to$ infinity

The asymptotic variance of $\hat{\alpha^*}$ is given by

$$V(\hat{\alpha^*}) = E(\hat{\alpha^*}^2) - [E(\hat{\alpha^*})]^2$$

Similarly, the mean of $\hat{\alpha_1^*}$ is given by

$E(\hat{\alpha_1^*}) = \frac{\sigma\left(\frac{\partial g(T'_m)}{\partial \mu'}\right)}{\sqrt{m}}\Big[\{\phi(a') - \phi(b')\}\Big] + \alpha\Big[\Phi(b') - \Phi(a')\Big]$ since $b[1 - \Phi(b')] \to 0$ as $m \to$ infinity

$E(\hat{\alpha_1^*}^2) = \frac{\sigma^2\left(\frac{\partial g(T'_m)}{\partial \mu'}\right)^2}{m}\Big[\{a'\phi(a') - b'\phi(b') + \Phi(b') - \Phi(a')\}\Big] + \alpha^2\{\Phi(b') - \Phi(a')\} + 2\alpha\frac{\sigma\left(\frac{\partial g(T'_m)}{\partial \mu'}\right)}{\sqrt{m}}\{\phi(a') - \phi(b')\}$ since $b^2.\, [1 - \Phi(b')] \to 0$ as $m \to$ infinity

The asymptotic variance of $\hat{\alpha_1^*}$ is given by

$$V(\hat{\alpha_1^*}) = E(\hat{\alpha_1^*}^2) - [E(\hat{\alpha_1^*})]^2$$

where a'=$-\alpha\Big/\frac{\sigma\frac{\partial g(T'_m)}{\partial \mu'}}{\sqrt{m}}$ and b'=$(2-\alpha)\Big/\frac{\sigma\frac{\partial g(T'_m)}{\partial \mu'}}{\sqrt{m}}$

The mean of $\hat{\alpha_2^*}$ is given by

$E(\hat{\alpha_2^*}) = \frac{\sqrt{\gamma_1{}'\Sigma'\gamma_1}}{\sqrt{m}}\left[\{\phi(a'') - \phi(b'')\}\right] + \alpha\left[\Phi(b'') - \Phi(a'')\right]$ since $b[1-\Phi(b'')] \to 0$ as $m \to$ infinity

$E(\hat{\alpha_2^*}^2) = \frac{\gamma_1{}'\Sigma'\gamma_1}{m}\left[\{a''\phi(a'') - b''\phi(b'') + \Phi(b'') - \Phi(a'')\}\right] + \alpha^2\{\Phi(b'') - \Phi(a'')\} + 2\alpha\sqrt{\frac{\gamma_1{}'\Sigma'\gamma_1}{m}}\{\phi(a'') - \phi(b'')\}$ since $b^2.[1-\Phi(b'')] \to 0$ as $m \to$ infinity

The asymptotic variance of $\hat{\alpha_2^*}$ is given by

$$V(\hat{\alpha_2^*}) = E(\hat{\alpha_2^*}^2) - [E(\hat{\alpha_2^*})]^2$$

where $a'' = \frac{-\alpha}{\sqrt{\frac{\gamma_1{}'\Sigma'\gamma_1}{m}}}$ and $b'' = \frac{2-\alpha}{\sqrt{\frac{\gamma_1{}'\Sigma'\gamma_1}{m}}}$

Thus, the following theorem is established

Theorem 4.

$$(\hat{\alpha^*} - \alpha) \xrightarrow{L} N(0, V(\hat{\alpha^*}))$$

where $V(\hat{\alpha^*})$ is as derived above.

Now, assuming the sample size m is large, say 100, the asymptotic variances of the modified truncated estimator $\hat{\alpha^*}$ for different values of α and different values of ρ (ranging from 0 to 1) are displayed in Table 2.

Table 2. Comparison of RMSEs of modified truncated estimator (RMSE1) and Hill estimator (RMSE2, RMSE3, and RMSE4) with relocations of true mean, sample mean and median.

					Relocation		
					True Mean = 0	**Sample Mean**	**Sample Median**
α	ρ	**Sample Size**	**RMSE1**	k^*	**RMSE2**	**RMSE3**	**RMSE4**
1.01	0.2	100	0.483	12	0.486	0.529	0.514
1.01	0.3	100	0.468	12	0.414	0.429	0.423
1.01	0.4	100	0.412	12	0.408	0.409	0.411
1.01	0.5	100	0.320	12	0.428	0.415	0.432
1.01	0.6	100	0.277	12	0.438	0.409	0.441
1.01	0.7	100	0.272	12	0.395	0.380	0.404
1.01	0.8	100	0.299	12	0.418	0.414	0.427
1.01	0.9	100	0.403	12	0.419	0.465	0.424
1.01	0.2	250	0.395	22	0.254	0.255	0.254
1.01	0.3	250	0.353	22	0.258	0.261	0.258
1.01	0.4	250	0.242	22	0.253	0.255	0.254
1.01	0.5	250	0.189	22	0.251	0.252	0.253
1.01	0.6	250	0.168	22	0.255	0.250	0.256
1.01	0.7	250	0.165	22	0.259	0.252	0.260
1.01	0.8	250	0.179	22	0.247	0.240	0.248
1.01	0.9	250	0.238	22	0.256	0.256	0.257
1.01	0.2	500	0.360	37	0.181	0.181	0.181
1.01	0.3	500	0.251	37	0.180	0.181	0.180
1.01	0.4	500	0.161	37	0.178	0.179	0.179
1.01	0.5	500	0.131	37	0.180	0.181	0.180

Table 2. *Cont.*

					Relocation		
					True Mean = 0	Sample Mean	Sample Median
α	ρ	Sample Size	RMSE1	k^*	RMSE2	RMSE3	RMSE4
1.01	0.6	500	0.118	37	0.176	0.176	0.177
1.01	0.7	500	0.115	37	0.181	0.179	0.181
1.01	0.8	500	0.125	37	0.180	0.176	0.180
1.01	0.9	500	0.162	37	0.183	0.181	0.183
1.01	0.2	1000	0.295	64	0.131	0.131	0.131
1.01	0.3	1000	0.161	64	0.132	0.132	0.132
1.01	0.4	1000	0.113	64	0.132	0.132	0.132
1.01	0.5	1000	0.092	64	0.132	0.133	0.132
1.01	0.6	1000	0.081	64	0.131	0.131	0.131
1.01	0.7	1000	0.080	64	0.131	0.131	0.131
1.01	0.8	1000	0.086	64	0.133	0.131	0.134
1.01	0.9	1000	0.110	64	0.131	0.129	0.131
1.01	0.2	2000	0.220	83	0.114	0.114	0.114
1.01	0.3	2000	0.110	83	0.117	0.117	0.117
1.01	0.4	2000	0.078	83	0.116	0.116	0.116
1.01	0.5	2000	0.064	83	0.115	0.115	0.115
1.01	0.6	2000	0.058	83	0.114	0.114	0.114
1.01	0.7	2000	0.057	83	0.115	0.115	0.115
1.01	0.8	2000	0.062	83	0.114	0.114	0.114
1.01	0.9	2000	0.078	83	0.116	0.115	0.116
1.01	0.2	5000	0.125	193	0.074	0.074	0.074
1.01	0.3	5000	0.068	193	0.073	0.073	0.073
1.01	0.4	5000	0.049	193	0.073	0.073	0.073
1.01	0.5	5000	0.040	193	0.073	0.073	0.073
1.01	0.6	5000	0.037	193	0.073	0.073	0.073
1.01	0.7	5000	0.036	193	0.073	0.074	0.073
1.01	0.8	5000	0.039	193	0.074	0.074	0.074
1.01	0.9	5000	0.049	193	0.073	0.072	0.073
1.01	0.2	10000	0.083	282	0.060	0.060	0.060
1.01	0.3	10000	0.047	282	0.061	0.061	0.061
1.01	0.4	10000	0.035	282	0.061	0.061	0.061
1.01	0.5	10000	0.029	282	0.061	0.061	0.061
1.01	0.6	10000	0.026	282	0.061	0.061	0.061
1.01	0.7	10000	0.026	282	0.062	0.062	0.062
1.01	0.8	10000	0.027	282	0.061	0.061	0.061
1.01	0.9	10000	0.034	282	0.061	0.061	0.061
1.25	0.2	100	0.549	18	0.360	0.390	0.368
1.25	0.3	100	0.450	18	0.364	0.352	0.377
1.25	0.4	100	0.398	18	0.357	0.321	0.364
1.25	0.5	100	0.333	18	0.362	0.319	0.366
1.25	0.6	100	0.269	18	0.358	0.325	0.368
1.25	0.7	100	0.252	18	0.362	0.342	0.370
1.25	0.8	100	0.264	18	0.363	0.362	0.370
1.25	0.9	100	0.346	18	0.376	0.425	0.380
1.25	0.2	250	0.413	42	0.202	0.213	0.206
1.25	0.3	250	0.355	42	0.205	0.202	0.208
1.25	0.4	250	0.282	42	0.203	0.194	0.208
1.25	0.5	250	0.201	42	0.199	0.189	0.205
1.25	0.6	250	0.163	42	0.207	0.193	0.210
1.25	0.7	250	0.154	42	0.201	0.191	0.203
1.25	0.8	250	0.161	42	0.203	0.201	0.207
1.25	0.9	250	0.207	42	0.205	0.219	0.208
1.25	0.2	500	0.337	82	0.140	0.148	0.142
1.25	0.3	500	0.290	82	0.140	0.139	0.142
1.25	0.4	500	0.192	82	0.141	0.135	0.143

Table 2. *Cont.*

					Relocation		
					True Mean = 0	Sample Mean	Sample Median
α	ρ	Sample Size	RMSE1	k *	RMSE2	RMSE3	RMSE4
1.25	0.5	500	0.135	82	0.141	0.134	0.144
1.25	0.6	500	0.115	82	0.141	0.134	0.143
1.25	0.7	500	0.106	82	0.140	0.134	0.142
1.25	0.8	500	0.112	82	0.139	0.137	0.141
1.25	0.9	500	0.143	82	0.140	0.147	0.143
1.25	0.2	1000	0.296	159	0.099	0.105	0.101
1.25	0.3	1000	0.222	159	0.101	0.101	0.102
1.25	0.4	1000	0.128	159	0.099	0.097	0.101
1.25	0.5	1000	0.095	159	0.099	0.095	0.100
1.25	0.6	1000	0.079	159	0.098	0.093	0.100
1.25	0.7	1000	0.075	159	0.100	0.096	0.101
1.25	0.8	1000	0.079	159	0.098	0.097	0.100
1.25	0.9	1000	0.100	159	0.100	0.104	0.102
1.25	0.2	2000	0.300	314	0.314	0.316	0.313
1.25	0.3	2000	0.219	314	0.315	0.314	0.313
1.25	0.4	2000	0.088	314	0.070	0.068	0.071
1.25	0.5	2000	0.067	314	0.071	0.068	0.072
1.25	0.6	2000	0.056	314	0.070	0.066	0.071
1.25	0.7	2000	0.053	314	0.070	0.067	0.071
1.25	0.8	2000	0.055	314	0.069	0.068	0.071
1.25	0.9	2000	0.070	314	0.070	0.072	0.071
1.25	0.2	5000	0.206	314	0.044	0.047	0.045
1.25	0.3	5000	0.087	776	0.045	0.045	0.045
1.25	0.4	5000	0.055	776	0.044	0.043	0.045
1.25	0.5	5000	0.042	776	0.044	0.043	0.045
1.25	0.6	5000	0.035	776	0.045	0.043	0.045
1.25	0.7	5000	0.034	776	0.045	0.043	0.045
1.25	0.8	5000	0.036	776	0.045	0.044	0.045
1.25	0.9	5000	0.044	776	0.045	0.046	0.045
1.25	0.2	10000	0.141	1547	0.032	0.034	0.032
1.25	0.3	10000	0.061	1547	0.032	0.032	0.032
1.25	0.4	10000	0.039	1547	0.031	0.030	0.031
1.25	0.5	10000	0.030	1547	0.031	0.030	0.032
1.25	0.6	10000	0.025	1547	0.031	0.030	0.032
1.25	0.7	10000	0.024	1547	0.031	0.030	0.032
1.25	0.8	10000	0.025	1547	0.031	0.031	0.032
1.25	0.9	10000	0.031	1547	0.031	0.032	0.032
1.5	0.2	100	0.702	21	0.413	0.435	0.408
1.5	0.3	100	0.461	21	0.393	0.341	0.394
1.5	0.4	100	0.370	21	0.404	0.332	0.396
1.5	0.5	100	0.311	21	0.382	0.326	0.378
1.5	0.6	100	0.259	21	0.402	0.342	0.393
1.5	0.7	100	0.226	21	0.386	0.350	0.385
1.5	0.8	100	0.227	21	0.398	0.374	0.390
1.5	0.9	100	0.278	21	0.379	0.393	0.376
1.5	0.2	250	0.499	51	0.222	0.228	0.221
1.5	0.3	250	0.343	51	0.223	0.203	0.221
1.5	0.4	250	0.283	51	0.221	0.196	0.220
1.5	0.5	250	0.213	51	0.221	0.196	0.220
1.5	0.6	250	0.161	51	0.222	0.198	0.219
1.5	0.7	250	0.139	51	0.221	0.201	0.219
1.5	0.8	250	0.138	51	0.219	0.208	0.219
1.5	0.9	250	0.171	51	0.219	0.223	0.218
1.5	0.2	500	0.388	101	0.151	0.155	0.152

Table 2. *Cont.*

					Relocation		
					True Mean = 0	Sample Mean	Sample Median
α	ρ	Sample Size	RMSE1	k^*	RMSE2	RMSE3	RMSE4
1.5	0.3	500	0.285	101	0.152	0.140	0.151
1.5	0.4	500	0.226	101	0.152	0.137	0.152
1.5	0.5	500	0.153	101	0.155	0.139	0.156
1.5	0.6	500	0.113	101	0.150	0.136	0.151
1.5	0.7	500	0.098	101	0.152	0.140	0.151
1.5	0.8	500	0.097	101	0.152	0.147	0.153
1.5	0.9	500	0.121	101	0.153	0.156	0.152
1.5	0.2	1000	0.311	201	0.105	0.106	0.105
1.5	0.3	1000	0.245	201	0.106	0.099	0.106
1.5	0.4	1000	0.166	201	0.105	0.096	0.105
1.5	0.5	1000	0.105	201	0.106	0.095	0.105
1.5	0.6	1000	0.079	201	0.107	0.096	0.106
1.5	0.7	1000	0.069	201	0.106	0.099	0.106
1.5	0.8	1000	0.068	201	0.106	0.101	0.106
1.5	0.9	1000	0.084	201	0.106	0.109	0.107
1.5	0.2	2000	0.261	400	0.075	0.076	0.075
1.5	0.3	2000	0.204	400	0.075	0.070	0.075
1.5	0.4	2000	0.113	400	0.074	0.068	0.074
1.5	0.5	2000	0.072	400	0.075	0.068	0.075
1.5	0.6	2000	0.056	400	0.074	0.068	0.075
1.5	0.7	2000	0.048	400	0.073	0.068	0.074
1.5	0.8	2000	0.048	400	0.074	0.071	0.075
1.5	0.9	2000	0.059	400	0.075	0.076	0.075
1.5	0.2	5000	0.222	995	0.047	0.048	0.047
1.5	0.3	5000	0.133	995	0.047	0.044	0.047
1.5	0.4	5000	0.069	995	0.047	0.043	0.048
1.5	0.5	5000	0.046	995	0.047	0.042	0.047
1.5	0.6	5000	0.035	995	0.047	0.043	0.047
1.5	0.7	5000	0.031	995	0.047	0.044	0.047
1.5	0.8	5000	0.030	995	0.047	0.045	0.047
1.5	0.9	5000	0.037	995	0.047	0.048	0.047
1.5	0.2	10000	0.201	1991	0.033	0.034	0.034
1.5	0.3	10000	0.089	1991	0.033	0.031	0.033
1.5	0.4	10000	0.048	1991	0.033	0.030	0.033
1.5	0.5	10000	0.031	1991	0.033	0.030	0.033
1.5	0.6	10000	0.025	1991	0.033	0.030	0.033
1.5	0.7	10000	0.021	1991	0.033	0.031	0.033
1.5	0.8	10000	0.022	1991	0.033	0.032	0.033
1.5	0.9	10000	0.026	1991	0.033	0.033	0.033
1.75	0.2	100	0.890	22	0.469	0.478	0.443
1.75	0.3	100	0.590	22	0.471	0.388	0.448
1.75	0.4	100	0.378	22	0.489	0.378	0.457
1.75	0.5	100	0.276	22	0.479	0.378	0.441
1.75	0.6	100	0.222	22	0.493	0.399	0.446
1.75	0.7	100	0.183	22	0.470	0.408	0.449
1.75	0.8	100	0.169	22	0.491	0.430	0.463
1.75	0.9	100	0.201	22	0.466	0.439	0.442
1.75	0.2	250	0.652	54	0.264	0.253	0.252
1.75	0.3	250	0.400	54	0.265	0.225	0.255
1.75	0.4	250	0.266	54	0.263	0.219	0.252
1.75	0.5	250	0.201	54	0.260	0.219	0.249
1.75	0.6	250	0.153	54	0.262	0.226	0.251
1.75	0.7	250	0.120	54	0.258	0.229	0.250
1.75	0.8	250	0.108	54	0.264	0.239	0.251

Table 2. *Cont.*

					Relocation		
					True Mean = 0	Sample Mean	Sample Median
α	ρ	Sample Size	RMSE1	k *	RMSE2	RMSE3	RMSE4
1.75	0.9	250	0.125	54	0.263	0.250	0.251
1.75	0.2	500	0.520	107	0.181	0.173	0.173
1.75	0.3	500	0.308	107	0.181	0.154	0.173
1.75	0.4	500	0.213	107	0.179	0.152	0.172
1.75	0.5	500	0.159	107	0.180	0.152	0.172
1.75	0.6	500	0.114	107	0.180	0.156	0.174
1.75	0.7	500	0.084	107	0.179	0.157	0.171
1.75	0.8	500	0.077	107	0.179	0.164	0.173
1.75	0.9	500	0.088	107	0.180	0.171	0.171
1.75	0.2	1000	0.404	214	0.123	0.119	0.119
1.75	0.3	1000	0.242	214	0.123	0.107	0.119
1.75	0.4	1000	0.172	214	0.122	0.104	0.118
1.75	0.5	1000	0.118	214	0.125	0.107	0.120
1.75	0.6	1000	0.080	214	0.124	0.108	0.119
1.75	0.7	1000	0.060	214	0.122	0.109	0.118
1.75	0.8	1000	0.054	214	0.123	0.112	0.118
1.75	0.9	1000	0.062	214	0.123	0.118	0.118
1.75	0.2	2000	0.324	428	0.088	0.083	0.084
1.75	0.3	2000	0.199	428	0.087	0.077	0.084
1.75	0.4	2000	0.141	428	0.085	0.073	0.082
1.75	0.5	2000	0.086	428	0.086	0.074	0.082
1.75	0.6	2000	0.057	428	0.087	0.076	0.083
1.75	0.7	2000	0.043	428	0.086	0.077	0.083
1.75	0.8	2000	0.038	428	0.087	0.079	0.083
1.75	0.9	2000	0.045	428	0.087	0.083	0.084
1.75	0.2	5000	0.244	1070	0.054	0.052	0.052
1.75	0.3	5000	0.159	1070	0.055	0.047	0.053
1.75	0.4	5000	0.094	1070	0.054	0.046	0.052
1.75	0.5	5000	0.054	1070	0.055	0.047	0.053
1.75	0.6	5000	0.035	1070	0.054	0.047	0.052
1.75	0.7	5000	0.027	1070	0.054	0.048	0.052
1.75	0.8	5000	0.024	1070	0.054	0.050	0.052
1.75	0.9	5000	0.028	1070	0.055	0.052	0.053
1.75	0.2	10000	0.199	2139	0.038	0.037	0.037
1.75	0.3	10000	0.133	2139	0.039	0.034	0.037
1.75	0.4	10000	0.067	2139	0.039	0.033	0.037
1.75	0.5	10000	0.038	2139	0.038	0.033	0.037
1.75	0.6	10000	0.025	2139	0.038	0.034	0.037
1.75	0.7	10000	0.019	2139	0.038	0.034	0.037
1.75	0.8	10000	0.017	2139	0.038	0.035	0.037
1.75	0.9	10000	0.020	2139	0.038	0.037	0.037
1.9	0.2	100	1.038	22	0.568	0.542	0.504
1.9	0.3	100	0.672	22	0.563	0.432	0.507
1.9	0.4	100	0.428	22	0.531	0.416	0.488
1.9	0.5	100	0.274	22	0.549	0.430	0.498
1.9	0.6	100	0.189	22	0.562	0.449	0.510
1.9	0.7	100	0.139	22	0.586	0.460	0.492
1.9	0.8	100	0.114	22	0.585	0.493	0.512
1.9	0.9	100	0.125	22	0.566	0.502	0.511
1.9	0.2	250	0.761	55	0.287	0.267	0.264
1.9	0.3	250	0.462	55	0.287	0.232	0.264
1.9	0.4	250	0.280	55	0.292	0.232	0.268
1.9	0.5	250	0.179	55	0.287	0.233	0.264
1.9	0.6	250	0.127	55	0.288	0.238	0.264
1.9	0.7	250	0.092	55	0.292	0.247	0.268

Table 2. *Cont.*

					Relocation		
					True Mean = 0	Sample Mean	Sample Median
α	ρ	Sample Size	RMSE1	k *	RMSE2	RMSE3	RMSE4
1.9	0.8	250	0.077	55	0.288	0.253	0.265
1.9	0.9	250	0.082	55	0.288	0.262	0.268
1.9	0.2	500	0.601	110	0.193	0.179	0.177
1.9	0.3	500	0.350	110	0.196	0.162	0.182
1.9	0.4	500	0.213	110	0.197	0.162	0.184
1.9	0.5	500	0.137	110	0.195	0.162	0.181
1.9	0.6	500	0.095	110	0.192	0.163	0.180
1.9	0.7	500	0.070	110	0.193	0.168	0.181
1.9	0.8	500	0.056	110	0.193	0.172	0.180
1.9	0.9	500	0.058	110	0.192	0.176	0.180
1.9	0.2	1000	0.487	220	0.133	0.123	0.125
1.9	0.3	1000	0.272	220	0.134	0.113	0.126
1.9	0.4	1000	0.161	220	0.133	0.110	0.124
1.9	0.5	1000	0.105	220	0.134	0.112	0.125
1.9	0.6	1000	0.073	220	0.136	0.115	0.126
1.9	0.7	1000	0.052	220	0.134	0.117	0.126
1.9	0.8	1000	0.041	220	0.135	0.119	0.124
1.9	0.9	1000	0.042	220	0.136	0.124	0.128
1.9	0.2	2000	0.399	438	0.095	0.087	0.088
1.9	0.3	2000	0.212	438	0.094	0.079	0.088
1.9	0.4	2000	0.126	438	0.095	0.078	0.088
1.9	0.5	2000	0.082	438	0.093	0.078	0.087
1.9	0.6	2000	0.055	438	0.094	0.080	0.087
1.9	0.7	2000	0.038	438	0.094	0.081	0.087
1.9	0.8	2000	0.029	438	0.093	0.083	0.087
1.9	0.9	2000	0.030	438	0.095	0.086	0.088
1.9	0.2	5000	0.303	1093	0.059	0.054	0.055
1.9	0.3	5000	0.153	1093	0.059	0.050	0.056
1.9	0.4	5000	0.091	1093	0.059	0.049	0.055
1.9	0.5	5000	0.058	1093	0.059	0.049	0.055
1.9	0.6	5000	0.037	1093	0.059	0.051	0.056
1.9	0.7	5000	0.024	1093	0.059	0.051	0.056
1.9	0.8	5000	0.018	1093	0.060	0.053	0.056
1.9	0.9	5000	0.019	1093	0.059	0.054	0.055
1.9	0.2	10000	0.245	2187	0.041	0.038	0.039
1.9	0.3	10000	0.123	2187	0.042	0.035	0.040
1.9	0.4	10000	0.072	2187	0.042	0.035	0.039
1.9	0.5	10000	0.043	2187	0.041	0.035	0.039
1.9	0.6	10000	0.025	2187	0.041	0.035	0.039
1.9	0.7	10000	0.017	2187	0.041	0.036	0.039
1.9	0.8	10000	0.013	2187	0.042	0.038	0.040
1.9	0.9	10000	0.013	2187	0.041	0.038	0.039

* The value of k is obtained by linear interpolation from Dufour and Kurz-Kim (2010).

6. Comparison of the Proposed Estimator With the Hill Estimator and the Characteristic Function Based Estimator

Next, we want to compare the performance of this modified truncated estimator with that of a popular estimator known as Hill-estimator Dufour and Kurz-Kim (2010); Hill (1975), which is a simple non-parametric estimator based on order statistic. Given a sample of n observations $X_1, X_2, ...X_n$, the Hill-estimator is defined as,

$$\hat{\alpha}_H = [(k^{-1} \sum_{j=1}^{k} \ln X_{n+1-j} : n) - \ln X_{n-k:n}]^{-1}$$

with standard error

$$SD(\hat{\alpha}_H) = \frac{k\hat{\alpha}_H}{(k-1)\sqrt{k-2}}$$

where k is the number of observations which lie on the tails of the distribution of interest and is to be optimally chosen depending on the sample size, n, tail thickness α, as $k = k(n, \alpha)$ and $X_{j:n}$ denotes j-order statistic of the sample of size n.

The asymptotic normality of the Hill estimator is provided by Goldie and Richard (1987) as,

$$\sqrt{k}(\hat{\alpha_H^{-1}} - \alpha^{-1}) \xrightarrow{L} N(0, \alpha^{-2}) \tag{3}$$

Lemma 4.

$$(\hat{\alpha_H} - \alpha) \xrightarrow{L} N\Big(0, \frac{1}{\alpha^2 k}\Big)$$

Proof. Assuming $g\Big(\hat{\alpha_H^{-1}}\Big) = \frac{1}{\hat{\alpha_H^{-1}}} = \hat{\alpha_H}$ (since $g'(.)$ exists and is non-zero valued) and using Equation (3), we get

$$\begin{aligned}
&(\hat{\alpha_H^{-1}} - \alpha^{-1}) \xrightarrow{L} N\Big(0, \frac{\alpha^{-2}}{k}\Big) \\
&\Rightarrow (\hat{\alpha_H} - \alpha) \xrightarrow{L} N\Big(0, \frac{(g'^{-1}))^2\alpha^{-2}}{k}\Big) \\
&\Leftrightarrow \hat{\alpha_H} \xrightarrow{L} N\Big(\alpha, \frac{1}{\alpha^2 k}\Big)
\end{aligned}$$

□

We need this result for comparing the performances of the estimators for α.

In addition, we make a comparison of the performance of the modified truncated estimator $\hat{\alpha_2}$ with that of the characteristic function based estimator Anderson and Arnold (1993), which is obtained by minimization of the objective function (where $\mu = 0$ and $\sigma = 1$) given by

$$\hat{I}_s(\alpha) = \sum_{i=1}^{n} w_i(\hat{\eta}(z_i) - \exp(-|z_i|^{\alpha}))^2 \tag{4}$$

The performance of the modified truncated estimator $\hat{\alpha_3}$ is compared with that of the characteristic function-based estimator Anderson and Arnold (1993), which is obtained by minimization of the objective function (where $\mu = 0$ and σ unknown) given by,

$$\hat{I'}_s(\alpha) = \sum_{i=1}^{n} w_i(\hat{\eta}(z_i) - \exp(-|\sigma z_i|^{\alpha}))^2 \tag{5}$$

where

$$\hat{\eta}(t) = \frac{1}{n} \sum_{j} \cos(tx_j).$$

$x_1, x_2, ..., x_n$ are realizations from symmetric stable (α) distribution, z_i is the ith zero of the mth degree Hermite polynomial $H_m(z)$ and

$$w_i = \frac{2^{m-1} m! \sqrt{m}}{(mH_{m-1}(z_i))^2}$$

It is to be noted that, for the estimator of $\alpha < 1$, we do not know any explicit form of the probability density function. However, for value of the estimator between 1 and 2, i.e., for $1 < \hat{\alpha^*} < 2$, we may compare the fit with the stable family by modeling a mixture of normal and Cauchy distribution and then using the method as proposed in Anderson and Arnold (1993) by the objective function given by

$$\sum_{i=1}^{n} w_i (\hat{\eta}(z_i) - \psi_{NC})^2$$

where $\hat{\eta}(t)$ is the same as defined above with the realizations taken from the mixture distribution. ψ_{NC} denotes the corresponding theoretical characteristic function given by

$$\psi_{NC} = p \exp(-\sigma_1^2 t^2 / 2) + (1 - p) \exp(-\sigma_2 |t|)$$

where p denotes the mixture proportion, σ_1 and σ_2 are taken as the scale parameters of the normal and Cauchy distributions, respectively (the location parameters are taken as zeros, the reason for which is mentioned above). Finally, a measure for the goodness of fit is proposed as:

Index of Objective function (I.O.) = Objective function + Number of parameters estimated

The distribution for which I.O. is minimum gives the best fit to the data.

The modified truncated estimator based on the moment estimator is free of the location parameter since it is defined in terms of $\bar{R}_j = \frac{1}{m} \sum_{i=1}^{m} \cos j(\theta_i - \bar{\theta}), j = 1, 2$, that is in terms of the quantity $(\theta_i - \bar{\theta})$, which is centered with respect to the mean direction $\bar{\theta}$, although it is not free of the nuisance parameter that is the concentration parameter ρ. The Hill estimator is scale invariant since it is defined in terms of log of ratios but not location invariant. Therefore, centering needs to be done in order to take care of the location invariance.

7. Computational Studies

The analytical variance of the untruncated moment estimator was compared with that of the modified truncated estimator, as presented in Table 1, for values of $\alpha < 1$, which is more applicable in practical situations for volatile data.

The comparison of the performances of the two estimators is shown in Table 2. The parameter configurations were chosen as given by Hill (1975) and Dufour and Kurz-Kim (2010). The simulation is presented in Table 2 for the values of α = 1.01, 1.25, 1.5, 1.75, and 1.9 each with sample size n = 100, 250, 500, 1000, 2000, 5000, and 10,000 and for different values of ρ = 0.2, 0.4, 0.6, and 0.8 when skewness parameter β = 0, location parameter μ = 0, and scale parameter $\sigma = (-\ln(\rho))^{(1/\alpha)}$, i.e., concentration parameter $\rho = e^{-\sigma^\alpha}$. For each combination of α and n, 10,000 replications were performed. In this simulation, the sample was relocated by three different relocations, viz. true mean = 0, estimated sample mean, and estimated sample median, and comparison of the root mean square errors (RMSEs) was made.

Next, in Table 3, comparison of the performance of the modified truncated estimator $\hat{\alpha_2}$ with that of the characteristic function-based estimator where the simulation is presented for the values of α = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, and 2.0 each with sample size n = 20, 30, 40, and 50, and the values of σ were taken as 3, 5, and 10 . For each combination of α and n, 10,000 replications were performed.

Table 3. Comparison of the RMSEs of the modified truncated estimator $\hat{\alpha_3}$ (RMSE3) and the characteristic function-based estimator (RMSE4) when $\mu = 0$ and σ unknown.

α	σ	Sample Size	RMSE3	RMSE4
0.2	3.0	20	0.514	1.477
0.4	3.0	20	0.495	1.293
0.6	3.0	20	0.421	1.134
0.8	3.0	20	0.401	1.012
1.0	3.0	20	0.446	0.912
1.2	3.0	20	0.510	0.823
1.4	3.0	20	0.588	0.757
1.6	3.0	20	0.680	0.733
1.8	3.0	20	0.763	0.746
2.0	3.0	20	0.851	0.798
0.2	5.0	20	0.512	1.424
0.4	5.0	20	0.421	1.245
0.6	5.0	20	0.346	1.110
0.8	5.0	20	0.354	0.989
1.0	5.0	20	0.411	0.882
1.2	5.0	20	0.497	0.776
1.4	5.0	20	0.572	0.687
1.6	5.0	20	0.668	0.635
1.8	5.0	20	0.763	0.625
2.0	5.0	20	0.859	0.623
0.2	3.0	30	0.471	1.486
0.4	3.0	30	0.468	1.299
0.6	3.0	30	0.416	1.127
0.8	3.0	30	0.407	1.006
1.0	3.0	30	0.453	0.895
1.2	3.0	30	0.521	0.817
1.4	3.0	30	0.605	0.753
1.6	3.0	30	0.686	0.734
1.8	3.0	30	0.767	0.748
2.0	3.0	30	0.859	0.803
0.2	5.0	30	0.476	1.433
0.4	5.0	30	0.419	1.234
0.6	5.0	30	0.339	1.103
0.8	5.0	30	0.354	0.987
1.0	5.0	30	0.422	0.885
1.2	5.0	30	0.494	0.782
1.4	5.0	30	0.583	0.709
1.6	5.0	30	0.685	0.658
1.8	5.0	30	0.770	0.669
2.0	5.0	30	0.874	0.673
0.2	3.0	40	0.426	1.494
0.4	3.0	40	0.467	1.300
0.6	3.0	40	0.418	1.123
0.8	3.0	40	0.414	0.996
1.0	3.0	40	0.462	0.891
1.2	3.0	40	0.519	0.806
1.4	3.0	40	0.595	0.750
1.6	3.0	40	0.689	0.724
1.8	3.0	40	0.784	0.757
2.0	3.0	40	0.887	0.807
0.2	5.0	40	0.444	1.439
0.4	5.0	40	0.412	1.242
0.6	5.0	40	0.338	1.100
0.8	5.0	40	0.354	0.989
1.0	5.0	40	0.422	0.880
1.2	5.0	40	0.487	0.784

Table 3. *Cont.*

α	σ	Sample Size	RMSE3	RMSE4
1.4	5.0	40	0.584	0.720
1.6	5.0	40	0.680	0.674
1.8	5.0	40	0.769	0.692
2.0	5.0	40	0.881	0.711
0.2	3.0	50	0.393	1.500
0.4	3.0	50	0.447	1.292
0.6	3.0	50	0.411	1.117
0.8	3.0	50	0.414	0.989
1.0	3.0	50	0.466	0.885
1.2	3.0	50	0.530	0.805
1.4	3.0	50	0.612	0.737
1.6	3.0	50	0.698	0.719
1.8	3.0	50	0.778	0.751
2.0	3.0	50	0.870	0.828
0.2	5.0	50	0.411	1.451
0.4	5.0	50	0.402	1.235
0.6	5.0	50	0.344	1.098
0.8	5.0	50	0.357	0.983
1.0	5.0	50	0.415	0.879
1.2	5.0	50	0.502	0.788
1.4	5.0	50	0.598	0.716
1.6	5.0	50	0.677	0.691
1.8	5.0	50	0.782	0.703
2.0	5.0	50	0.858	0.729

The asymptotic variance of the characteristic function-based estimator, unlike that of the modified truncated estimator, is not available in any closed analytical form. We are thus unable to present the Asymptotic Relative Efficiency (ARE) of these estimators of α analytically. Instead, we compared these through their MSEs based on extensive simulations over all reasonable small, moderate, and large sample sizes.

8. Applications

8.1. Inference on the Gold Price Data (In US Dollars) (1980–2013)

Gold price data, say x_t, were collected per ounce in US dollars over the years 1980–2013. These were transformed as $z_t = 100(\ln(x_t) - \ln(x_{t-1}))$, which were then "wrapped" to obtain $\theta_t = z_t mod 2\pi$ and finally transformed to $\widehat{\theta} = (\theta_t - \bar{\theta})$ mod 2π, where $\bar{\theta}$ denotes the mean direction of θ_t and $\widehat{\theta}$ denotes the variable thetamod as used in the graphs. The Durbin–Watson test performed on the log ratio transformed data shows that the autocorrelation is zero. The test statistic of Watson's goodness of fit Jammalamadaka and SenGupta (2001) for wrapped stable distribution was obtained as 0.01632691 and the corresponding p-value was obtained as 0.9970284, which is greater than 0.05, indicating that the wrapped stable distribution fits the transformed gold price data (in US dollars). The modified truncated estimate $\widehat{\alpha_1^*}$ is 0.3752206 while the estimate by characteristic function method is 0.401409. The value of the objective function using the characteristic function estimate is 2.218941 while that using our modified truncated estimate is 2.411018.

8.2. Inference on the Silver Price Data (In US Dollars) (1980–2013)

Data on the price of silver in US dollars collected per ounce over the same time period also underwent the same transformation. The Durbin–Watson test performed on the log ratio transformed data shows that the autocorrelation is zero. Here, the Watson's goodness of fit test for wrapped stable distribution was also performed and the value of the statistic was obtained as 0.02530653 and the corresponding p-value is 0.9639666, which is greater than 0.05, indicating that the wrapped stable distribution also fits the transformed silver price data (in US dollars). The modified truncated estimate of the index parameter α is 0.4112475 while the estimate by characteristic function method is 0.644846. The value of the objective function using the characteristic function estimate is 2.234203 while that using our modified truncated estimate is 2.234432.

8.3. Inference on the Silver Price Data (In INR) (1970–2011)

Data on the price of silver in INR were also collected per 10 grams over the same time period. The p-value for the Durbin–Watson test performed on the log ratio transformed data is 0.3437, which indicates that the autocorrelation is zero. Here, the Watson's goodness of fit test was also performed on the transformed data and the value of the statistic was obtained as 0.03382334 and the corresponding p-value is 0.8919965, which is greater than 0.05, indicating that the wrapped stable distribution also fits the silver price data (in INR). The estimate $\hat{\alpha_1^*}$ is 1.142171, which is the same as the characteristic function estimate. The value of the objective function using the characteristic function estimate is 2.813234 while that using our modified truncated estimate is 2.665166. Since the estimate of α lies between 1 and 2, a mixture of normal and Cauchy distributions is used in Anderson and Arnold (1993) to estimate the respective parameters. The initial values of the scale parameter (σ_1) for the normal distribution is taken as the sample standard deviation and that for the Cauchy distribution (σ_2) is taken as the sample quartile deviation. In addition, different initial values of the mixing parameter p yield the same estimate of the parameters, viz. $\hat{p} = 0.165$, $\hat{\sigma_1} = 14.38486$, and $\hat{\sigma_2} = 0.077$, and the value of the objective function was found to be 0.9308165. Then, the value of I.O. using modified truncated estimate (assuming stable distribution) is 4.665166 (2.665166 + 2), using the characteristic function estimate (assuming stable distribution) is 4.813234 (2.813234 + 2), and using the characteristic function estimate (assuming mixture of normal and Cauchy distribution) is 3.9308165 (0.9308165 + 3). Thus, it can be observed using the I.O. measure that a mixture of normal and Cauchy distribution gives the best fit to the data. The maximum likelihood estimate of α assuming wrapped stable distribution is 1.1421361. Akaike's information criterion (AIC) value assuming wrapped stable distribution is 153.5426 and that assuming a mixture of normal and Cauchy distribution is 201.4.

8.4. Inference on the Box and Jenkins Stock Price Data

Series B Box and Jenkins (IBM) common stock closing price data obtained from Box et al. (2016) were also transformed similarly as for the preceding one. The Durbin–Watson test performed on the log ratio transformed data shows that the autocorrelation is zero. Watson's test statistic for the goodness of fit test was obtained as 0.0554223 and the corresponding p-value was obtained as 0.6442058, which is greater than 0.05, indicating that the wrapped stable distribution fits the stock price data. The estimates of the index parameter α and the concentration parameter ρ as obtained by modified truncation method are 1.102854 and 0.4335457, respectively.

9. Findings and Concluding Remarks

It can be observed from Table 1 that the asymptotic variance of the untruncated estimator is reduced for the corresponding truncated estimator, indicating the efficiency of the truncated estimator.

It can also be noted from Table 2 that, for α = 1.01, the RMSE of the modified truncated estimator is less than that of the Hill estimator when the sample is relocated by three different relocations, viz. true mean = 0, sample mean, and sample median, for higher values of the concentration parameter ρ = 0.5, 0.6, 0.8, and 0.9 for sample sizes n = 100, 250, 500, and 1000 and for ρ = 0.3, 0.4, 0.6, 0.8, and 0.9 for sample sizes n = 2000, 5000, and 10,000. Furthermore, it can be observed that, for α = 1.25, 1.5, 1.75 and 1.9, the RMSE of the modified truncated is less than that of the Hill estimator for different relocations for ρ = 0.6, 0.7, 0.8, and 0.9 for smaller sample size and even for ρ = 0.5 for larger sample size. This clearly indicates the efficiency of the modified truncated estimator over the Hill estimator for higher values of the concentration parameter ρ.

It can be observed in Table 3 that the RMSE of the modified truncated estimator is less than that of the characteristic function-based estimator for almost all values of α corresponding to all values of σ.

The Hill estimator (Dufour and Kurz-Kim (2010)) is defined for $1 \leq \alpha \leq 2$, whereas the modified truncated estimator is defined for the whole range $0 \leq \alpha \leq 2$. In addition, the overall performance of the modified truncated estimator is quite good in terms of efficiency and consistency over both the Hill estimator and the characteristic function-based estimator.

Thus, we have established an estimator of the index parameter α that strongly supports its parameter space $(0, 2]$. It can be observed from the above real life data applications that the modified truncated estimator is quite close to that of the characteristic function-based estimator. In addition, it is simpler and computationally easier than that of the estimator defined in Anderson and Arnold (1993). Thus, it may be considered as a better estimator.

Again, when the estimator of α lies between 1 and 2, is attempted to model a mixture of two distributions with the value of the index parameter as that of the two extreme tails that is modeling a mixture of Cauchy ($\alpha = 1$) and normal ($\alpha = 2$) distributions when $1 < \alpha < 2$ or modeling a mixture of Double Exponential ($\alpha = \frac{1}{2}$) and Cauchy ($\alpha = 1$) distributions when $\frac{1}{2} < \alpha < 1$. Then, it is compared with that of the stable family of distributions for goodness of fit.

We could have used the usual technique of non-linear optimization as used in Salimi et al. (2018) for estimation, but it is computationally demanding and also the (statistical) consistency of the estimators obtained by such method is unknown. In contrast, our proposed methods of trigonometric moment and modified truncated estimation are much simpler, computationally easier and also possess useful consistency properties and, even their asymptotic distributions can be presented in simple and elegant forms as already proved above.

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Appendix A

Proof of Lemma 1. Putting $p = 1$ in Equation (2) and using the expansion of the characteristic function of θ, we get (henceforth, we denote $E(X)$ and $V(X)$ as expectation and variance of a random variable X, respectively, as usual)

$$E \cos\theta = \rho \cos\mu_0 \tag{A1}$$
$$\Rightarrow E(\bar{C}_1) = E\left[\frac{1}{m}\sum_{i=1}^{m}\cos\theta_i\right] = \rho\cos\mu_0$$

Again, putting $p = 2$ in Equation (2) and using the expansion of the characteristic function of θ, we get

$$E \cos 2\theta = \rho^{2^\alpha} \cos 2\mu_0 \tag{A2}$$
$$\Rightarrow E(\bar{C}_2) = E\left[\frac{1}{m}\sum_{i=1}^{m}\cos 2\theta_i\right] = \rho^{2^\alpha}\cos 2\mu_0$$

In addition, Equation (A2) implies that,

$$E\cos^2\theta = \frac{(\rho^{2^\alpha}\cos 2\mu_0 + 1)}{2}$$
$$\Rightarrow V(\cos\theta) = E\cos^2\theta - E^2\cos\theta$$
$$= \frac{\rho^{2^\alpha}\cos 2\mu_0 + 1}{2} - \rho^2\cos^2\mu_0$$

Hence,

$$V(\bar{C_1}) = V\left[\frac{1}{m}\sum_{i=1}^{m}\cos\theta_i\right] \tag{A3}$$
$$= \frac{\rho^{2^\alpha}\cos 2\mu_0 + 1 - 2\rho^2\cos^2\mu_0}{2m} \tag{A4}$$

Now, putting $p = 4$ in Equation (2) and using the expansion of the characteristic function of θ, we get,

$$E\cos 4\theta = \rho^{4^\alpha}\cos 4\mu_0 \tag{A5}$$
$$\Rightarrow E\left[\cos^2 2\theta\right] = \frac{(\rho^{4^\alpha}\cos 4\mu_0 + 1)}{2}$$

Hence,

$$V(\cos 2\theta) = E\cos^2 2\theta - E^2\cos 2\theta$$
$$= \frac{\rho^{4^\alpha}\cos 4\mu_0 + 1}{2} - (\rho^{2^\alpha})^2\cos^2 2\mu_0$$

Therefore,

$$V(\bar{C_2}) = V\left[\frac{1}{m}\sum_{i=1}^{m}\cos 2\theta_i\right]$$
$$= \frac{\rho^{4^\alpha}\cos 4\mu_0 + 1 - 2(\rho^{2^\alpha})^2\cos^2 2\mu_0}{2m} \tag{A6}$$

Now, putting $p = 1$ in Equation (2) and using the expansion of the characteristic function of θ, we get

$$E\sin\theta = \rho\sin\mu_0 \tag{A7}$$
$$\Rightarrow E(\bar{S_1}) = E\left[\frac{1}{m}\sum_{i=1}^{m}\sin\theta_i\right] = \rho\sin\mu_0$$

Again, putting $p = 2$ in Equation (2) and using the expansion of the characteristic function of θ, we get

$$E\sin 2\theta = \rho^{2^\alpha}\sin 2\mu_0 \tag{A8}$$
$$\Rightarrow E(\bar{S_2}) = E\left[\frac{1}{m}\sum_{i=1}^{m}\sin 2\theta_i\right] = \rho^{2^\alpha}\sin 2\mu_0$$

Now, using Equation (A2),

$$E\sin^2\theta = \frac{(1-\rho^{2^\alpha}\cos 2\mu_0)}{2}$$

Hence,

$$\begin{aligned} V(\sin\theta) &= E\sin^2\theta - E^2\sin\theta \\ &= \frac{1-\rho^{2^\alpha}\cos 2\mu_0}{2} - \rho^2\sin^2\mu_0 \end{aligned}$$

Therefore,

$$\begin{aligned} V(\bar{S_1}) &= V\left[\frac{1}{m}\sum_{i=1}^{m}\sin\theta_i\right] \\ &= \frac{1-\rho^{2^\alpha}\cos 2\mu_0 - 2\rho^2\sin^2\mu_0}{2m} \end{aligned}$$

Now, using Equation (A5),

$$E\sin^2 2\theta = \frac{(1-\rho^{4^\alpha}\cos 4\mu_0)}{2}$$

Hence,

$$\begin{aligned} V(\sin 2\theta) &= E\sin^2 2\theta - E^2\sin 2\theta \\ &= \frac{1-\rho^{4^\alpha}\cos 4\mu_0}{2} - (\rho^{2^\alpha})^2\sin^2 2\mu_0 \end{aligned}$$

Therefore,

$$\begin{aligned} V(\bar{S_2}) &= V\left[\frac{1}{m}\sum_{i=1}^{m}\sin 2\theta_i\right] \\ &= \frac{1-\rho^{4^\alpha}\cos 4\mu_0 - 2(\rho^{2^\alpha})^2\sin^2 2\mu_0}{2m} \end{aligned}$$

Now, using Equations (A1), (A7) and (A8),

$$\begin{aligned} Cov(\cos\theta,\sin\theta) &= E\cos\theta\sin\theta - E\cos\theta E\sin\theta \\ &= \frac{\rho^{2^\alpha}\sin 2\mu_0}{2} - \rho\cos\mu_0\rho\sin\mu_0 \end{aligned}$$

Therefore,

$$E\sum_{i=1}^{m}\cos\theta_i\sum_{i=1}^{m}\sin\theta_i = E\sum_{i=1}^{m}\cos\theta_i\sin\theta_i + \sum_{i}^{m}\sum_{j\neq i}^{m}\cos\theta_i\sin\theta_j$$

Thus,

$$Cov(\bar{C_1}, \bar{S_1}) = Cov\left[\frac{1}{m}\sum_{i=1}^{m}\cos\theta_i, \frac{1}{m}\sum_{i=1}^{m}\sin 2\theta_i\right]$$
$$= \frac{\rho^{2^\alpha}\sin 2\mu_0 - 2\rho^2\cos\mu_0\sin\mu_0}{2m}$$

Putting $p = 3$ in Equation (2) and using the expansion of characteristic function of θ, we get

$$E\big(\sin 3\theta\big) = \rho^{3^\alpha}\sin 3\mu_0 \tag{A9}$$

Now,

$$Cov(\bar{C_1}, \bar{S_2}) = Cov\left[\frac{1}{m}\sum_{i=1}^{m}\cos\theta_i, \frac{1}{m}\sum_{i=1}^{m}\sin 2\theta_i\right]$$
$$Cov(\cos\theta, \sin 2\theta) = E\cos\theta\sin 2\theta - E\cos\theta E\sin 2\theta$$

Now, using Equations (A7) and (A9),

$$E\cos\theta\sin 2\theta = E\left[\frac{\sin 3\theta + \sin\theta}{2}\right]$$
$$= \frac{\rho^{3^\alpha}\sin 3\mu_0 + \rho\sin\mu_0}{2}$$

Thus, using Equations (A1) and (A8),

$$Cov(\cos\theta, \sin 2\theta) = \frac{\rho^{3^\alpha}\sin 3\mu_0 + \rho\sin\mu_0}{2} - \rho\cos\mu_0\rho^{2^\alpha}\sin 2\mu_0$$

Hence,

$$Cov(\bar{C_1}, \bar{S_2}) = \frac{\rho^{3^\alpha}\sin 3\mu_0 + \rho\sin\mu_0 - 2\rho^{2^\alpha+1}\cos\mu_0\sin 2\mu_0}{2m}$$

Similarly, it can be shown that,

$$Cov(\bar{C_1}, \bar{C_2}) = \frac{\rho\cos\mu_0 + \rho^{3^\alpha}\cos 3\mu_0 - 2\rho^{2^\alpha+1}\cos\mu_0\cos 2\mu_0}{2m} \tag{A10}$$

$$Cov(\bar{C_2}, \bar{S_1}) = \frac{\rho^{3^\alpha}\sin 3\mu_0 - \rho\sin\mu_0 - 2\rho^{2^\alpha+1}\cos 2\mu_0\sin\mu_0}{2m}$$

$$Cov(\bar{C_2}, \bar{S_2}) = \frac{\rho^{4^\alpha}\sin 4\mu_0 - 2(\rho^{2^\alpha})^2\cos 2\mu_0\sin 2\mu_0}{2m}$$

$$Cov(\bar{S_1}, \bar{S_2}) = \frac{\rho\cos\mu_0 - \rho^{3^\alpha}\cos 3\mu_0 - 2\rho^{2^\alpha+1}\sin\mu_0\sin 2\mu_0}{2m}$$

□

Appendix B

Proof of Lemma 2. This proof follows simply by putting $\mu_0 = 0$ into Equations (A2) and (A6) in the proof for Lemma 1. □

Appendix C

Proof of Lemma 3. This proof follows simply by putting $\mu_0 = 0$ in Equations (A1), (A2), (A3), (A6) and (A10) in the proof for Lemma 1. □

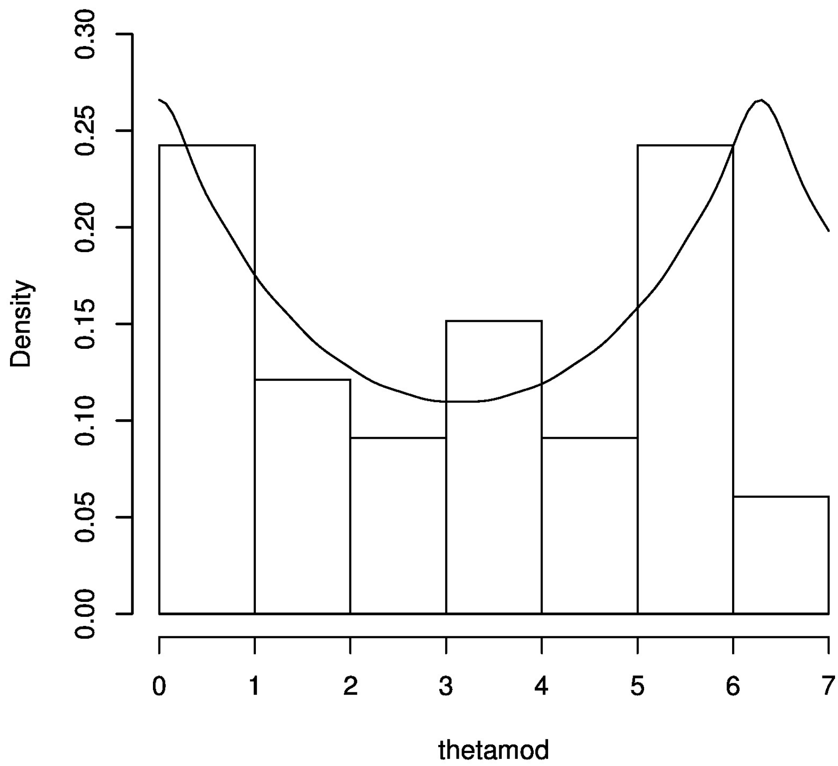

Figure A1. Histogram of wrapped log-ratio transformed gold price data (in US dollars) with wrapped stable density.

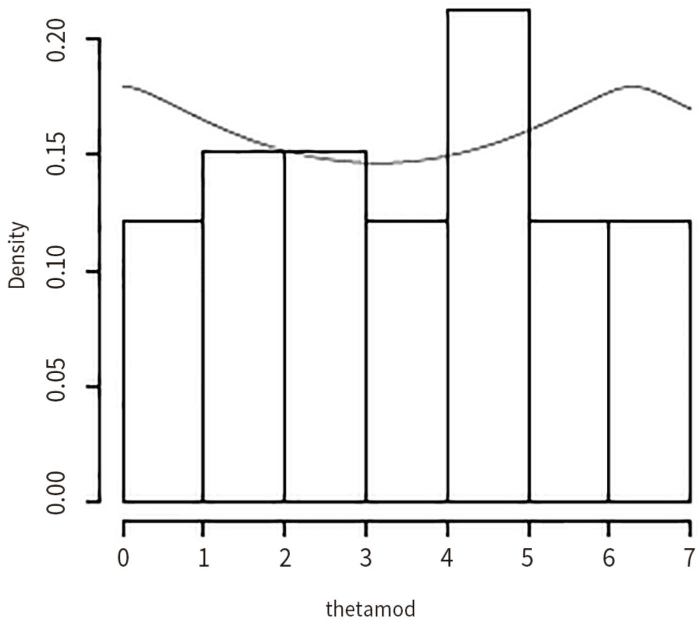

Figure A2. Histogram of wrapped log-ratio transformed silver price data(in US dollars)with wrapped stable density.

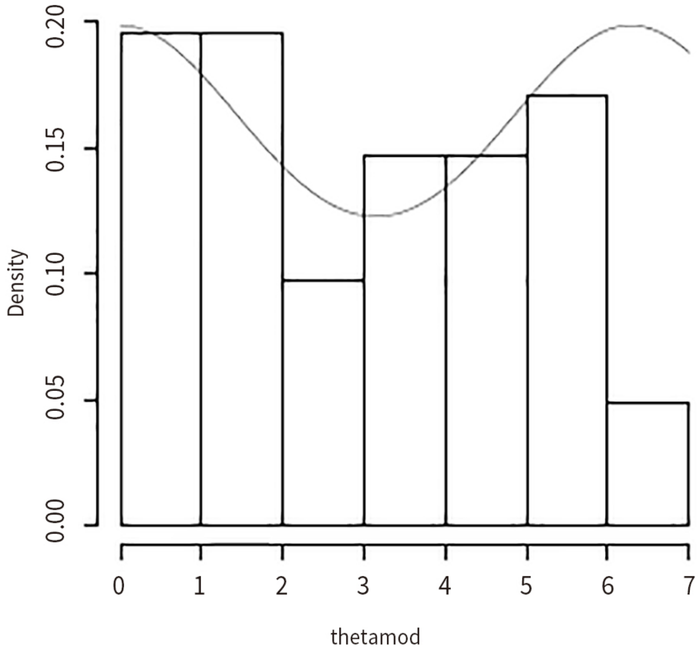

Figure A3. Histogram of wrapped log-ratio transformed gold price data(in INR)with wrapped stable density.

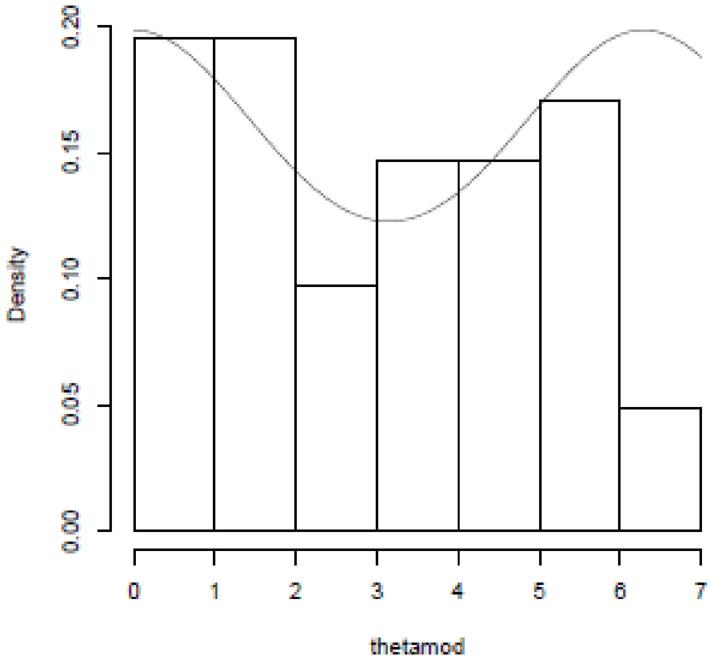

Figure A4. Histogram of wrapped log-ratio transformed silver price data (in INR) with wrapped stable density.

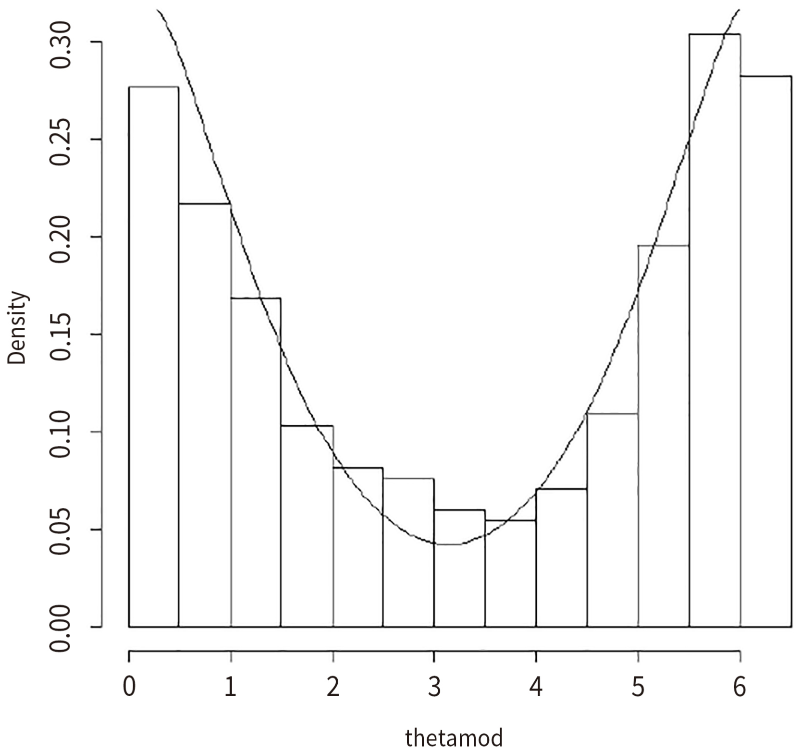

Figure A5. Histogram of wrapped log-ratio transformed Box and Jenkins data with wrapped stable density.

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