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Cohen Macaulayness and Arithmetical Rank of Generalized Theta Graphs

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Abstract: In this paper, we study some algebraic invariants of the edge ideal of generalized theta graphs, such as arithmetical rank, big height and height. We give an upper bound for the difference between the arithmetical rank and big height. Moreover, all Cohen-Macaulay (and unmixed) graphs of this type will be characterized.

Keywords: arithmetical rank; Cohen-Macaulay; height

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1. Introduction

For an ideal I of a commutative ring R with identity , the arithmetical rank (ara I) of the ideal I is defined as the minimum number s of elements $a_1, \ldots, a_s$ of R such that $\sqrt{I} = \sqrt{(a_1, \ldots, a_s)}$. For a squarefree monomial ideal I, it is known that $pd_R(R/I) \leq ara(I)$ and $bight(I) \leq pd_R(R/I)$ (see, for example, [1,2]). Thus,

$$ht(I) \leq bight(I) \leq pd_R(R/I) \leq ara(I) \leq \mu(I)$$

where $\mu(I)$ is the minimum number of generators of I. R/I is Cohen–Macaulay if and only if $ht(I) = pd_R(R/I)$. An ideal I is called a set-theoretic complete intersection whenever $ht(I) = ara(I)$. If I is not unmixed, then I is not a set-theoretic complete intersection. However, it is possible to have $bight(I) = pd_R(R/I) = ara(I)$. The question then arises, "For which ideal does the previous equality hold?"

Given a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K and a simple graph G with the vertex set $V_G = \{x_1, \ldots, x_n\}$ and the edge set E_G, the edge ideal of G, denoted by $I(G)$, is the ideal of R generated by $x_i x_j$ such that $\{x_i, x_j\} \in E_G$. The graph G is called Cohen–Macaulay over the field K if the ring $R/I(G)$ is Cohen–Macaulay.

It is still an open problem to find an explicit formula for the arithmetical rank of the edge ideal of a graph. For the edge ideal of a forest, it is shown that $bight(I(G)) = araI(G) = pd(R/I(G))$ by Barile [3] and Kimura and Terai [4]. In [5], Barile et al. proved that $araI(G) = pd(R/I(G))$ when G is a cyclic or bicyclic graph. In [6], Mohammadi and Kiani investigated the graphs consisting of some cycles and lines that have a common vertex. It is shown that the projective dimension equals the arithmetical rank for all such graphs. A graph G is called an *n-cyclic graph with a common edge* if G is a graph consisting of n cycles $C_{3r_1+1}, \ldots, C_{3r_{k_1}+1}, C_{3t_1+2}, \ldots, C_{3t_{k_2}+2}, C_{3s_1}, \ldots, C_{3s_{k_3}}$ connected through a common edge, where $k_1 + k_2 + k_3 = n$. Zhu, Shi and Gu proved that $pd(R/I(G)) = bightI(G) = ara(I(G))$ for some special n-cyclic graphs with a common edge [7]. For the class of generalized theta graphs, $G = \theta_{n_1,\ldots,n_k}$, the authors in [8] showed that $pd(R/I(G)) = bightI(G)$ except in the following two cases:

1. $n_i \equiv 0 \pmod 3$ for any $1 \leq i \leq k$;
2. there exists exactly one n_j such that $n_j \equiv 1 \pmod 3$, and for any $1 \leq i \neq j \leq k$, we have $n_i \equiv 2 \pmod 3$.

For these cases, they show that $pd(R/I(G)) = bightI(G) + 1$.

Since $bightI(G) \leq ara(I(G))$, it can be interesting to compare these invariants for the generalized theta graphs. In the sequel, we compute the height of the edge ideal of generalized theta graphs based on the number of vertices being even or odd in any path. Moreover, we show that $G = \theta_{n_1,\ldots,n_k}$ is Cohen–Macaulay (and unmixed) if and only if $G = \theta_{2,3,4}$.

2. Arithmetical Rank of the Edge Ideal of a Generalized Theta Graph

Let $k \geq 3$ be a positive integer and $n_1, \ldots, n_k$ be a sequence of positive integers. Let $\theta_{n_1,\ldots,n_k}$ be the graph constructed by k paths with $n_1, \ldots, n_k$ vertices with only the endpoints in common. Since the graphs are assumed to be simple, at most one of $n_1, \ldots, n_k$ can be equal to two. Throughout this paper, we assume that x and y are the common vertices. We define the projective dimension of G to be the projective dimension of the R-module $R/I(G)$ and we will write $pd(G) = pd(R/I(G))$. The edge ideal of a cycle of length n with the vertex set $\{x_1, \ldots, x_n\}$ is $I(C_n) = (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1)$. The edge ideal of a line (path) with the vertex set $\{x_1, \ldots, x_n\}$ is $I(L_n) = (x_1x_2, x_2x_3, \ldots, x_{n-1}x_n)$. In the following, we consider the labeling below:

$$I(L_{3s_i}) = (x_{1,i}x_{2,i}, x_{2,i}x_{3,i}, \ldots, x_{3s_i-1,i}x_{3s_i,i});\ for\ i = 1, \ldots, k_3$$

$$I(L_{3r_j+1}) = (y_{1,j}y_{2,j}, y_{2,j}y_{3,j}, \ldots, y_{3r_j,j}y_{3r_j+1,j});\ for\ j = 1, \ldots, k_1$$

$$I(L_{3t_l+2}) = (z_{1,l}z_{2,l}, z_{2,l}z_{3,l}, \ldots, z_{3t_l+1,l}z_{3t_l+2,l});\ for\ l = 1, \ldots, k_2$$

Suppose that $min\{n_1, \ldots, n_k\} = n_t$. One can consider the graph $\theta_{n_1,\ldots,n_k}$ as a $(k-1)$-cyclic graph with common path L_{n_t} consisting of $k-1$ cycles of lengths $n_i + n_t - 2$ for any $1 \leq i \neq t \leq k$. This generalizes the concept of n-cyclic graphs with a common edge.

For instance, let $G = \theta_{n_1,\ldots,n_{k_1+k_3}}$ be the graph consisting of lines $L_{3r_1+1}, \ldots, L_{3r_{k_1}+1}, L_{3s_1}, \ldots, L_{3s_{k_3}}$ such that $k_1, k_3 > 0$. Without loss of generality, suppose that $min\{n_1, \ldots, n_{k_1+k_3}\} = 3s_1$. One can consider G as a $(k_1 + k_3 - 1)$-cyclic graph with common path L_{3s_1} that the cycles are of lengths $3s_1 + 3r_j + 1 - 2 = 3(s_1 + r_j - 1) + 2$ or $3s_1 + 3s_i - 2 = 3(s_1 + s_i - 1) + 1$ for $1 \leq j \leq k_1$ and $2 \leq i \leq k_3$. Consider the following labeling for $I(C_{3(s_1+r_j-1)+2})$:

$$\begin{aligned} I(C_{3(s_1+r_j-1)+2}) = (&x_{1,1}x_{2,1}, \ldots, x_{3s_1-1,1}x_{3s_1,1}, x_{3s_1,1}y_{3s_1+1,j}, y_{3s_1+1,j}y_{3s_1+2,j}, \ldots, \\ &y_{3(s_1+r_j-1)+1,j}y_{3(s_1+r_j-1)+2,j}, y_{3(s_1+r_j-1)+2,j}x_{1,1}) \end{aligned}$$

In this section, we obtain an upper bound for the arithmetical rank of the edge ideal of generalized theta graphs. Using the big height of the edge ideal of these graphs computed in [8], we estimate an upper bound for $araI(\theta_{n_1,\ldots,n_k}) - bightI(\theta_{n_1,\ldots,n_k})$. For this purpose, we consider seven cases that are treated separately in the following theorems.

Theorem 1. *Let $G = \theta_{n_1,\ldots,n_{k_3}}$ be the graph consisting of lines $L_{3s_1}, \ldots, L_{3s_{k_3}}$,* i.e., *$n_i = 3s_i$ for $1 \leq i \leq k_3$. Then,*

$$0 \leq araI(G) - bightI(G) \leq k_3 - 1$$

Proof. By definition, one can consider G as a $(k_3 - 1)$-cyclic graph with common path of length $min\{n_1, \ldots, n_{k_3}\}$. Without loss of generality, we may assume $min\{n_1, \ldots, n_{k_3}\} = 3s_1$.

Since $3s_1 + 3s_i - 2 = 3(s_1 + s_i - 1) + 1$ for any $i \neq 1$, by ([5], Proposition 2.3), we can construct $Q_2, \ldots, Q_{k_3}$ as follows: for any $2 \leq i \leq k_3$, set $Q_i = (q_0, q_{1,i}, q_2, \ldots, q_{2(s_1+s_i-1)+1,i})$ where

$$
\begin{aligned}
q_0 &= x_{1,1}x_{2,1} \\
q_{1,i} &= x_{1,1}x_{3(s_1+s_i-1)+1,i} + x_{2,1}x_{3,1} \\
q_2 &= x_{4,1}x_{5,1} \\
q_3 &= x_{3,1}x_{4,1} + x_{5,1}x_{6,1} \\
&\vdots \\
q_{2(s_1-1)} &= x_{3s_1-2,1}x_{3s_1-1,1} \\
q_{2s_1-1} &= x_{3s_1-3,1}x_{3s_1-2,1} + x_{3s_1-1,1}x_{3s_1,1} \\
q_{2s_1,i} &= x_{3s_1+1,i}x_{3s_1+3,i} \\
&\vdots \\
q_{2(s_1+s_i-1),i} &= x_{3(s_1+s_i-1),i}x_{3(s_1+s_i-1)+1,i}
\end{aligned}
$$

Observe that the sequences $Q_2, \ldots, Q_{k_3}$ have $2s_1 - 1$ common elements, namely $q_0, q_2, \ldots, q_{2s_1-1}$. On the other hand, by ([9] p. 249), we have $\sqrt{Q_i} = I(C_{3(s_1+s_i-1)+1})$. Therefore, we deduce

$$
araI(G) \leq (2s_1 - 1) + \sum_{i=2}^{k_3}((2s_1 + 2s_i - 2 - 2s_1 + 1) + 1) = 2\sum_{i=1}^{k_3} s_i - 1
$$

Similar to the proof of Theorem 2.11 of [8], we obtain that $bightI(G) = 2\sum_{i=1}^{k_3} s_i - k_3$, it follows that

$$
0 \leq araI(G) - bightI(G) \leq k_3 - 1
$$

as desired. □

Theorem 2. *Let $G = \theta_{n_1,\ldots,n_{k_1}}$ be the graph consisting of lines $L_{3r_1+1}, \ldots, L_{3r_{k_1}+1}$,* i.e., *$n_i = 3r_i + 1$ for $1 \leq i \leq k_1$. Then,*

$$
pd(G) = bightI(G) = araI(G) = 2\sum_{i=1}^{k_1} r_i
$$

Proof. For $1 \leq i \leq k_1$, a similar argument as in ([3], p. 4701), Put Q_i is generated up to radical by

$$
\begin{aligned}
&y_{2,i}y_{3,i} \\
&y_{1,i}y_{2,i} + y_{3,i}y_{4,i} \\
&\vdots \\
&y_{3j-1,i}y_{3j,i} \\
&y_{3j-2,i}y_{3j-1,i} + y_{3j,i}y_{3j+1,i} \\
&\vdots \\
&y_{3r_i-2,i}y_{3r_i-1,i} + y_{3r_i,i}y_{3r_i+1,i}
\end{aligned}
$$

we have $I(G) = \sqrt{Q_1 + \ldots + Q_{k_1}}$ by ([9], p. 249). Then, $araI(G) \leq 2\sum_{i=1}^{k_1} r_i$. Similar to the proof of Theorem 2.6 of [8], we obtain that $2\sum_{i=1}^{k_1} r_i$, it follows that

$$pd(G) = bightI(G) = araI(G) = 2\sum_{i=1}^{k_1} r_i$$

□

Theorem 3. *Let $G = \theta_{n_1,\dots,n_{k_2}}$ be the graph consisting of lines $L_{3t_1+2},\dots,L_{3t_{k_2}+2}$,* i.e., *$n_i = 3t_i + 2$ for $1 \le i \le k_2$. Then,*

$$0 \le araI(G) - bightI(G) \le k_2 - 2$$

Proof. We can assume, without loss of generality, that $min\{n_1,\dots,n_{k_2}\} = n_1$. By definition, one may consider G as a (k_2-1)-cyclic graph with common path L_{n_1} that any cycle contains only $3(t_1+t_i)+2$ vertices. Applying Proposition 2.4 of [5], we construct $Q_i = (q_0, q_1, \dots, q_{2(t_1-1)+1}, q_{2t_1,i}, q_{2t_1+1,i}, \dots, q_{2(t_1+t_i),i})$ for any $2 \le i \le k_2$ as follows:

$$\begin{aligned}
&q_0 = z_{1,1}z_{2,1}\\
&q_1 = z_{2,1}z_{3,1} + z_{4,1}z_{5,1}\\
&\vdots\\
&q_{2l} = z_{3l,1}z_{3l+1,1} + z_{3l+2,1}z_{3l+3,1}\\
&q_{2l+1} = z_{3l+2,1}z_{3l+3,1} + z_{3l+4,1}z_{3l+5,1}\\
&\vdots\\
&q_{2(t_1-1)+1} = z_{3t_1-1,1}z_{3t_1,1} + z_{3t_1+1,1}z_{3t_1+2,1}\\
&q_{2t_1,i} = z_{3t_1,1}z_{3t_1+1,1} + z_{3t_1+2,1}z_{3t_1+3,i}\\
&\vdots\\
&q_{2(t_1+h),i} = z_{3(t_1+h),i}z_{3(t_1+h)+1,i} + z_{3(t_1+h)+2,i}z_{3(t_1+h)+3,i}\\
&q_{2(t_1+h)+1,i} = z_{3(t_1+h)+2,i}z_{3(t_1+h)+3,i} + z_{3(t_1+h)+4,i}z_{3(t_1+h)+5,i}\\
&\vdots\\
&q_{2(t_1+t_i),i} = z_{1,1}z_{3(t_1+t_i)+2,i} + z_{3(t_1+t_i),i}z_{3(t_1+h)+1,i}
\end{aligned}$$

We have $I(G) = \sum_{i=2}^{k_2} I(C_{3(t_1+t_i)+2})$, and it follows from ([9], p. 249) that $I(G) = \sqrt{\sqrt{(Q_2)} + \dots + \sqrt{(Q_{k_2})}}$. It is easily seen that the sequences $Q_2,\dots,Q_{k_2}$ have the terms $q_0, q_1, \dots, q_{2(t_1-1)+1}$ in common. Hence,

$$araI(G) \le 2\sum_{i=1}^{k_3} t_i + k_2 - 1$$

Similar to the proof of Theorem 2.7 of [8], we obtain that $bightI(G) = 2\sum_{i=1}^{k_2} t_i + 1$, and it follows that

$$0 \le araI(G) - bightI(G) \le 2\sum_{i=1}^{k_2} t_i + k_2 - 1 - (2\sum_{i=1}^{k_2} t_i + 1) = k_2 - 2$$

as required. □

Theorem 4. *Let $G = \theta_{n_1,\dots,n_{k_1+k_3}}$ be the graph consisting of lines $L_{3r_1+1},\dots,L_{3r_{k_1}+1}, L_{3s_1},\dots,L_{3s_{k_3}}$,* i.e., *$n_i = 3r_i + 1$ for $1 \le i \le k_1$ and $n_i = 3s_i$ for $k_1 + 1 \le i \le k_1 + k_3$ such that $k_1, k_3 > 0$. Then,*

1. *If there exists* $1 \leq i \leq k_1$ *such that* $min\{n_1, \ldots, n_{k_1+k_3}\} = 3r_i + 1$, *then* $0 \leq araI(G) - bightI(G) \leq min\{k_3, r_i + k_1 + k_3 - 3\}$;
2. *If there exists* $1 \leq j \leq k_3$ *such that* $min\{n_1, \ldots, n_{k_1+k_3}\} = 3s_j$, *then* $0 \leq araI(G) - bightI(G) \leq min\{k_3, s_j + k_3 - 2\}$.

Proof. We have $araI(L_{3r_i+1}) = 2r_i$ and $araI(L_{3s_j}) = 2s_j$ by ([3], p. 4701). It follows that

$$araI(G) \leq 2\sum_{i=1}^{k_1} r_i + 2\sum_{j=1}^{k_3} s_j \tag{1}$$

1. Without loss of generality, assume that $min\{n_1, \ldots, n_{k_1+k_3}\} = 3r_1 + 1$. One can consider G as a $(k_1 + k_3 - 1)$-cyclic graph with common path L_{3r_1+1} that the cycles are of lengths $3r_1 + 1 + 3r_i + 1 - 2 = 3(r_1 + r_i)$ or $3r_1 + 1 + 3s_j - 2 = 3(r_1 + s_j - 1) + 2$ for $2 \leq i \leq k_1$ and $1 \leq j \leq k_3$. Now, suppose that

$$\begin{aligned}
&q_0 = y_{1,1}y_{2,1} \\
&q_{1,i} = y_{1,1}y_{3(r_1+r_i),i} + y_{2,1}y_{3,1} \\
&\vdots \\
&q_{2(r_1-1)+1} = y_{3(r_1-1),1}y_{3(r_1-1)+1,1} + y_{3(r_1-1)+2,1}y_{3(r_1-1)+3,1} \\
&q_{2r_1,i} = y_{3r_1+1,1}y_{3r_1+2,i} \\
&\vdots \\
&q_{2h,i} = y_{3h+1,i}y_{3h+2,i} \\
&q_{2h+1,i} = y_{3h,i}y_{3h+1,i} + y_{3h+2,i}y_{3h+3,i} \\
&\vdots \\
&q_{2(r_1+r_i-1)+1,i} = y_{3(r_1+r_i-1),i}y_{3(r_1+r_i-1)+1,i} + y_{3(r_1+r_i-1)+2,i}y_{3(r_1+r_i-1)+3,i}
\end{aligned}$$

Which generate up to radical $I(C_{3(r_1+r_i)})$. Note that the terms $q_0, q_2, \ldots, q_{2(r_1-1)+1}$ are in common for any sequences generating ideal $I(C_{3(r_1+r_i)})$ up to radical and $2 \leq i \leq k_1$. For any $1 \leq j \leq k_3$, we define:

$$\begin{aligned}
&q'_0 = y_{1,1}y_{2,1} \\
&q'_1 = y_{2,1}y_{3,1} + y_{4,1}y_{5,1} \\
&\vdots \\
&q'_{2l} = y_{3l,1}y_{3l+1,1} + y_{3l+2,1}y_{3l+3,1} \\
&q'_{2l+1} = y_{3l+2,1}y_{3l+3,1} + y_{3l+4,1}y_{3l+5,1} \\
&\vdots \\
&q'_{2(r_1-1)} = y_{3r_1-3,1}y_{3r_1-2,1} + y_{3r_1-1,1}y_{3r_1,1} \\
&q'_{2r_1-1,j} = y_{3r_1-1,1}y_{3r_1,1} + y_{3r_1+1,1}x_{3r_1+2,j} \\
&\vdots \\
&q'_{2h,j} = x_{3h,j}x_{3h+1,j} + x_{3h+2,j}x_{3h+3,j} \\
&q'_{2h+1,j} = x_{3h+2,j}x_{3h+3,j} + x_{3h+4,j}x_{3h+5,j} \\
&\vdots \\
&q'_{2(r_1+s_j-1),i} = y_{1,1}x_{3(r_1+s_j-1)+2,j} + x_{3(r_1+s_j-1),j}x_{3(r_1+s_j-1)+1,j}
\end{aligned}$$

and we can obtain that $I(C_{3(r_1+s_j-1)+2}) = \sqrt{(q'_0, q'_1, \ldots, q_{2(r_1+s_j-1),j})}$. We can obtain that there are the common terms $q'_0, q'_1, \ldots, q'_{2(r_1-1)}$ in any of sequences generating ideal $I(C_{3(r_1+s_j-1)+2})$ up to radical for any $1 \le j \le k_3$. On the other hand, we have $q_0 = q'_0$ and $q_{2m+1} = q'_{2m}$, for any $1 \le m \le r_1 - 1$. Applying these arguments, we obtain

$$araI(G) \le 2\sum_{i=1}^{k_1} r_i + 2\sum_{j=1}^{k_3} s_j + r_1 + k_1 - 3 \tag{2}$$

Thus, the inequalities Equations (1) and (2), together with ([8], Theorem 2.8), imply that

$$0 \le araI(G) - bightI(G) \le min\{k_3, r_1 + k_1 + k_3 - 3\}$$

2. We may assume, without loss of generality, that $min\{n_1, \ldots, n_{k_1+k_3}\} = 3s_1$. One can consider G as a $(k_1 + k_3 - 1)$-cyclic graph with common path L_{3s_1} that the cycles are of lengths $3s_1 + 3s_i - 2 = 3(s_1 + s_i - 1) + 1$ for any $2 \le i \le k_3$ or $3s_1 + 3r_j + 1 - 2 = 3(s_1 + r_j - 1) + 2$ for any $1 \le j \le k_1$. Applying Proposition 2.3 of [5], we construct the following sequences:

$$\begin{aligned}
&q_0 = x_{1,1}x_{2,1}\\
&q_{1,i} = x_{1,1}x_{3(s_1+s_i-1)+1,i} + x_{2,1}x_{3,1}\\
&\vdots\\
&q_{2(s_1-1)} = x_{3s_1-2,1}x_{3s_1-1,1}\\
&q_{2s_1-1} = x_{3s_1-3,1}x_{3s_1-2,1} + x_{3s_1-1,1}x_{3s_1,1}\\
&q_{2s_1,i} = x_{3s_1+1,i}x_{3s_1+2,i}\\
&\vdots\\
&q_{2h,i} = x_{3h+1,i}x_{3h+2,i}\\
&q_{2h+1,i} = x_{3h,i}x_{3h+1,i} + x_{3h+2,i}x_{3h+3,i}\\
&\vdots\\
&q_{2(s_1+s_i-1),i} = x_{3(s_1+s_i-1),i}x_{3(s_1+s_i-1)+1,i}
\end{aligned}$$

We have $I(C_{3(s_1+s_i-1)+1}) = \sqrt{(q_0, q_{1,i}, \ldots, q_{2(s_1+s_i-1),i})}$ for any $2 \le i \le k_3$. It is easily seen that the above constructed sequences have $2s_1 - 1$ terms in common. Now, suppose that $I(C_{3(s_1+r_j-1)+2}) = \sqrt{(q'_0, \ldots, q'_{2(s_1+r_j-1),j})}$ where

$$
\begin{aligned}
q'_0 &= x_{1,1}x_{2,1} \\
q'_1 &= x_{2,1}x_{3,1} + x_{4,1}x_{5,1} \\
&\vdots \\
q'_{2l} &= x_{3l,1}x_{3l+1,1} + x_{3l+2,1}x_{3l+3,1} \\
q'_{2l+1} &= x_{3l+2,1}x_{3l+3,1} + x_{3l+4,1}x_{3l+5,1} \\
&\vdots \\
q'_{2(s_1-1)} &= x_{3s_1-3,1}x_{3s_1-2,1} + x_{3s_1-1,1}x_{3s_1,1} \\
q'_{2s_1-1,j} &= x_{3s_1-1,1}x_{3s_1,1} + y_{3s_1+1,j}y_{3s_1+2,j} \\
&\vdots \\
q'_{2h,j} &= y_{3h,j}y_{3h+1,j} + y_{3h+2,j}y_{3h+3,j} \\
q'_{2h+1,j} &= y_{3h+2,j}y_{3h+3,j} + y_{3h+4,j}y_{3h+5,j} \\
&\vdots \\
q'_{2(s_1+r_j-1),j} &= x_{1,1}y_{3(s_1+r_j-1)+2,j} + y_{3(s_1+r_j-1),j}y_{3(s_1+r_j-1)+1,j}
\end{aligned}
$$

for all $1 \leq j \leq k_1$. One can check that the above constructed sequences have $2s_1 - 1$ terms in common. On the other hand, we have $q_0 = q'_0$ and $q_{2m+1} = q'_{2m}$ for any $1 \leq m \leq s_1 - 1$. Using the preceding arguments and the fact that $I(G) = \sum_{i=2}^{k_3} I(C_{3(s_1+s_i-1)+1}) + \sum_{j=1}^{k_1} I(C_{3(s_1+r_j-1)+2})$, we get

$$ araI(G) \leq 2\sum_{i=1}^{k_3} s_i + 2\sum_{j=1}^{k_1} r_j + s_1 - 2 \tag{3} $$

Thus, the inequalities Equations (1) and (3), together with ([8], Theorem 2.8), yield the inequality

$$ 0 \leq araI(G) - bightI(G) \leq min\{k_3, s_1 + k_3 - 2\} $$

□

Theorem 5. *Let $G = \theta_{n_1,\ldots,n_{k_2+k_3}}$ be the graph consisting of lines $L_{3s_1}, \ldots, L_{3s_{k_3}}, L_{3t_1+2}, \ldots, L_{3t_{k_2}+2}$,* i.e., *$n_i = 3s_i$ for $1 \leq i \leq k_3$ and $n_i = 3t_i + 2$ for $k_3 + 1 \leq i \leq k_3 + k_2$ such that $k_2, k_3 > 0$. Then,*

1. *If there exists $1 \leq i \leq k_3$ such that $min\{n_1, \ldots, n_{k_2+k_3}\} = 3s_i$, then $0 \leq araI(G) - bightI(G) \leq k_2 + k_3 - 2$;*
2. *If there exists $1 \leq j \leq k_2$ such that $min\{n_1, \ldots, n_{k_2+k_3}\} = 3t_j + 2$, then $0 \leq araI(G) - bightI(G) \leq min\{k_2 + k_3 + t_j - 2, k_2 + k_3 - 1\}$.*

Proof.

1. Without loss of generality, one may assume $min\{n_1, \ldots, n_{k_2+k_3}\} = 3s_1$. We can consider G as a $(k_2 + k_3 - 1)$-cyclic graph with common path L_{3s_1} of which the cycles are of lengths $3s_1 + 3s_i - 2 = 3(s_1 + s_i - 1) + 1$ for any $2 \leq i \leq k_3$ or $3s_1 + 3t_j + 2 - 2 = 3(s_1 + t_j)$ for any $1 \leq j \leq k_2$. Applying Proposition 2.2 of [5], we have $I(C_{3(s_1+t_j)}) = \sqrt{(q'_0, q'_{1,j}, \ldots, q'_{2(s_1+t_j)-1,j})}$, where

$$
\begin{aligned}
&q'_0 = x_{1,1}x_{2,1} \\
&q'_{1,j} = x_{1,1}z_{3(s_1+t_j),j} + x_{2,1}x_{3,1} \\
&\vdots \\
&q'_{2(s_1-1)+1} = x_{3(s_1-1),1}x_{3(s_1-1)+1,1} + x_{3(s_1-1)+2,1}x_{3(s_1-1)+3,1} \\
&q'_{2s_1,j} = z_{3s_1+1,j}z_{3s_1+2,j} \\
&\vdots \\
&q'_{2h,j} = z_{3h+1,j}z_{3h+2,j} \\
&q'_{2h+1,j} = z_{3h,j}z_{3h+1,j} + z_{3h+2,j}z_{3h+3,j} \\
&\vdots \\
&q'_{2(s_1+t_j-1)+1,j} = z_{3(s_1+t_j-1),j}z_{3(s_1+t_j-1)+1,j} + z_{3(s_1+t_j-1)+2,j}z_{3(s_1+t_j-1)+3,j}
\end{aligned}
$$

for any $1 \leq j \leq k_2$. Note that, for the above constructed sequences, the elements $q'_0, q'_2, \ldots, q'_{2s_1-1}$ are in common. With the same argument as in the proof of Theorem 4, we have $I(C_{3(s_1+r_i-1)+1}) = \sqrt{(q_0, \ldots, q_{2(s_1+s_i-1),i})}$ for any $2 \leq i \leq k_3$. On the other hand, we have $q_0 = q'_0$ and $q_m = q'_m$ for any $2 \leq m \leq 2s_1 - 1$. In addition, $I(G) = \sum_{i=2}^{k_3} I(C_{3(s_1+s_i-1)+1}) + \sum_{j=1}^{k_2} I(C_{3(s_1+t_j)})$. Thus, it follows that

$$araI(G) \leq 2\sum_{i=1}^{k_3} s_i + 2\sum_{j=1}^{k_2} t_j + k_2 - 1 \tag{4}$$

Furthermore, ([3], p. 4701) implies that $araI(L_{3t_j+2}) = 2t_j + 1$ and $araI(L_{3s_i}) = 2s_i$, and hence

$$araI(G) \leq 2\sum_{i=1}^{k_3} s_i + 2\sum_{j=1}^{k_2} t_j + k_2 \tag{5}$$

From the Equations (4) and (5), together with ([8], Theorem 2.9), we get

$$0 \leq araI(G) - bightI(G) \leq k_2 + k_3 - 2$$

as desired.

2. Without loss of generality, one may assume $min\{n_1, \ldots, n_{k_2+k_3}\} = 3t_1 + 2$. One can consider G as a $(k_2 + k_3 - 1)$-cyclic graph with common path L_{3t_1+2} which the cycles are of lengths $3(t_1 + t_i) + 2$ for any $2 \leq i \leq k_2$ or $3(t_1 + s_j)$ for any $1 \leq j \leq k_3$. Using the proof of Theorem 3, we get $I(C_{3(t_1+t_i)+2}) = \sqrt{(q_0, \ldots, q_{2(t_1+t_i),i})}$ for any $2 \leq i \leq k_2$. Assume that

$$
\begin{aligned}
&q'_0 = z_{1,1}z_{2,1}\\
&q'_{1,j} = z_{1,1}x_{3(t_1+r_j),j} + z_{2,1}z_{3,1}\\
&\vdots\\
&q'_{2(t_1-1)+1} = z_{3(t_1-1),1}z_{3(t_1-1)+1,1} + z_{3(t_1-1)+2,1}z_{3(t_1-1)+3,1}\\
&q'_{2t_1,j} = z_{3t_1+1,1}x_{3t_1+2,j}\\
&\vdots\\
&q'_{2h,j} = x_{3h+1,j}x_{3h+2,j}\\
&q'_{2h+1,j} = x_{3h,j}x_{3h+1,j} + x_{3h+2,j}x_{3h+3,j}\\
&\vdots\\
&q'_{2(t_1+s_j-1)+1,j} = x_{3(t_1+s_j-1),j}x_{3(t_1+s_j-1)+1,j} + x_{3(t_1+s_j-1)+2,j}x_{3(t_1+s_j-1)+3,j}
\end{aligned}
$$

which generate up to radical $I(C_{3(t_1+s_j)})$. Observe that in all sequences generating ideal $I(C_{3(t_1+s_j)})$ up to the radical, the elements $q'_0, q'_2, \ldots, q'_{2(t_1-1)+1}$ are in common, for any $1 \leq j \leq k_3$. We have $q_0 = q'_0$ and $q_{2m} = q'_{2m+1}$ for any $1 \leq m \leq t_1 - 1$. Since $I(G) = \sqrt{\sum_{i=2}^{k_2} I(C_{3(t_1+t_i)+2}) + \sum_{j=1}^{k_3} I(C_{3(t_1+s_j)})}$,

$$ araI(G) \leq 2\sum_{j=1}^{k_3} s_j + 2\sum_{i=1}^{k_2} t_i + t_1 + k_2 - 1 \tag{6} $$

Thus, the inequalities Equations (4) and (6), together with ([8], Theorem 2.9), yield the asserted inequality.

□

Theorem 6. *Let $G = \theta_{n_1,\ldots,n_{k_1+k_2}}$ be the graph consisting of lines $L_{3r_1+1}, \ldots, L_{3r_{k_1+1}}, L_{3t_1+2}, \ldots, L_{3t_{k_2}+2}$,* i.e., *$n_i = 3r_i + 1$ for $1 \leq i \leq k_1$ and $n_i = 3t_i + 2$ for $k_1 + 1 \leq i \leq k_1 + k_2$ such that $k_1, k_2 > 0$. Then,*

1. *If there exists $1 \leq i \leq k_1$ such that $min\{n_1, \ldots, n_{k_1+k_2}\} = 3r_i + 1$, then $0 \leq araI(G) - bightI(G) \leq min\{2k_2 + k_1 - 2, k_2\}$;*
2. *If there exists $1 \leq j \leq k_2$ such that $min\{n_1, \ldots, n_{k_1+k_2}\} = 3t_j + 2$, then $0 \leq araI(G) - bightI(G) \leq min\{t_j + k_2 + k_1 - 1, k_2\}$.*

Proof.

1. Without loss of generality, suppose that $min\{n_1, \ldots, n_{k_1+k_2}\} = 3r_1 + 1$. One can consider G as a $(k_1 + k_2 - 1)$-cyclic graph with common path L_{3r_1+1} where the cycles are of lengths $3r_1 + 1 + 3t_j + 2 - 2 = 3(r_1 + t_j) + 1$ for any $1 \leq j \leq k_2$ or $3r_1 + 1 + 3r_i + 1 - 2 = 3(r_1 + r_i)$ for any $2 \leq i \leq k_1$. Applying the same argument in the proof of Theorem 5 (1), we get

$$ araI(G) \leq 2\sum_{i=1}^{k_1} r_i + 2\sum_{j=1}^{k_2} t_j + 2k_2 + k_1 - 2 \tag{7} $$

Hence, ([8], Theorem 2.12), ([3], p. 4701) and Equation (7) imply that $0 \leq araI(G) - bightI(G) \leq min\{2k_2 + k_1 - 2, k_2\}$.
2. We may assume, without loss of generality, that $min\{n_1, \ldots, n_{k_1+k_2}\} = 3t_1 + 2$. One can consider G as a $(k_1 + k_2 - 1)$-cyclic graph with common path L_{3t_1+2}. Therefore, the cycles are of lengths

$3t_1 + 2 + 3t_i + 2 - 2 = 3(t_1 + t_i) + 2$ for any $2 \le i \le k_2$ or $3t_1 + 2 + 3r_j + 1 - 2 = 3(t_1 + r_j) + 1$ for any $1 \le j \le k_1$. The same argument as in the proof of Theorem 4 (2) shows that

$$araI(G) \le 2\sum_{i=1}^{k_1} r_i + 2\sum_{i=1}^{k_2} t_i + t_1 + k_2 + k_1 - 1 \tag{8}$$

Using ([8], Theorem 2.12), ([3], p. 4701) and Equation (8), one derives that $0 \le araI(G) - bightI(G) \le min\{t_1 + k_2 + k_1 - 1, k_2\}$.

□

Theorem 7. *Let $G = \theta_{n_1,\dots,n_{k_1+k_2+k_3}}$ consist of lines $L_{3s_1}, \dots, L_{3s_{k_3}}, L_{3r_1+1}, \dots, L_{3r_{k_1}+1}, L_{3t_1+2}, \dots, L_{3t_{k_2}+2}$,* i.e., *$n_i = 3s_i$ for $1 \le i \le k_3$, $n_i = 3r_i + 1$ for $k_3 + 1 \le i \le k_3 + k_1$ and $n_i = 3t_i + 2$ for $k_1 + k_3 + 1 \le i \le k_1 + k_3 + k_2$ such that $k_1, k_2, k_3 > 0$. Then,*

1. *If there exists $1 \le i \le k_3$ such that $min\{n_1, \dots, n_{k_1+k_2+k_3}\} = 3s_i$, then $0 \le araI(G) - bightI(G) \le min\{k_2 + k_3 + s_i - 2, k_2 + k_3\}$;*
2. *If there exists $1 \le j \le k_1$ such that $min\{n_1, \dots, n_{k_1+k_2+k_3}\} = 3r_j + 1$, then $0 \le araI(G) - bightI(G) \le min\{k_1 + 2k_2 + k_3 + r_i - 3, k_2 + k_3\}$;*
3. *If there exists $1 \le l \le k_2$ such that $min\{n_1, \dots, n_{k_1+k_2+k_3}\} = 3t_l + 2$, then $0 \le araI(G) - bightI(G) \le min\{k_1 + k_2 + k_3 + t_l - 1, k_2 + k_3\}$.*

Proof.

1. Without loss of generality, suppose that $min\{n_1, \dots, n_{k_1+k_2+k_3}\} = 3s_1$. One can consider that G is a $(k_1 + k_2 + k_3 - 1)$-cyclic graph with common path L_{3s_1} where the cycles are of lengths $3s_1 + 3s_i - 2 = 3(s_1 + s_i - 1) + 1$ for any $2 \le i \le k_3$ or $3s_1 + 3r_j + 1 - 2 = 3(s_1 + r_j - 1) + 2$ for any $1 \le j \le k_1$ or $3s_1 + 3t_l + 2 - 2 = 3(s_1 + t_l)$ for any $1 \le l \le k_2$. Using the same argument as in the proof of Theorems 1, 4 (2) and 5 (1), we get

$$\begin{aligned} araI(G) \le & (2s_1 - 1) + 2\sum_{l=1}^{k_2} t_l + k_2 \\ & + (2s_1 - 1) + 2\sum_{i=2}^{k_3} s_i + (2s_1 - 1) + 2\sum_{j=1}^{k_1} r_j \\ & - (1 + 2s_1 - 2) - (1 + s_1 - 1) \\ & = 2\sum_{l=1}^{k_2} t_l + 2\sum_{j=1}^{k_1} r_j + 2\sum_{i=1}^{k_3} s_i + k_2 + s_1 - 2 \end{aligned} \tag{9}$$

It follows from Equation (9), ([8], Theorem 2.10) and ([3], p. 4701) that

$$0 \le araI(G) - bightI(G) \le min\{k_2 + k_3 + s_1 - 2, k_2 + k_3\}$$

2. Without loss of generality, assume that $min\{n_1, \dots, n_{k_1+k_2+k_3}\} = 3r_1 + 1$. One can consider that G is a $(k_1 + k_2 + k_3 - 1)$-cyclic graph with common path L_{3r_1+1} where the cycles are of lengths $3(r_1 + r_j)$ for any $2 \le j \le k_1$, $3(r_1 + t_l) + 1$ for any $1 \le l \le k_2$ or $3(r_1 + s_i - 1) + 2$ for any $1 \le i \le k_3$. The same argument as in the proof of Theorems 4 (1) and 6 (1) shows that

$$
\begin{aligned}
araI(G) \leq & (2r_1 - 1) + 2\sum_{l=1}^{k_1} r_l + k_1 - 1 \\
& + (2r_1 - 1) + 2\sum_{j=2}^{k_2} t_j + 2k_2 + (2r_1 - 1) + 2\sum_{i=1}^{k_3} s_i \\
& - (2r_1 - 1) - (1 + r_1 - 1) \\
& = 2\sum_{l=1}^{k_2} t_l + 2\sum_{j=1}^{k_1} r_j + 2\sum_{i=1}^{k_3} s_i + 2k_2 + k_1 + r_1 - 3
\end{aligned} \tag{10}
$$

Therefore, by Equation (10), ([8], Theoerem 2.10) and ([3], p. 4701), we conclude that

$$0 \leq araI(G) - bightI(G) \leq min\{k_1 + 2k_2 + k_3 + r_1 - 3, k_2 + k_3\}$$

3. Without loss of generality, assume that $min\{n_1, \ldots, n_{k_1+k_2+k_3}\} = 3t_1 + 2$. One can consider that G is a $(k_1 + k_2 + k_3 - 1)$-cyclic graph with common path L_{3t_1+2}, where the cycles are of lengths $3(t_1 + s_i)$ for any $1 \leq i \leq k_3$, $3(t_1 + r_j) + 1$ for any $1 \leq j \leq k_1$ or $3(t_1 + t_l) + 2$ for any $2 \leq l \leq k_2$. We can use the same argument as in the proof of Theorems 3, 5 (2) and 6 (2) to obtain

$$
\begin{aligned}
araI(G) \leq & 2t_1 + 2\sum_{i=1}^{k_3} s_i \\
& + 2t_1 + 2\sum_{j=1}^{k_1} r_j + k_1 + 2t_1 + 2\sum_{l=2}^{k_2} t_l + (k_2 - 1) \\
& - 2t_1 - (1 + t_1 - 1) \\
& = 2\sum_{l=1}^{k_2} t_l + 2\sum_{j=1}^{k_1} r_j + 2\sum_{i=1}^{k_3} s_i + t_1 + k_2 + k_1 - 1
\end{aligned} \tag{11}
$$

Applying Equation (11), ([8], Theorem 2.10) and ([3], p. 4701), we get

$$0 \leq araI(G) - bightI(G) \leq min\{k_1 + k_2 + k_3 + t_1 - 1, k_2 + k_3\}$$

as desired.

□

3. Cohen-Macaulayness of Generalized Theta Graph

In [2], Mohammadi and Kiani investigated some properties of graphs of the form $\theta_{n_1,\ldots,n_k}$, such as shellability, vertex decomposability and sequential Cohen-Macaulayness. The present section is devoted to study Cohen-Macaulayness and unmixedness of these graphs, especially the height of generalized theta graphs. The most important motivation to study this property comes from the fact that R/I is Cohen-Macaulay if and only if $ht(I) = pd(R/I)$. We check the equality $htI(G) = pd(R/I(G))$ to verify Cohen-Macaulayness of the graph $G = \theta_{n_1,\ldots,n_k}$ in some cases. Since the projective dimension of a graph in this class is computed in [8], it only remains to obtain the value of $htI(G)$.

Let us fix some notations that will be used throughout this section. By $G_1 \cup G_2$, we mean the graph obtaind by the disjoint union of G_1 and G_2. Furthermore, we suppose that the vertices of a line graph L_{n_i} are labeled by $x_{1,i}, x_{2,i}, \ldots, x_{n_i,i}$ where $x_{1,i} = x$ and $x_{n_i,i} = y$. Note that

$$ht(I(L_n)) = \begin{cases} k & n = 2k \\ k & n = 2k+1 \end{cases}$$

Lemma 8. *Let $G = \theta_{n_1,\ldots,n_k}$ such that $n_i = 2m_i$ for any $1 \leq i \leq k$. Then,*

$$htI(G) = (\sum_{i=1}^{k} m_i) - k + 1$$

Proof. Assume that A is a minimal vertex cover of G. One of the following cases may happen: $(x \in A, y \notin A)$, $(x \notin A, y \in A)$, $(x, y \in A)$, or $(x, y \notin A)$.

1. Suppose that $x \in A, y \notin A$. We are going to find the minimum cardinality of minimal vertex cover of G which do not contain y, so it suffices to cover the disjoint lines $L_{n_1-2}, \ldots, L_{n_k-2}$ with the minimum number of vertices such that $N_G(y) \subseteq A$. We claim that

$$htI(L_{n_1-2} \cup \ldots \cup L_{n_k-2}) = \sum_{i=1}^{k}(m_i - 1)$$

   Note that $n_i - 2 = 2(m_i - 1)$. Furthermore, there exists a minimal vertex cover B_i for L_{n_i-2} with the minimum cardinality $m_i - 1$ such that $x_{2,i} \notin B_i$ and $x_{3,i}, x_{n_i-1,i} \in B_i$ for any $1 \leq i \leq k$. Hence, in this case, the minimum number of vertices of such A to be equal to

$$1 + \sum_{i=1}^{k}(m_i - 1) = (\sum_{i=1}^{k} m_i) - k + 1$$

   The number 1 appears in the above equality because $x \in A$.
2. Suppose that $y \in A, x \notin A$. We can apply the same argument as in the previous case.
3. Assume that $x, y \in A$. To obtain a minimal vertex cover of G with minimum cardiality, we may cover the disjoint lines $L_{n_1-2}, \ldots, L_{n_k-2}$ with the minimum number of vertices such that $N_G(x)$ and $N_G(y)$ are not contained in A. Since $n_i - 2 = 2(m_i - 1)$ and there exists a minimal vertex cover B_i for L_{n_i-2} with the minimum cardinality $m_i - 1$ such that $x_{2,i}, x_{n_i-1,i} \notin B_i$ for any $1 \leq i \leq k$, we deduce that

$$htI(L_{n_1-2} \cup \ldots \cup L_{n_k-2}) = \sum_{i=1}^{k}(m_i - 1)$$

   It follows that the minimum cardinality of such A to be equal to

$$2 + \sum_{i=1}^{k}(m_i - 1) = (\sum_{i=1}^{k} m_i) - k + 2$$

   The number 2 appears in the above equality because $x, y \in A$.
4. Assume that $x, y \notin A$. Applying the same argument, we may cover the disjoint lines $L_{n_1-2}, \ldots, L_{n_k-2}$ with the minimum number of vertices such that A contains $N_G(x)$ and $N_G(y)$. There exists a minimal vertex cover B_i of cardinality m_i for the line L_{n_i-2} having an even number of vertices such that $x_{2,i}, x_{n_i-1,i} \in B_i$ for any $1 \leq i \leq k$; therefore, we obtain the minimum number of vertices of such A to be equal to $\sum_{i=1}^{k} m_i$ because $x, y \notin A$.
   Now, by comparing the above cases, we get

$$htI(G) = (\sum_{i=1}^{k} m_i) - k + 1$$

as desired.

□

Lemma 9. *Let $G = \theta_{n_1,\ldots,n_k}$ such that $n_i = 2m_i + 1$ for any $1 \leq i \leq k$. Then,*

$$htI(G) = (\sum_{i=1}^{k} m_i) - k + 2$$

Proof. The techniques used in this proof are similar to the previous lemma. Suppose that A is a minimal vertex cover of G. The only possible cases for x, y are $(x \in A, y \notin A)$, $(x \notin A, y \in A)$, $(x, y \in A)$, or $(x, y \notin A)$.

1. Suppose that $x \in A, y \notin A$. We have $N_G(y) \subseteq A$. There exists a minimal vertex cover B_i for the line L_{n_i-2} having an odd number of vertices such that $x_{2,i} \notin B_i$ and $x_{n_i-1,i} \in B_i$ where the minimum cardinality of B_i is m_i for any $1 \leq i \leq k$. Therefore,

$$htI(L_{n_1-2} \cup \ldots \cup L_{n_k-2}) = \sum_{i=1}^{k} m_i$$

Hence, we get the minimum number of vertices of such A to be equal to

$$1 + \sum_{i=1}^{k} m_i$$

The number 1 appears in the above equality because $x \in A$.
2. Suppose that $y \in A, x \notin A$. We can apply the same argument as in the previous case.
3. Suppose that $x, y \in A$. There exists a minimal vertex cover B_i for L_{n_i-2} such that $x_{2,i}, x_{n_i-1,i} \notin B_i$, and, furthermore, the minimum number vertices of such B_i is $m_i - 1$ for any $1 \leq i \leq k$. Since $n_i - 2 = 2(m_i - 1) + 1$,

$$htI(L_{n_1-2} \cup \ldots \cup L_{n_k-2}) = (\sum_{i=1}^{k} m_i) - k$$

It follows that the minimum cardinality of such A is equal to

$$2 + \sum_{i=1}^{k} (m_i - 1) = (\sum_{i=1}^{k} m_i) - k + 2$$

The number 2 appears in the above equality because $x, y \in A$.
4. Suppose that $x, y \notin A$. Applying the same argument, we may cover the disjoint lines $L_{n_1-2}, \ldots, L_{n_k-2}$ with the minimum number of vertices such that A contains $N_G(x)$ and $N_G(y)$. There exists a minimal vertex cover B_i of cardinality m_i for the line L_{n_i-2} having an odd number of vertices such that $x_{2,i}, x_{n_i-1,i} \in B_i$ for any $1 \leq i \leq k$; therefore, we obtain the minimum number of vertices of such A to be equal to $\sum_{i=1}^{k} m_i$, because $x, y \notin A$.
Since $k \geq 3$, $2 - k < 0$. Now, we compare the results obtained from the cases above to get

$$htI(G) = (\sum_{i=1}^{k} m_i) - k + 2$$

as required.

□

Lemma 10. *Let $G = \theta_{n_1,\ldots,n_k}$ such that $n_i = 2m_i$ for any $1 \le i \le k_1$ and $n_i = 2l_i + 1$ for any $k_1 + 1 \le i \le k$. Then,*

$$htI(G) = (\sum_{i=1}^{k_1} m_i) + (\sum_{i=k_1+1}^{k} l_i) - k + 2$$

Proof. Assume that A is a minimal vertex cover for G. Applying the same argument in lemma 8, the only possible cases for the common vertices x, y are $(x \in A, y \notin A)$, $(x \notin A, y \in A)$, $(x, y \in A)$, or $(x, y \notin A)$.

1. Assume that $x \in A, y \notin A$. We have to cover the disjoint lines $L_{n_1-2}, \ldots, L_{n_k-2}$ with the minimum number of vertices such that $N_G(y) \subseteq A$. We have $n_i - 2 = 2(m_i - 1)$ for $1 \le i \le k_1$ and $n_i - 2 = 2(l_i - 1) + 1$ for $k_1 + 1 \le i \le k$. It is easily seen that there exists a minimal vertex cover B_i for L_{n_i-2} having an even number of vertices, and the minimal vertex cover C_i for L_{n_i-2} having an odd number of vertices such that $x_{2,i} \notin B_i$ for $1 \le i \le k_1$, $x_{2,i} \notin C_i$ for $k_1 + 1 \le i \le k$, $x_{n_i-1,i} \in B_i$ for $1 \le i \le k_1$ and $x_{n_i-1,i} \in C_i$ for $k_1 + 1 \le i \le k_1$. The minimum size for B_i and C_i is $m_i - 1$ and l_i, respectively. Hence, in this case, we need at least $\sum_{i=1}^{k_1}(m_i - 1) + \sum_{i=k_1+1}^{k} l_i$ vertices to create a minimal vertex cover for the disjoint lines $L_{n_1-2}, \ldots, L_{n_k-2}$. Then, the minimum size of such A is
$$(\sum_{i=1}^{k_1} m_i) + (\sum_{i=k_1+1}^{k} l_i) - k_1 + 1$$
2. Assume that $y \in A, x \notin A$. We can apply the same argument as in the previous case.
3. Assume that $x, y \in A$. To obtain the minimum cardinality of such A, it suffices to cover the disjoint graphs $L_{n_1-2}, \ldots, L_{n_k-2}$ with the minimum number of vertices such that $x_{2,i}, x_{n_i-1,i} \notin A$ for any $1 \le i \le k$. There exists a minimal vertex cover B_i for the line having an even number of vertices L_{n_i-2} with the minimum number of vertices $m_i - 1$ such that $x_{2,i}, x_{n_i-1,i}$ for $1 \le i \le k_1$ are not contained in A. Moreover, there exists a minimal vertex cover C_i for the line having an odd number of vertices L_{n_i-2} with the minimum number of vertices $l_i - 1$ which does not contain $x_{2,i}, x_{n_i-1,i}$ for $k_1 + 1 \le i \le k$. With this argument, to make a minimal vertex cover with the minimum number vertices for the disjoint lines $L_{n_1-2}, \ldots, L_{n_k-2}$ we may have
$$\sum_{i=1}^{k_1}(m_i - 1) + \sum_{i=k_1+1}^{k}(l_i - 1)$$
vertices. Then, the minimum cardinalty of such A is equal to
$$\begin{aligned}&\sum_{i=1}^{k_1}(m_i - 1) + \sum_{i=k_1+1}^{k}(l_i - 1) + 2 = \\ &(\sum_{i=1}^{k_1} m_i) + (\sum_{i=k_1+1}^{k} l_i) - k_1 - (k - k_1) + 2 = \\ &(\sum_{i=1}^{k_1} m_i) + (\sum_{i=k_1+1}^{k} l_i) - k + 2\end{aligned}$$
4. Assume that $x, y \notin A$. In this case, we may cover the disjoint lines $L_{n_1-2}, \ldots, L_{n_k-2}$ with the minimum number of vertices such that A contains $N_G(x)$ and $N_G(y)$. For any line L_{n_i-2} $(1 \le i \le k_1)$ containing even number of vertices, we can find the minimal vertex cover B_i with the minimum number of vertices m_i which contains $x_{2,i}, x_{n_i-1,i}$ for $1 \le i \le k_1$. In addition, there exists a minimal vertex cover C_i with the minimum number of vertices l_i which contains $x_{2,i}, x_{n_i-1,i}$ for

$k_1+1 \le i \le k$. Therefore, in this case, the minimum number of vertices for covering the disjoint lines $L_{n_1-2},\ldots,L_{n_k-2}$ is equal to $\sum_{i=1}^{k_1} m_i + \sum_{i=k_1+1}^{k} l_i$. Hence, the minimum cardinality of such A equals

$$\sum_{i=1}^{k_1} m_i + \sum_{i=k_1+1}^{k} l_i$$

Since $k \ge 3$ and $k_1 \ge 1$, $1-(k-k_1) \le 0$. We compare the results obtained from the above cases to obtain

$$htI(G) = (\sum_{i=1}^{k_1} m_i) + (\sum_{i=k_1+1}^{k} l_i) - k + 2$$

as desired.

□

To verify Cohen-Macaulayness and unmixedness of the generalized theta graphs, we consider only seven possible cases that are described in the following theorems.

Theorem 11. *Let G be the graph $\theta_{n_1,\ldots,n_{k_1}}$ consisting of lines $L_{3r_1+1},\ldots,L_{3r_{k_1}+1}$,* i.e., *$n_i = 3r_i + 1$ for $1 \le i \le k_1$. Then, G is not unmixed and hence not Cohen-Macaulay.*

Proof. Applying Theorem 2.6 of [2], one can conclude that G is not sequentially Cohen–Macaulay. This implies G is neither Cohen-Macaulay by ([10], Lemma 3.6) nor unmixed by ([8], Theorem 2.14). □

Theorem 12. *Let G be the graph $\theta_{n_1,\ldots,n_{k_2}}$ consisting of lines $L_{3t_1+2},\ldots,L_{3t_{k_2}+2}$,* i.e., *$n_i = 3t_i + 2$ for $1 \le i \le k_2$. Then, G is not unmixed and hence not Cohen-Macaulay.*

Proof. It suffices to show that G is not unmixed. We distinguish the three following cases:

1. There exist positive integers $m_1,\ldots,m_{k_2}$ such that $n_i = 2m_i$ for any $1 \le i \le k_2$;
2. There exist nonnegative integers $r_1,\ldots,r_{k_2}$ such that $n_i = 2r_i + 1$ for any $1 \le i \le k_2$;
3. There exist positive integers $m_1,\ldots,m_k$ such that $n_i = 2m_i$ for any $1 \le i \le k$ and nonnegative integer numbers $r_{k+1},\ldots,r_{k_2}$ such that $n_i = 2r_i + 1$ for any $k+1 \le i \le k_2$.

Using proof of Lemmas 8–10, it is readily seen that there exist two minimal vertex covers of different sizes in any case, and then G is not unmixed. □

Theorem 13. *Let G be the graph $\theta_{n_1,\ldots,n_{k_1+k_3}}$ consisting of lines $L_{3r_1+1},\ldots,L_{3r_{k_1}+1},L_{3s_1},\ldots,L_{3s_{k_3}}$,* i.e., *$n_i = 3r_i + 1$ for $1 \le i \le k_1$ and $n_i = 3s_i$ for $k_1+1 \le i \le k_1+k_3$ such that $k_1,k_3 > 0$. Then, G is not unmixed and hence not Cohen-Macaulay.*

Proof. We have to verify the following cases:

- The set $\{n_1,\ldots,n_{k_1+k_3}\}$ does not contain $\{3,4\}$. Applying Theorem 2.6 of [2], G is not sequentially Cohen-Macaulay, and then G is not Cohen-Macaulay by ([10], Lemma 3.6). We therefore get G is not unmixed by ([8], Theorem 2.14).
- $\{3,4\} \subseteq \{n_1,\ldots,n_{k_1+k_3}\}$. There exist positive integers m_i such that $n_i = 2m_i$ for any $1 \le i \le s$ and there exist nonnegative integers l_i such that $n_i = 2l_i + 1$ for any $s+1 \le i \le k_1+k_3$. Using cases 3 and 4 of Lemma 10, we obtain two minimal vertex covers A and B of cardinalities $2 + (\sum_{i=2}^{s} m_i) + 1 + (\sum_{i=s+2}^{k_1+k_3} l_i) - (k_1+k_3) + 2$ and $2 + (\sum_{i=2}^{s} m_i) + 1 + \sum_{i=s+2}^{k_1+k_3} l_i$, respectively. Since $k_1+k_3 \ge 3$, $5-(k_1+k_3) \ne 3$. Hence, G is not unmixed. Moreover, G is not Cohen-Macaulay.

□

Theorem 14. *Let G be the graph $\theta_{n_1,\ldots,n_{k_2+k_3}}$ consisting of lines $L_{3s_1},\ldots,L_{3s_{k_3}},L_{3t_1+2},\ldots,L_{3t_{k_2}+2}$,* i.e., *$n_i = 3s_i$ for $1 \le i \le k_3$ and $n_i = 3t_i + 2$ for $k_3+1 \le i \le k_3+k_2$ such that $k_2,k_3 > 0$. Then, G is not unmixed and hence not Cohen-Macaulay.*

Proof. It suffices to replace $\{3,4\}$ by $\{2,3\}$ in the proof of Theorem 13 and apply the same argument. □

Theorem 15. *Let G be the graph $\theta_{n_1,\ldots,n_{k_3}}$ consisting of lines $L_{3s_1},\ldots,L_{3s_{k_3}}$, i.e., $n_i = 3s_i$ for $1 \le i \le k_3$. Then, G is not unmixed and hence not Cohen-Macaulay.*

Proof. In order to show that G is not unmixed, we use the same argument of Theorem 12. □

Theorem 16. *Let G be the graph $\theta_{n_1,\ldots,n_{k_1+k_2}}$ consisting of lines $L_{3r_1+1},\ldots,L_{3r_{k_1}+1},L_{3t_1+2},\ldots,L_{3t_{k_2}+2}$, i.e., $n_i = 3r_i+1$ for $1 \le i \le k_1$ and $n_i = 3t_i+2$ for $k_1+1 \le i \le k_1+k_2$ such that $k_1,k_2 > 0$. Then, G is not unmixed and hence not Cohen-Macaulay.*

Proof. From ([2], Lemma 2.6), we obtain that G is not sequentially Cohen-Macaulay and hence G is not Cohen-Macaulay by ([10], Lemma 3.6). Applying Theorem 2.14 of [8], one concludes that G is not unmixed for $k_1 \ge 2$. To complete the proof, it remains to prove that G is not unmixed for $k_1 = 1$. In this case, the same argument of Theorem 12 holds. □

Theorem 17. *Let G be the graph $\theta_{n_1,\ldots,n_{k_1+k_2+k_3}}$ consisting of lines $L_{3r_1+1},\ldots,L_{3r_{k_1}+1},L_{3t_1+2},\ldots,L_{3t_{k_2}+2},L_{3s_1},\ldots,L_{3s_{k_3}}$, i.e., $n_i = 3r_i+1$ for $1 \le i \le k_1$, $n_i = 3t_i+2$ for $k_1+1 \le i \le k_1+k_2$ and $n_i = 3s_i$ for $k_1+k_2+1 \le i \le k_1+k_2+k_3$ such that $k_1,k_2,k_3 > 0$. Then, G is Cohen-Macaulay (unmixed) if and only if $G = \theta_{2,3,4}$.*

Proof.

$\Leftarrow$) Suppose that $G = \theta_{2,3,4}$. Set $I(L_2) = (xy)$, $I(L_3) = (xe,ey)$ and $I(L_4) = (xz,zt,ty)$. Using CoCoA, $I(G)$ has the minimal primary decomposition as

$$I(G) = (y,z,e) \cap (x,y,t) \cap (x,y,z) \cap (x,t,e)$$

Hence, G is unmixed and Cohen-Macaulay by ([8], Theorem 2.14).

$\Rightarrow$) We know that G is Cohen-Macaulay (and hence unmixed) if and only if $htI(G) = pd(G)$. It is not difficult to see that $htI(G)$ changes according to being even or odd the numbers r_i, t_j, s_m $(1 \le i \le k_1, 1 \le j \le k_2, 1 \le m \le k_3)$. By the description given above, there are only nine possible cases. By checking all cases, it is seen that the equality $htI(G) = pd(G)$ holds only for one case. In the following, we examine two cases that seem more important.

- Suppose that there are nonnegative integers r_i, t_j and s_m such that $r_i = 2l_i+1$, $t_j = 2g_j$ and $s_m = 2h_m+1$ for any $1 \le i \le k_1$, $1 \le j \le k_2$ and $1 \le m \le k_3$. By Lemma 10, we get

$$\begin{aligned} htI(G) &= \sum_{i=1}^{k_1}(3l_i) + \sum_{j=1}^{k_2}(3g_j) + \sum_{m=1}^{k_3}(3h_m) + k_1 + 2 \\ &= \frac{3}{2}\sum_{i=1}^{k_1} r_i + \frac{3}{2}\sum_{j=1}^{k_2} t_j + \frac{3}{2}\sum_{m=1}^{k_3} s_m - \frac{k_1}{2} - \frac{3}{2}k_3 + 2 \end{aligned}$$

Similar to the proof of Theorem 2.10 of [8], we obtain that $pd(G) = 2\sum_{i=1}^{k_1} r_i + 2\sum_{j=1}^{k_2} t_j + 2\sum_{m=1}^{k_3} s_m - k_3$. Applying Theorem 2.10 of [8], we have

$$
\begin{aligned}
htI(G) = pd(G) &\Longleftrightarrow \frac{3}{2}\sum_{i=1}^{k_1} r_i + \frac{3}{2}\sum_{j=1}^{k_2} t_j + \frac{3}{2}\sum_{m=1}^{k_3} s_m - \frac{k_1}{2} - \frac{3}{2}k_3 + 2 = 2\sum_{i=1}^{k_1} r_i + 2\sum_{j=1}^{k_2} t_j + 2\sum_{m=1}^{k_3} s_m - k_3 \\
&\Longleftrightarrow \frac{1}{2}\sum_{i=1}^{k_1} r_i + \frac{1}{2}\sum_{j=1}^{k_2} t_j + \frac{1}{2}\sum_{m=1}^{k_3} s_m = 2 - \frac{k_1}{2} - \frac{k_3}{2} \\
&\Longleftrightarrow \sum_{i=1}^{k_1} r_i + \sum_{j=1}^{k_2} t_j + \sum_{m=1}^{k_3} s_m = 4 - k_1 - k_3 \\
&\Longleftrightarrow k_1 = k_3 = 1 \Longleftrightarrow r_1 + \sum_{j=1}^{k_2} t_j + s_1 = 2 \\
&\Longleftrightarrow r_1 = s_1 = 1, k_2 = 1, t_1 = 0 \\
&\Longleftrightarrow G = \theta_{2,3,4}
\end{aligned}
$$

- Suppose that there exist nonnegative integers l_i, g_j and h_m such that $r_i = 2l_i$ for any $1 \leq i \leq \alpha$ and $r_i = 2l_i + 1$ for any $\alpha + 1 \leq i \leq k_1$, $t_j = 2g_j$ for any $1 \leq j \leq \beta$ and $t_j = 2g_j + 1$ for any $\beta + 1 \leq j \leq k_2$, $s_m = 2h_m$ for any $1 \leq m \leq \gamma$ and $s_m = 2h_m + 1$ for any $\gamma + 1 \leq m \leq k_3$ which at least one of α, β and γ is non zero. Note that we choose α, β and γ such that any of the other cases do not occur. Using Lemma 10, we obtain

$$
\begin{aligned}
htI(G) &= \sum_{i=1}^{\alpha}(3l_i) + \sum_{i=\alpha+1}^{k_1}(3l_i) + \sum_{j=1}^{\beta}(3g_j) + \sum_{j=\beta+1}^{k_2}(3g_j) + \sum_{m=1}^{\gamma}(3h_m) + \sum_{m=\gamma+1}^{k_3}(3h_m) \\
&+ k_1 + k_2 - 2\alpha - \beta - \gamma + 2 = \frac{3}{2}\sum_{i=1}^{k_1} r_i + \frac{3}{2}\sum_{j=1}^{k_2} t_j + \frac{3}{2}\sum_{m=1}^{k_3} s_m \\
&- \frac{k_1}{2} - \frac{k_2}{2} - \frac{3}{2}k_3 + 2 + \frac{\beta}{2} + \frac{\gamma}{2} - \frac{\alpha}{2}
\end{aligned}
$$

  Applying Theorem 2.10 of [8], we have

$$
\begin{aligned}
htI(G) = pd(G) &\Longleftrightarrow \frac{3}{2}\sum_{i=1}^{k_1} r_i + \frac{3}{2}\sum_{j=1}^{k_2} t_j + \frac{3}{2}\sum_{m=1}^{k_3} s_m - \frac{k_1}{2} - \frac{k_2}{2} - \frac{3}{2}k_3 + \frac{\beta}{2} + \frac{\gamma}{2} - \frac{\alpha}{2} + 2 \\
&= 2\sum_{i=1}^{k_1} r_i + 2\sum_{j=1}^{k_2} t_j + 2\sum_{m=1}^{k_3} s_m - k_3 \\
&\Longleftrightarrow \frac{1}{2}\sum_{i=1}^{k_1} r_i + \frac{1}{2}\sum_{j=1}^{k_2} t_j + \frac{1}{2}\sum_{m=1}^{k_3} s_m = 2 + \frac{(\beta + \gamma - \alpha)}{2} - \frac{(k_1 + k_2 + k_3)}{2} \\
&\Longleftrightarrow \sum_{i=1}^{k_1} r_i + \sum_{j=1}^{k_2} t_j + \sum_{m=1}^{k_3} s_m = 4 + (\beta - k_2) + (\gamma - k_3) - (\alpha + k_1)
\end{aligned}
$$

  By assumption, we have $\beta - k_2 \leq 0$, $\gamma - k_3 \leq 0$ and $\alpha + k_1 \geq 1$. Then, it follows that $4 + (\beta - k_2) + (\gamma - k_3) - (\alpha + k_1) \leq 3$. Furthermore, we know $\sum_{i=1}^{k_1} r_i + \sum_{j=1}^{k_2} t_j + \sum_{m=1}^{k_3} s_m \geq 2$. Assume $\sum_{i=1}^{k_1} r_i + \sum_{j=1}^{k_2} t_j + \sum_{m=1}^{k_3} s_m = 3$. Since $r_i > 0$, $t_j \geq 0$ and $s_m > 0$, we conclude $(r_1 = s_1 = t_1 = 1)$, $(r_1 = 1, t_1 = 0, s_1 = 2)$ or $(r_1 = 2, t_1 = 0, s_1 = 1)$ which are contradictions by assumption. Suppose that $\sum_{i=1}^{k_1} r_i + \sum_{j=1}^{k_2} t_j + \sum_{m=1}^{k_3} s_m = 2$, then we have $r_1 = 1$, $t_1 = 0$ and $s_1 = 1$. This implies that $\alpha = \gamma = 0$ and $\beta = 1$, a contradiction. Hence, G is not Cohen-Macaulay (unmixed).

□

By considering the nine previous theorems, we get the following result:

Corollary 18. *Let $G = \theta_{n_1,\ldots,n_k}$. Then, the following conditions are equivalent:*
(a) G is Cohen–Macaulay;
(b) G is unmixed;
(c) $G = \theta_{2,3,4}$.

4. Conclusions

We have shown that algebraic invariants of the ideals associated to combinatorial structers are computable.

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References

1. Lyubeznik, G. On the local cohomology modules $H^i_A(R)$ for ideals A generated by monomials in an R-sequence. *Lect. Notes Math.* **1984**, *1092*, 214–220.
2. Mohammadi, F.; Kiani, D. Sequentially Cohen-Macaulay graphs of form $\theta_{n_1,\ldots,n_k}$. *Bull. Iran. Math. Soc.* **2010**, *36*, 109–118.
3. Barile, M. On the arithmetical rank of the edge ideals of forests. *Commun. Algebra* **2008**, *36*, 4678–4703.
4. Kimura, K.; Terai, N. Binomial arithmetical rank of edge ideals of forests. *Proc. Am. Math. Soc.* **2013**, *141*, 1925–1932.
5. Barile, M.; Kiani, D.; Mohammadi, F.; Yassemi, S. Arithmetical rank of the cyclic and bicyclic graphs. *J. Algebra Appl.* **2012**, *11*, 1250039.
6. Mohammadi, F.; Kiani, D. On the arithmetical rank of the edge ideals of some graphs. *Algebra Colloq.* **2012**, *19*, 797–806.
7. Zhu, G.; Shi, F.; Gu, Y. Arithmetical rank of the edge ideals of some n-cyclic graphs with a common edge. *Turk. J. Math.* **2015**, *39*, 112–123.
8. Seyyedi, S.M.; Rahmati, F. Regularity and projective dimension of the edge ideal of generalized theta graph. *Turk. J. Math.*, submitted for publication, 2015.
9. Schmitt, T.; Vogel, W. Note on set-theoretic intersections of subvarieties of projective space. *Math. Ann.* **1979**, *245*, 247–253.
10. Francisco, C.A.; van Tuyl, A. Sequentially Cohen-Macaulay edge ideals. *Proc. Am. Math. Soc.* **2007**, *135*, 2327–2337.