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Sharing of Nonlocality of a Single Member of an Entangled Pair of Qubits Is Not Possible by More than Two Unbiased Observers on the Other Wing

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Abstract: We address the recently posed question as to whether the nonlocality of a single member of an entangled pair of spin 1/2 particles can be shared among multiple observers on the other wing who act sequentially and independently of each other. We first show that the optimality condition for the trade-off between information gain and disturbance in the context of weak or non-ideal measurements emerges naturally when one employs a one-parameter class of positive operator valued measures (POVMs). Using this formalism we then prove analytically that it is impossible to obtain violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality by more than two Bobs in one of the two wings using unbiased input settings with an Alice in the other wing.

1. Introduction

The quantum theory of measurement is counter-classical in a sense that in order to obtain any statistical information about the state, some disturbance of the state becomes unavoidable. Measurement does not change the state only if the system is in an eigenstate of the observable to be measured. A von Neumann type measurement [1] dubbed as strong measurement induces strong collapse transforming the initial state of the system into one of the eigenstates of the measured observable. This type of measurement yields maximum information about the measured system. On the other hand, there exist measurement schemes such as weak measurement [2–4] which provide less information about the system while affecting it minimally, thus indicating a trade-off. A pertinent question in this context is to what extent one may control the state change due to measurement while obtaining some information about it. Recently, it has been shown that such a trade-off between the degree of disturbance and the amount of information gain about the system may be optimized using suitably chosen pointer states by employing weak measurements without post-selection [5].

Another counter-classical feature of quantum theory is the presence of nonlocal correlations in the measurement outcomes of two or more parties sharing certain types of quantum states. This property of quantum systems manifested by the violation of local realist, e.g., Bell-CHSH [6–8] type inequalities has been well studied in in the literature in the context of multipartite or multilevel states [9–13]. However, a new fundamental question on the sharing of non-locality by multiple observers was posed recently [5]: Can the nonlocality pertaining to a single member of an entangled pair of particles be shared among more than two independent observers who sequentially perform measurements on the other member of the entangled pair? Note that the monogamy constraints [14–16] on entanglement and nonlocality do not apply in this scenario since the condition

of no-signalling is violated. Though the observers who perform consecutive measurements are independent of one another, the observer(s) who perform the prior measurement(s) implicitly signal to the latter one(s) through the choice of their measurement(s).

The motivation of the present work is to investigate the above question of sharing of nonlocality of an entangled pair of two particles by multiple observers. The experimental scenario considered here [5] is as follows. One of the particles of an entangled pair is measured by a single observer Alice who performs projective measurements on one side. There exist multiple observers (Bobs) on the other side who act sequentially and independently. Using weak measurements with optimized pointer settings it was shown [5] that Bell-CHSH inequalities between Alice and an arbitrary number of sequential Bobs can be consecutively violated in case of biased observation settings used by the various Bobs. In other words, the protocol works when one of the inputs to the various independent observers occurs a lot more often than the other input. Though, in the unbiased input scenario numerical evidence indicated that violation of the CHSH inequality is impossible by more than two Bobs, it was left as an open problem to show this analytically [5].

In this paper we study this problem analytically using the framework of unsharp measurements [17] or POVMs with a single parameter, based upon the notion of generalised observables beyond the usual framework projective valued measures (PVM) or sharp measurements. In the measurement process after interaction of the physical system with the apparatus the latter indicates a particular value corresponding to the former. This indication is modelled by means of pointer observable assuming an actual value corresponding to a value of the physical quantity of interest. Actual measurements in which the apparatus are represented by broad meter states, are seldom compatible with PVMs. On the other hand, the generalised notion of POVMs turns out to be very helpful not only in explaining some of the conceptual problems of quantum theory, such as joint measurability of non-commutative observables [18,19], but in also performing tasks such as probing non-locality in the case of mixed states. There are non-separable mixed states for which the Bell-CHSH inequalities are violated not for the usual pair of sharp spins but only for suitable families of POVMs. For two outcome measurements only projective measurements are sufficient for Bell violation [20] whereas advantages of POVMs are discernible for measurements with more outcomes [21]. This is an illustration of the fact that optimisation of information gain in measurements can under certain conditions only be achieved with POVMs but not with PVMs. A comprehensive introduction to the topic of POVMs and their application in quantum foundations and experiments can be found in the monographs [17,22] and references therein.

Using the formalism of POVMs we show here that unsharp observables characterized by a single unsharpness parameter saturate the optimal pointer condition with respect to the trade-off between disturbance and information gain, a condition that was earlier obtained using numerical optimization [5]. We then apply this formalism to the case of the problem of sharing nonlocality by multiple observers, as mentioned above. We prove analytically that more than two consecutive violations of the CHSH inequality are impossible in the unbiased scenario. The plan of the paper is as follows. In Section 2 we provide a brief discussion on the quantum theory of measurement and POVMs. In Section 3 we show how the optimality condition for the information gain versus disturbance trade-off emerges naturally within the unsharp measurement formalism. In Section 4 we prove analytically that nonlocality of an entangled pair of spin 1/2 particles cannot be shared between Alice and more than two Bobs. We conclude with a brief summary in Section 5.

2. Quantum Measurements

The minimal content of the notion of measurement in quantum mechanics [23] is given by the probability reproducibility requirement, according to which a particular measurement scheme qualifies as a measurement of a given observable E if for all initial states of the system the associated probability distributions of E are reproduced in the resulting statistics of pointer readings. The information available by a given measurement depends on the statistical dependencies

established by the interaction between the system and the apparatus. Let S be the system with associated Hilbert space $\mathcal{H}_S$, and $\mathcal{A}$ be the measuring apparatus with Hilbert space $\mathcal{H}_A$. The initial joint state of system and the apparatus is transformed unitarily during *pre-measurement*, and is given by

$$\mathcal{U}(\rho_S \otimes \rho_A) \mapsto \mathcal{U}\rho_S \otimes \rho_A \mathcal{U}^* \tag{1}$$

An explicit construction of pre-measurement for discrete sharp observables has been known since the work of von Neumann [1]. Any pre-measurement of an observable determines a state transformer on a measurable space $(\Omega, \mathcal{F})$, $\mathcal{I} : \mathcal{F} \mapsto \mathcal{L}(T(H_s))$ through the relation

$$\mathcal{I}_M(X)(\rho) := tr[\mathbb{I} \otimes Z(\mathcal{U}\rho \otimes \rho_A \mathcal{U}^*)\mathbb{I} \otimes Z] \tag{2}$$

where, $X \in \mathcal{F}$, and $\mathcal{L}(T(H_s))$ is the set of operators acting on a set of density states. The state transformer summarizes all the features of the pre-measurement. It recovers the observable via the relation

$$tr[E(X)\rho] = tr[\mathcal{I}_M(X)\rho]\forall X \in \mathcal{F}, \rho \in T(H_S) \tag{3}$$

The state transformer for projective measurement of a discrete observable A with eigenvalues a_is is given by

$$\mathcal{I}_M(\rho) = \sum_{a_i \in X} P_i \rho P_i \tag{4}$$

For an observable $A = \sum a_i P_i$ with eigenvalue a_i and eigenprojectors P_i, pre-measurement is given by

$$\mathcal{U}(\varphi \otimes \phi) = \sum P_i \varphi \otimes \phi_i, \varphi \in \mathcal{H}_S, \phi \in \mathcal{H}_A \tag{5}$$

Let $\mathcal{Z} = \sum z_i \mathcal{Z}$ be an observable of apparatus $\mathcal{A}$, known as pointer observable. The reduced state of the apparatus is given by $W(\varphi) = \sum_{a_i} p_\varphi^A(a_i)P[\phi_i]$ (all the $P[\phi_i]$ are not necessarily mutually orthogonal) with the probability reproducibility condition given by

$$p_\varphi^A(a_i) = p_{W(\varphi)}^Z(z_i) \tag{6}$$

where $p_\varphi^A(a_i)$ is the probability distribution of outcomes of the observable A and $p_{W(\varphi)}^Z(z_i)$ is that of the pointer observable. Equation (6) implies that the outcome probabilities for observable A are recovered as the distribution of the pointer values in the final apparatus state. The emerging observable out of this measurement scheme is given by

$$E_i = \sum p_{W(\varphi)}^Z(z_i)P_j \tag{7}$$

Now, following [22] let us see how POVM emerges quite naturally in an actual measurement on a two level system. Let us take the system-apparatus coupling as

$$\mathcal{U} = \exp^{i\lambda\sigma.\hat{n}\otimes\mathcal{P}} \tag{8}$$

where $\mathcal{P}$ is the momentum operator and the pre-measurement is given by

$$\begin{aligned}|\Psi\rangle = \mathcal{U}(\varphi \otimes \phi) &= \sum P_i \varphi \otimes \exp^{i\lambda a_i \mathcal{P}} \phi \\ &= C_+ \varphi_+ \otimes \phi_+ + C_- \varphi_- \otimes \phi_- \end{aligned} \tag{9}$$

Vectors $\exp^{i\lambda a_i \mathcal{P}} \phi$ or $\phi_\pm$ need not be mutually orthogonal. Next, to describe registration of spots on the screen, the pointer observable is modelled by $P_\pm$, projectors onto the upper and lower half of the screen. For unsharp observables the state transformer is given by the generalised Luder transformer as

$$\mathcal{I}_M(\rho) = \sum_{a_i \in X} \sqrt{E_i} \rho \sqrt{E_i} \tag{10}$$

This measurement scheme yields

$$< \Psi | \mathbb{I} \otimes P_\pm \Psi >= |C_+|^2 \langle \phi_+ | P_+ \phi_+ \rangle + |C_-|^2 \langle \phi_- | P_+ \phi_- \rangle$$
$$:= \langle \varphi | E_\pm \varphi \rangle \tag{11}$$

where the effects $E_\pm$ constitute the unsharp spin observables actually measured in this experiment, given by

$$E_\pm = \langle \phi_+ | P_+ \phi_+ \rangle P[\varphi_+] + \langle \phi_- | P_- \phi_- \rangle P[\varphi_-] \tag{12}$$

with $E_+ + E_- = \mathbb{I}$, and $E_\pm^2 \neq E_\pm$, i.e., $\langle \varphi_+ | E_+ \varphi_+ \rangle, \langle \varphi_- | E_- \varphi_- \rangle \neq 0$ or 1. If the center of mass of the wave-packets $\phi_\pm$ were well separated and localized in the appropriate half planes, i.e., if $\langle \phi_\pm | P_\pm \phi_\pm \rangle = 1$, then $\langle \phi_\pm | P_\mp \phi_\pm \rangle = 0$, in which case $E_\pm$ coincides with $P[\varphi_\pm]$. However, due to spreading of this wave-packet this coincidence is achieved only approximately. This provides a possible source of inaccuracy due to quantum indeterminism inherent in the center of mass wave-function. Even if spin is prepared sharply *a priori*, its value can only be ascribed with some uncertainty.

3. Optimality Condition for Information Gain Versus Disturbance Trade-off Using Unsharp Measurements

Following the work of [5], let us consider a spin 1/2 particle whose initial state is described by the state $|\psi\rangle (= \alpha|0\rangle + \beta|1\rangle)$. Considering von Neumann type measurements, after interaction with a meter with the state $\phi(q)$, the joint state of system and apparatus goes to $\alpha|0\rangle \otimes \phi(q-1) + \beta|1\rangle \otimes \phi(q+1)$. On tracing out the pointer state the reduced state of the system is given by

$$\rho' = F\rho + (1-F)(\pi^+ \rho \pi^+ + \pi^- \rho \pi^-) \tag{13}$$

where $\rho = |\psi\rangle\langle\psi|$, and $\pi^\pm$ are projectors onto states $|0\rangle, |1\rangle$, and $F(\phi) = \int_{-\infty}^{\infty} \langle \phi(q+1)|\phi(q-1)\rangle dq$, is called the quality factor of the measurement. The probability of getting outcomes '$\pm$' is given by

$$p(\pm) = G\langle\psi|\pi^\pm|\psi\rangle + (1-G)\frac{1}{2} \tag{14}$$

Here, $G = \int_{-1}^{1} \phi^2(q) dq$, which quantifies the precision of the measurement. It is independent of the state of the spin and depends on the width of the pointer compared to the distance between the eigenvalues. These two parameters F and G characterize a weak measurement (the case with $F = 0$ and $G = 1$ corresponds to a strong measurement). It was found in [5] that a square pointer yields the relation $G = 1 - F$. However, such a pointer is not optimal. An optimal pointer is defined as the one which gives the best trade-off, i.e., for a given quality factor F, it provides the largest precision G. It was shown that the optimal information-disturbance trade-off condition given by

$$F^2 + G^2 = 1 \tag{15}$$

emerges using an optimal pointer that is not a Gaussian wave-packet.

For two outcome measurements the notion of unsharp measurement discussed in section-II is captured by the effect operator, $E^\lambda = (\mathbb{I} + \lambda n_i \sigma_i)/2$, $i = 1, 2, 3.$, with $\lambda \in (0, 1]$. Thus, the set of effects can be written as a linear combination of sharp projectors with white noise, $E^\lambda \equiv \{E_+^\lambda, E_-^\lambda | E_+^\lambda + E_-^\lambda = \mathbb{I}\}$, given by

$$E_\pm^\lambda = \lambda P_\pm + \frac{1 - \lambda}{2} \mathbb{I} \tag{16}$$

In the unsharp formalism the non-selective un-normalised state of the system after pre-measurement according to the Luder transformation rule (10) is given by $\rho' = \sqrt{E_+^\lambda} \rho \sqrt{E_+^\lambda} + \sqrt{E_-^\lambda} \rho \sqrt{E_-^\lambda}$. From this we get

$$\rho' = \sqrt{1 - \lambda^2} \rho + (1 - \sqrt{1 - \lambda^2})(P_+ \rho P_+ + P_- \rho P_-) \tag{17}$$

The probabilities of getting the outcomes $\pm$ are given by

$$p(\pm) = tr[E_\pm^\lambda \rho] = \lambda tr[P_\pm \rho] + \frac{1 - \lambda}{2} \tag{18}$$

Comparing the two formalisms, i.e., comparing Equation (13) with Equation (17) and Equation (14) with Equation (18), one sees that $G = \lambda$ and $F = \sqrt{1 - \lambda^2}$. Hence, λ characterises the precision of the measurement. For $G = \lambda = 1$, F becomes zero, this being the limit of sharp measurement. We thus find that the optimal pointer state condition, $F^2 + G^2 = 1$ given by Equation (15) and derived through an optimization in [5] emerges explicitly within the formalism of unsharp measurements. In other words, unsharp measurement yields the maximum information about the system while disturbing the original state minimally.

4. Sharing of Non-Locality

We now show that an application of the formalism of unsharp measurements in the context of sharing of non-locality enables us to resolve an open issue stated in [5]. We show here analytically that using a pair of entangled spin $1/2$ particles Alice cannot share non-locality with more than two Bobs. To address this question let us consider the following Bell-CHSH scenario. All the observers have two measurement choices which they perform one at a time. Tsirelson's bound is achieved when Alice measures in the direction $\hat{X}$ and $\hat{Z}$ and Bob measures in directions $\frac{-(\hat{Z} + \hat{X})}{\sqrt{2}}$, $\frac{-\hat{Z} + \hat{X}}{\sqrt{2}}$. As we want to see how many Bobs can have measurement statistics violating the CHSH inequality with a single Alice, the 1st Bob cannot measure sharply. This would destroy the entanglement between the particles shared by Alice and Bob and there would be no possibility of violation of the CHSH inequality for the 2nd Bob. Hence, in order to share nonlocality among n Bobs, $n - 1$ of them have to measure weakly. Here, it is important to note that each Bob measures independently of the previous Bobs on the particle of his possession. Hence, any Bob has to consider the average effect of possible choices of measurements done by previous Bobs [5]. Here, we do not consider multiple Alices, and thus it is not required to consider unsharp measurement for Alice as it reduces the violation of the CHSH inequality thereby reducing the possibility of sharing nonlocality for multiple Bobs. However, there may be a possibility of getting an advantage if Alice performs sharp nonorthogonal measurements. We comment on this issue later.

The joint probability of getting the outcome 'a' and 'b_n' by Alice and Bob^n (the n-th Bob) respectively, is given by

$$p(a, b_n) = p(a)p(b_n | a) = \frac{1}{2} Tr\left[\frac{\mathbb{I} + \lambda_n b_n \hat{y}_n.\vec{\sigma}}{2} \rho_{n | y_1 \dots y_{n-1}}\right] \tag{19}$$

where $\rho_{n|y_1...y_{n-1}}$ is the state of the pair of spin-$\frac{1}{2}$ particles before the measurements of Alice and Bob^n, and y_i is the measurement done by the i-th Bob. For two Bob measuring in succession, the joint probability is given by

$$p(a, b_2) = \frac{\sqrt{1-\lambda_1^2}}{2}\frac{1-ab_2\lambda_2\hat{y_2}.\hat{x}}{2}+$$

$$\frac{1-\sqrt{1-\lambda_1^2}}{2}\frac{1-ab_2\lambda_2\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_2}}{2} \tag{20}$$

The measurement directions chosen for Alice are $\hat{X}, \hat{Z}$, and those for Bob are $\frac{-(\hat{Z}+\hat{X})}{\sqrt{2}}, \frac{-\hat{Z}+\hat{X}}{\sqrt{2}}$. For the first Bob measuring weakly and the second Bob measuring sharply, the CHSH values are given by $CHSH_{AB_1} = 2\sqrt{2}\lambda_1$, and $CHSH_{AB_2} = \sqrt{2}(1 + \sqrt{1-\lambda_1^2})$ respectively. This result coincides with that obtained in [5]. In this case both Bobs obtain violation of the Bell-CHSH inequality when the precision of the 1st Bob remains within the range $1/\sqrt{2}$ and $\sqrt{2(\sqrt{2}-1)}$.

Now consider the case of three Bobs with a single Alice. In this case the 1st and 2nd Bobs both measure weakly, while the last Bob measures sharply. We get

$$p(a, b_3) = \frac{1}{2}[\sqrt{1-\lambda_1^2}\sqrt{1-\lambda_2^2}\frac{1-ab_3\lambda_3\hat{y_3}.\hat{x}}{2}+$$

$$(1-\sqrt{1-\lambda_1^2})\sqrt{1-\lambda_2^2}\frac{1-ab_3\lambda_3\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_3}}{2}+$$

$$\sqrt{1-\lambda_1^2}(1-\sqrt{1-\lambda_2^2})\frac{1-ab_3\lambda_3\hat{x}.\hat{y_2}\hat{y_2}.\hat{y_3}}{2}+$$

$$(1-\sqrt{1-\lambda_1^2})(1-\sqrt{1-\lambda_2^2})\frac{1-ab_3\lambda_3\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_2}\hat{y_2}.\hat{y_3}}{2}] \tag{21}$$

Here λ_2 is precision of measurement by the 2nd Bob. The correlation between Alice and Bob^3 is given by

$$C_3 = \lambda_3[\sqrt{1-\lambda_1^2}\sqrt{1-\lambda_2^2}\hat{y_3}.\hat{x}+$$

$$(1-\sqrt{1-\lambda_1^2})\sqrt{1-\lambda_2^2}\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_3}+$$

$$\sqrt{1-\lambda_1^2}(1-\sqrt{1-\lambda_2^2})\hat{x}.\hat{y_2}\hat{y_2}.\hat{y_3}+$$

$$(1-\sqrt{1-\lambda_1^2})(1-\sqrt{1-\lambda_2^2})\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_2}\hat{y_2}.\hat{y_3}] \tag{22}$$

As any Bob is ignorant about inputs of previous Bobs, this correlation has to be averaged over all possible earlier inputs. Hence, one has

$$\bar{C}_3 = \sum_{y_1 y_2} C_3 P(y_1)P(y_2) \tag{23}$$

With this average correlation we find the CHSH sum between Alice and Bob^3 given by

$$\mathcal{I}^3 = \frac{(1+\sqrt{1-\lambda_1^2})(1+\sqrt{1-\lambda_2^2})}{\sqrt{2}} \tag{24}$$

For the 1st and 2nd Bobs the corresponding CHSH values are given by $CHSH_{AB_1} = 2\sqrt{2}\lambda_1$ and $CHSH_{AB_2} = \lambda_2\sqrt{2}(1 + \sqrt{1-\lambda^2})$, respectively. In order for the 1st Bob to obtain violation of the Bell-CHSH inequality, his measurement precision λ_1 has to be greater than $1/\sqrt{2}$. For the 2nd Bob to get the violation, it is required that $\lambda_2 > \frac{2}{\sqrt{2}+1}$. Thus, it follows from Equation (24) that if the first two

Bobs obtain violation, the subsequent CHSH value corresponding to Bob[3] cannot be greater than 2. In the worst case scenario of violation of CHSH by Bob[1] and Bob[2], i.e., when both of them obtain minimal violation, $\mathcal{I}^3$ can not becomes greater than 1.88.

The above arguments were based on the assumption of orthogonality for the measurements performed by the Bobs. Let us now consider the scenario when the orthogonality condition for the Bobs' measurements is relaxed. In this case the meausurements performed by the *n*-th Bob may be denoted as

$$y_n^0 = \cos\theta_{n0}\hat{Z} + \sin\theta_{n0}\hat{X}, y_n^1 = \cos\theta_{n1}\hat{Z} + \sin\theta_{n1}\hat{X} \tag{25}$$

Alice's measurements are the same as before, i.e., $\hat{X}, \hat{Z}$. In the case of non-orthogonal measurements done by all the Bobs CHSH violation is not possible by more than two of them. With the non-orthogonal measurements defined above, we have

$$\mathcal{I}^1(xy_1) = \lambda_1 \tilde{\mathcal{I}}^1(xy_1) \tag{26}$$

where $\tilde{\mathcal{I}}^1(xy_1) = (\cos[\theta_{10}] - \cos[\theta_{11}] + \sin[\theta_{10}] + \sin[\theta_{11}])$. After the 1st and 2nd Bob measuring weakly the CHSH expression between Alice and 2nd Bob is given by

$$\mathcal{I}^2(x(y_1)y_2) = \lambda_2[(1 - F_1)\tilde{\mathcal{I}}^2(x(y_1)y_2) + F_1\tilde{\mathcal{I}}^1(xy_2)] \tag{27}$$

In the above expression $\tilde{\mathcal{I}}^2(x(y_1)y_2) = \frac{1}{2}\sum_{i,j=0}^1((-1)^j\cos\theta_{1i} + \sin\theta_{1i})\cos(\theta_{1i} - \theta_{2j})$, and in the argument of $\tilde{\mathcal{I}}^2$, (y_1) implies averaging over the inputs of Bob[1]. Now denoting $F_i = \sqrt{1 - \lambda_i^2}$, and with the 3rd Bob measuring sharply, we have

$$\mathcal{I}^3(x(y_1y_2)y_3) = \frac{F_1+F_2}{2}\tilde{\mathcal{I}}^1(xy_3) +$$

$$\frac{(1-F_1)F_2}{2}\left(\tilde{\mathcal{I}}^2(x(y_1)y_3) - \tilde{\mathcal{I}}^2(x(y_1 + \frac{\pi}{2})y_3)\right) +$$

$$\frac{(1-F_2)F_1}{2}\left(\tilde{\mathcal{I}}^2(x(y_2)y_3) - \tilde{\mathcal{I}}^2(x(y_2 + \frac{\pi}{2})y_3)\right) + \tag{28}$$

$$\frac{(1-F_1)(1-F_2)}{16}\tilde{\mathcal{I}}^3(x(y_1y_2)y_3)$$

Here $(y_i + \frac{\pi}{2})$ means averaging over the measurement directions of Bob[i] after rotation by $\pi/2$ and $\tilde{\mathcal{I}}^3(x(y_1y_2)y_3) = \sum_{k=0}^1[\sum_{i=1,2,j=0,1} 2\sin(2\theta_{ij} - \theta_{3k}) + 4\sin\theta_{3k} + \sum_{i,j=0,1}\sin(2\theta_{1i} - 2\theta_{2j} + \theta_{3k})] + \sum_{k=0}^1(-1)^k[\sum_{i=1,2,j=0,1} 2\cos(2\theta_{ij} - \theta_{3k}) + 4\cos\theta_{3k} + \sum_{i,j=0,1}\cos(2\theta_{1i} - 2\theta_{2j} + \theta_{3k})]$.

With the above expressions for most general settings we find that when $\mathcal{I}^1 = 2$, and $\mathcal{I}^2 = 2$, then one obtains $\mathcal{I}^3 = 2$. When both the 1st and 2nd Bobs get 5 percent violation, i.e., having CHSH expression equalling 2.1, then $\mathcal{I}^3 \to 1.89$ at most, with the settings $y_1^0 \approx 0.19\hat{Z} + 0.98\hat{X}, y_1^1 \approx -0.19\hat{Z} + 0.98\hat{X}$, $y_2^0 \approx 0.19\hat{Z} + 0.98\hat{X}, y_2^1 \approx -0.19\hat{Z} + 0.98\hat{X}$ and $y_3^0 \approx 0.04\hat{Z} + 0.99\hat{X}, y_3^1 \approx -0.04\hat{Z} + 0.99\hat{X}$. It should be noted here that even if we consider nonorthogonal measurements by Alice along with all the Bobs we get similar results with expressions involving more variables. There exists settings for which $\mathcal{I}^3$ becomes at most 2 when $\mathcal{I}^1 = 2$, and $\mathcal{I}^2 = 2$. Again, when the 1st and 2nd Bobs get 5 percent violation, $\mathcal{I}^3 \to 1.89$ at most, and for these settings we find that Alice's measurements are orthogonal. It is thus clear that more than two Bobs can never share the nonlocality of a pair of spin 1/2 particles with a single Alice, a result that was numerically conjectured in [5]. One may note that the sequence of the particular Bobs is not important in this scenario. For example, Bob[3] may obtain violation if the sharpness of the 2nd Bob's measurement is sufficiently weak for the latter not to get a violation. There exists a range of unsharpness parameters for each Bob so that any one pair of Bobs in the combinations (Bob[1], Bob[2]), (Bob[1], Bob[3]), or (Bob[2], Bob[3]) can simultaneously demonstrate non-locality.

5. Discussions

In this work we have considered the question of sharing the nonlocality of a single member of an entangled pair of spin 1/2 particles by multiple observers on the other side acting sequentially and independently of each other. This fundamental issue has been recently studied by Silva et al. [5] within the context of the trade-off between information gain and disturbance in weak measurements. It was found that multiple Bobs having sequential access to one particle from the entangled pair can indeed violate Bell-CHSH inequalities with a single Alice on the other side, provided biased measurement settings were used. It was also observed therein using numerical evidence that in the case of unbiased input settings, it was not possible for more than two Bobs to obtain CHSH violations, and it was left as an open problem to prove this analytically [5]. In the present work we address this issue by considering usharp measurements within the framework of POVMs [17]. We first show that the optimality condition for the information gain versus disturbance trade-off derived in [5] emerges naturally using one-parameter POVMs. Applying this framework to the problem of sharing of nonlocality then enables us to show analytically that more than two Bobs cannot violate the CHSH inequality with a single Alice.

6. Conclusions

Generalised notion of quantum measurements modelled by POVM has various kind of advantages over notion of sharp von Neumann type measurements in explaining many quantum features. In the context of sharing nonlocality of a single member of an entangled pair of qubits by multiple observers on the other wing it is shown analytically that more than tow observers can not share nonlocality by considering unsharp measurements which is one parameter POVM.

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