mathematics

MDPI

Article

A New Approach to Study Fixed Point of Multivalued Mappings in Modular Metric Spaces and Applications

Dilip Jain [1], Anantachai Padcharoen [2], Poom Kumam [2,*] and Dhananjay Gopal [1]

1 Department of Applied Mathematics & Humanities, S.V. National Institute of Technology, Surat-395007 Gujarat, India; dilip18pri@gmail.com (D.J.); dg@ashd.svnit.ac.in (D.G.)

2 Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand; apadcharoen@yahoo.com

* Correspondence: poom.kum@kmutt.ac.th; Tel.: +66-2-470-8994

Academic Editor: Pasquale Vetro

Received: 5 June 2016; Accepted: 1 August 2016; Published: 8 August 2016

Abstract: The purpose of this paper is to present a new approach to study the existence of fixed points for multivalued F-contraction in the setting of modular metric spaces. In establishing this connection, we introduce the notion of multivalued F-contraction and prove corresponding fixed point theorems in complete modular metric space with some specific assumption on the modular. Then we apply our results to establish the existence of solutions for a certain type of non-linear integral equations.

Keywords: fixed point; multivalued F-contractive; modular metric space; non-linear integral equations

MSC: 47H10; 54H25; 37C25

1. Introduction

The Banach contraction principle [1] is one of the most important analytical results and considered as the main source of metric fixed point theory. It is the most widely applied fixed point result in many branches of mathematics. This result has been generalized in many different directions. Subsequently, in 2012, Wordowski [2] introduced the concept of F-contraction which generalized the Banach contraction principal in many ways. Further, Sgroi et al. [3] obtained a multivalued version of Wordowski's result.

On other hand, Chistyakov [4] introduced the concept of modular metric spaces and gave some fundamental results on this topic. The fixed point property in this space has been defined and investigated by many authors [5–9]. It is important to note that in the classical Banach contraction, the contractive condition of the mapping implies that any orbit is bounded (see [10]). In case of modular metric space, due to failure of triangle inequality, it is not always true that the contractive condition of the mapping implies the boundedness of the orbit. Therefore, it is very important to handle this obstacle when dealing with a fixed point in modular metric space. Keeping the above facts in mind, in this paper, we define multivalued F-contraction in the setting of modular metric spaces with specific modular situations. Our result is a partial extension of Nadler [11], Wardowski [2] and Sgroi [3] to modular metric spaces. We also give an application of our main results to establish the existence of the solution of a non-linear integral equation.

2. Preliminaries

Throughout the article $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}$ will denote the set of natural numbers, positive real numbers and real numbers respectively.

Let X be a nonempty set. Throughout this paper, for a function $w : (0,\infty) \times X \times X \to [0,\infty]$, we write

$$w_\lambda(x,y) = w(\lambda,x,y)$$

for all $\lambda > 0$ and $x,y \in X$.

Definition 1. *[4,5] Let X be a nonempty set. A function $w : (0,\infty) \times X \times X \to [0,\infty]$ is said to be a metric modular on X, it satisfies, for all $x,y,z \in X$ the following conditions:*

(*i*) $w_\lambda(x,y) = 0$ *for all* $\lambda > 0$ *if and only if* $x = y$;
(*ii*) $w_\lambda(x,y) = w_\lambda(y,x)$ *for all* $\lambda > 0$;
(*iii*) $w_{\lambda+\mu}(x,y) \leq w_\lambda(x,z) + w_\mu(z,y)$ *for all* $\lambda,\mu > 0$.

If instead of (i) we have only the condition $(i^{'})$

$$w_\lambda(x,x) = 0 \ \text{ for all } \lambda > 0,\ x \in X$$

then w is said to be a pseudomodular (metric) on X. A modular metric w on X is said to be regular if the following weaker version of (i) is satisfied:

$$x = y \text{ if and only if } w_\lambda(x,y) = 0 \text{ for some } \lambda > 0$$

This condition play a significant role to insure the existence of fixed point for contractive type mapping in the setting of modular metric.

Example 2. *Let $X = \mathbb{R}$ and w is defined by $w_\lambda(x,y) = \infty$ if $\lambda < 1$, and $w_\lambda(x,y) = \frac{1}{\lambda}|x-y|$ if $\lambda \geq 1$, it is easy to verify that w is regular but not metric modular on X.*

Finally, w is said to be convex if, for $\lambda,\mu > 0$ and $x,y,z \in X$, it satisfies the inequality

$$w_{\lambda+\mu}(x,y) \leq \frac{\lambda}{\lambda+\mu} w_\lambda(x,z) + \frac{\mu}{\lambda+\mu} w_\mu(z,y)$$

Note that for a metric pseudomodular w on a set X, and any $x,y \in X$, the function $\lambda \longrightarrow w_\lambda(x,y)$ is nonincreasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$w_\lambda(x,y) \quad \leq w_{\lambda-\mu}(x,x) + w_\mu(x,y) \quad = w_\mu(x,y)$$

Definition 3. *[4,5] Let w be a pseudomodular on X. Fix $x_0 \in X$. The set*

$$X_w = X_w(x_0) = \{x \in X : w_\lambda(x,x_0) \longrightarrow 0 \text{ as } \lambda \longrightarrow \infty\}$$

is said to be modular spaces (around x_0).

Definition 4. *[5] Let X_w be a modular metric space.*

(*i*) *The sequence $(x_n)_{n\in\mathbb{N}}$ in X_w is said to be w-convergent to $x \in X_w$ if and only if $w_1(x_n,x) \to 0$, as $n \to \infty$.*
(*ii*) *The sequence $(x_n)_{n\in\mathbb{N}}$ in X_w is said to be w-Cauchy if $w_1(x_m,x_n) \to 0$, as $m,n \to \infty$.*
(*iii*) *A subset D of X_w is said to be w-complete if any w-Cauchy sequence in D is a convergent sequence and its limit is in D.*
(*iv*) *A subset D of X_w is said to be w-closed if the w-limit of a w-convergent sequence of D always belongs to D.*
(*v*) *A subset D of X_w is said to be w-bounded if for some $\lambda > 0$, we have*

$$\delta_w(D) = \sup\{w_1(x,y); x,y \in D\} < \infty$$

(vi) *A subset D of X_w is said to be w-compact if for any $\{x_n\}$ in D there exists a subset sequence $\{x_{n_k}\}$ and $x \in D$ such that $w_1(x_{n_k}, x) \longrightarrow 0$.*

In general, if $\lim\limits_{n\longrightarrow\infty} w_\lambda(x_n, x) = 0$, for some $\lambda > 0$, then we may not have $\lim\limits_{n\longrightarrow\infty} w_\lambda(x_n, x) = 0$, for all $\lambda > 0$. Therefore, as is done in modular function spaces, we will say that w satisfies the Δ_2-condition (see page 4 in [5]) if this the case, i.e., $\lim\limits_{n\longrightarrow\infty} w_\lambda(x_n, x) = 0$, for some $\lambda > 0$ implies $\lim\limits_{n\longrightarrow\infty} w_\lambda(x_n, x) = 0$, for all $\lambda > 0$.

The motivation of the following definition can easily be predicted from the last step of proof of Cauchy sequence in Theorems 13 and 15 (given below).

Definition 5. *[12] Let X_w be a modular metric space and $\{x_n\}_{n\in\mathbb{N}}$ be sequence in X_w. We will say that w satisfies the Δ_M-condition if this the case, i.e., $\lim\limits_{m,n\longrightarrow\infty} w_{m-n}(x_n, x_m) = 0$ for $(m, n \in \mathbb{N},\ m > n)$ implies $\lim\limits_{m,n\longrightarrow\infty} w_\lambda(x_n, x_m) = 0$ for some $\lambda > 0$.*

Let $CB(D) := \{C : C$ is nonempty w-closed and w-bounded subsets of $D\}$, $K(D) := \{C : C$ is nonempty w-compact subsets of $D\}$ and the Hausdorff metric modular defined on $CB(D)$ by

$$H_w(A, B) := \max\{\sup_{x\in A} w_1(x, B), \sup_{y\in B} w_1(A, y)\}$$

where $w_1(x, B) = \inf\limits_{y\in B} w_1(x, y)$.

Lemma 6. *[5] Let (X, w) be a modular metric space. Assume that w satisfies Δ_2-condition. Let D be a nonempty subset of X_w. Let A_n be a sequence of sets in $CB(D)$, and suppose $\lim\limits_{n\longrightarrow\infty} H_w(A_n, A_0) = 0$ where $A_0 \in CB(D)$. Then if $x_n \in A_n$ and $\lim\limits_{n\longrightarrow\infty} x_n = x_0$, it follows that $x_0 \in A_0$.*

3. Fixed Point Results for Multivalued F-Contractions

Definition 7. *[2] Let $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfying the following condition:*

$(F1)$ *F is strictly increasing on $\mathbb{R}^+$,*
$(F2)$ *for every sequence $\{s_n\}$ in $\mathbb{R}^+$, we have $\lim\limits_{n\longrightarrow\infty} s_n = 0$ if and only if $\lim\limits_{n\longrightarrow\infty} F(s_n) = -\infty$,*
$(F3)$ *there exists a number $k \in (0, 1)$ such that $\lim\limits_{s\longrightarrow0^+} s^k F(s) = 0$.*

We denote by $\mathcal{F}$ the family of all function that satisfy the conditions (F1)–(F3).

Example 8. *The following functions $F : \mathbb{R}^+ \to \mathbb{R}$ belong to $\mathcal{F}$:*

(i) $F(s) = \ln s$, *with* $s > 0$,
(ii) $F(s) = -\frac{1}{\sqrt{s}}, s > 0$

Definition 9. *Let (X, w) be a modular metric space. Let D be non empty bounded subset of X. A multivalued mapping $T : D \longrightarrow CB(D)$ is called F-contraction on X if $F \in \mathcal{F}$, and $\tau \in \mathbb{R}^+$, for all $x, y \in D$ with $y \in Tx$ there exists $z \in Ty$ such that $w_1(y, z) > 0$, the following inequality holds:*

$$\tau + F(w_1(y, z)) \leq F(\mathcal{M}(x, y)) \tag{3.1}$$

where $\mathcal{M}(x, y) = \max\Big\{w_1(x, y), w_1(x, Tx), w_1(y, Ty), w_1(y, Tx)\Big\}$.

Definition 10. *Let (X, w) be a modular metric space. Let D be a nonempty subset of X_w. A multivalued mapping $T : D \longrightarrow CB(D)$ is said to be F-contraction of Hardy-Rogers-type if $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that,*

$$2\tau + F(H_w(Tx, Ty)) \leq F(\alpha w_1(x, y) + \beta w_1(x, Tx) + \gamma w_1(y, Ty) + L w_1(y, Tx)) \tag{3.2}$$

for all $x, y \in D$ with $H_w(Tx, Ty) > 0$, where $\alpha, \beta, \gamma, L \geq 0$, $\alpha + \beta + \gamma = 1$ and $\gamma \neq 1$.

Example 11. *Let $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be given by $F(s) = \ln s$. For each multivalued mapping $T : D \longrightarrow CB(D)$ satisfying Equation (3.1) we have*

$$w_1(y, z) \leq e^{-\tau} \mathcal{M}(x, y), \ \textit{for all } x, y \in D, \ y \neq z$$

It is clear that for $z, y \in D$ such that $y = z$ the previous inequality also holds.

Example 12. *Let $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be given by $F(s) = \ln s$. It is clear that F satisfies $(F1) - (F3)$ for any $k \in (0, 1)$. Each mapping $T : D \longrightarrow CB(D)$ satisfying Equation (3.2) is an F-contraction such that*

$$H_w(Tx, Ty) \leq e^{-\tau} w_1(x, y), \ \textit{for all } x, y \in D, \ Tx \neq Ty$$

It is clear that for $x, y \in D$ such that $Tx = Ty$ the previous inequality also holds and hence T is a contraction.

Theorem 13. *Let (X, w) be a modular metric space. Assume that w is a regular modular satisfying Δ_M-condition and Δ_2-condition. Let D be a nonempty w-bounded and w-complete subset of X_w. Let $T : D \longrightarrow CB(D)$ be a continuous F-contraction. Then T has a fixed point.*

Proof. Let $x_0 \in D$ be an arbitrary point of D and choose $x_1 \in Tx_0$. If $x_1 = x_0$, then x_1 is a fixed point of T and the proof is completed. Suppose that $x_1 \neq x_0$. Since T is an F-contraction, then there exists $x_2 \in Tx_1$ such that

$$\tau + F(w_1(x_1, x_2) \leq F(\mathcal{M}(x_0, x_1)) \ \text{ and } x_1 \neq x_2$$

Therefore, we have that there exists $x_3 \in Tx_2$ such that

$$\tau + F(w_1(x_2, x_3) \leq F(\mathcal{M}(x_1, x_2)) \ \text{ and } x_2 \neq x_3$$

Repeating this process, we find that there exists a sequence $\{x_n\}$ with initial point x_0 such that $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$ and

$$\tau + F(w_1(x_n, x_{n+1})) \leq F(\mathcal{M}(x_{n-1}, x_n)) \ \text{ for all } n \in \mathbb{N}$$

This implies

$$F(w_1(x_n, x_{n+1})) < F(\mathcal{M}(x_{n-1}, x_n)) \ \text{ for all } n \in \mathbb{N}$$

Consequently,

$$\begin{aligned} w_1(x_n, x_{n+1}) &< \mathcal{M}(x_{n-1}, x_n) \ \text{(Since F is strictly increasing.)} \\ &= \max\{w_1(x_{n-1}, x_n), w_1(x_{n-1}, Tx_{n-1}), w_1(x_n, Tx_n), w_1(x_n, Tx_{n-1})\} \\ &= \max\{w_1(x_{n-1}, x_n), w_1(x_n, Tx_n)\} \\ &\leq \max\{w_1(x_{n-1}, x_n), w_1(x_n, x_{n+1})\} \end{aligned}$$

Obviously, if $\max\{w_1(x_{n-1}, x_n), w_1(x_n, x_{n+1})\} = w_1(x_n, x_{n+1})$, we have a contradiction and so $\max\{w_1(x_{n-1}, x_n), w_1(x_n, x_{n+1})\} = w_1(x_{n-1}, x_n)$.

Consequently, By $(F1)$ we have

$$\tau + F(w_1(x_n, x_{n+1}) \leq F(w_1(x_{n-1}, x_n)) \ \text{ for all } n \in \mathbb{N} \tag{3.3}$$

By Equation (3.3), we have

$$F(w_1(x_n, x_{n+1})) \leq F(w_1(x_{n-1}, x_n)) - \tau \leq \cdots \leq F(w_1(x_0, x_1)) - n\tau, \text{ for all } n \in \mathbb{N} \tag{3.4}$$

and hence $\lim_{n\to\infty} F(w_1(x_n, x_{n+1})) = -\infty$. By $(F2)$ we have that $w_1(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Now, let $k \in (0,1)$ such that $\lim_{n\to\infty}((w_1(x_n, x_{n+1}))^k F(w_1(x_n, x_{n+1}))$. By Equation (3.4), the following holds for all $n \in \mathbb{N}$:

$$\left(w_1(x_{n+1}, x_n)\right)^k \Big(F\big(w_1(x_{n+1}, x_n)\big) - F\big(w_1(x_0, x_1)\big)\Big) \leq -\big(w_1(x_{n+1}, x_n)\big)^k n\tau \leq 0 \tag{3.5}$$

Taking $n \to \infty$ in Equation (3.5), we deduce

$$\lim_{n\to\infty} \left(n\big(w_1(x_{n+1}, x_n)\big)^k\right) = 0$$

Then there exists $n_1 \in \mathbb{N}$ such that $n\big(w_1(x_{n+1}, x_n)\big)^k \leq 1$ for all $n \geq n_1$, that is,

$$w_1(x_n, x_{n+1}) \leq \frac{1}{n^{1/k}} \text{ for all } n \geq n_1$$

Now, For all $m, n \geq n_1$ with $m > n$, we have

$$\begin{aligned} w_{m-n}(x_n, x_m) &\leq w_1(x_n, x_{n+1}) + w_1(x_{n+1}, x_{n+2}) + \cdots + w_1(x_{m-1}, x_m) \\ &\leq \tfrac{1}{n^{1/k}} + \tfrac{1}{(n+1)^{1/k}} + \cdots + \tfrac{1}{m^{1/k}} \\ &< \Sigma_{i=n}^{\infty} \tfrac{1}{i^{1/k}} \end{aligned}$$

Since the series $\Sigma_{i=n}^{\infty} \frac{1}{i^{1/k}}$ is convergent, this implies

$$\lim_{m,n\to\infty} \left(w_{m-n}(x_n, x_m)\right) = 0$$

Since w satisfies Δ_M-condition. Hence, we have

$$\lim_{m,n\longrightarrow\infty} w_1(x_n, x_m) = 0$$

This shows that $\{x_n\}$ is a w-Cauchy sequence. D is w-complete, there exists $v \in D$ such that $x_n \longrightarrow v$ as $n \longrightarrow \infty$. Now, we prove that v is a fixed point of T.

Let Tx_n be a sequence in CB(D). Since T is continuous then we have $Tx_n \longrightarrow Tv$ so $\lim_{n\longrightarrow\infty} H_w(Tx_n, Tv) = 0$, where $Tv \in CB(D)$. Then if $x_{n+1} \in Tx_n$ and $\lim_{n\longrightarrow\infty} x_{n+1} = v$, it follows from Lemma 6 that $v \in Tv$. Hence v is a fixed point of T. □

Example 14. *Let $X_w = D = \{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\}$ and $w_1(x,y) = \frac{1}{\lambda}|x-y|,\ \ x, y \in D$. Then (X, w) is a w-complete modular metric space. Define the mapping $T : D \to CB(D)$ by the:*

$$T(x) = \begin{cases} \{x_1\} & , \quad x = x_1 \\ \{x_1, x_2, \cdots, x_{n-1}\} & , \quad x = x_n \end{cases}$$

Then, as shown in Example 3 of [13], T is a multivalued F-contraction with respect to $F(s) = \ln s + s$ and $\tau = 1$. Therefore, Theorem 13 are satisfied and so T has a fixed point in X_w.

On the other hand, since

$$\lim_{n\to\infty} \frac{H_w(Tx_n, Tx_1)}{\mathcal{M}(x_n, x_1)} = \lim_{n\to\infty} \frac{x_{n-1} - 1}{x_n - 1} = 1$$

then T is not multivalued contraction.

Next, we give a fixed point result for multivalued F-contractions of Hardy-Rogers-type in modular metric space.

Theorem 15. *Let (X, w) be a modular metric space. Assume that w is a regular modular satisfying Δ_M-condition and Δ_2-condition. Let D be a nonempty w-bounded and w-complete subset of X_w and $T : X \longrightarrow K(D)$ be an F-contractions of Hardy-Rogers-type. Then T has a fixed point.*

Proof. Let x_0 be an arbitrary point in D. As Tx is nonempty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and so the proof is complete. Assume $x_1 \notin Tx_1$. Then, since Tx_1 is closed, $w(x, Tx_1) > 0$. On the other hand, from $w(x_1, Tx_1) \leq H_w(Tx_0, Tx_1)$ and (F1)

$$F(w(x_1, Tx_1)) \leq F(H_w(Tx_0, Tx_1))$$

From Equation (3.2), we can write that

$$\begin{aligned} F(w(x_1, Tx_1) \leq F(H_w(Tx_0, Tx_1)) \quad &\leq \quad F(\alpha w_1(x_0, x_1) + \beta w_1(x_0, Tx_0) + \gamma w_1(x_1, Tx_1) \\ &\quad\quad + L w_1(x_1, Tx_0)) - 2\tau \end{aligned}$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that

$$w_1(x_1, x_2) = w_1(x_1, Tx_1)$$

Then,

$$\begin{aligned} F(w_1(x_1, x_2)) = F(w(x_1, Tx_1) \quad &\leq \quad F(H_w(Tx_0, Tx_1)) \\ &\leq \quad F(\alpha w_1(x_0, x_1) + \beta w_1(x_0, Tx_0) + \gamma w_1(x_1, Tx_1) \\ &\quad\quad + L w_1(x_1, Tx_0)) - 2\tau \end{aligned}$$

Thus,

$$\begin{aligned} F(w_1(x_1, x_2)) \quad &\leq \quad F(H_w(Tx_0, Tx_1)) \\ &\leq \quad F(\alpha w_1(x_0, x_1) + \beta w_1(x_0, Tx_0) + \gamma w_1(x_1, Tx_1) \\ &\quad\quad + L w_1(x_1, Tx_0)) - 2\tau \\ &\leq \quad F(\alpha w_1(x_0, x_1) + \beta w_1(x_0, x_1) + \gamma w_1(x_1, x_2)) - 2\tau \\ &\leq \quad F((\alpha + \beta) w_1(x_0, x_1) + \gamma w_1(x_1, x_2)) \end{aligned}$$

Thus,

$$F(w_1(x_1, x_2)) \leq F((\alpha + \beta) w_1(x_0, x_1) + \gamma w_1(x_1, x_2))$$

Since F is strictly increasing, we deduce that

$$w_1(x_1, x_2) \leq (\alpha + \beta) w_1(x_0, x_1) + \gamma w_1(x_1, x_2)$$

and hence

$$(1 - \gamma) w_1(x_1, x_2) < (\alpha + \beta) w_1(x_0, x_1)$$

From $\alpha + \beta + \gamma = 1$ and $\gamma \neq 1$, we deduce that $1 - \gamma > 0$ and so

$$w_1(x_1, x_2) < \frac{\alpha + \beta}{1 - \gamma} w_1(x_0, x_1) = w_1(x_0, x_1)$$

Consequently,

$$\tau + F(w_1(x_1, x_2)) \leq F(w_1(x_0, x_1))$$

Continuing in this manner, we can define a sequence $\{x_n\} \subset D$ such that $x_n \notin Tx_n, x_{n+1} \in Tx_n$ and

$$\tau + F(w_1(x_{n+1}, x_{n+2})) \leq F(w_1(x_n, x_{n+1})) \text{ for all } n \in \mathbb{N} \cup \{0\}$$

Proceeding as in the proof of Theorem 13, we obtain that $\{x_n\}$ is a w-Cauchy sequence. Since D is a w-complete modular metric space, there exists $v \in D$ such that $x_n \longrightarrow v$ as $n \longrightarrow \infty$. Now, we prove that v is a fixed point of T. If there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that $x_{n_k} \in Tv$ for all $k \in \mathbb{N}$, since Tv is w-closed and $x_{n_k} \longrightarrow v$, we have $v \in Tv$ and the proof is completed. So we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \notin Tv$ for all $n \geq n_0$. This implies that $Tx_{n-1} \neq Tv$ for all $n \geq n_0$. Now, using Equation (3.2) with $x = x_n$ and $y = v$, we obtain

$$\begin{aligned} 2\tau + F(H_w(Tx_n, Tv)) &\leq F(\alpha w_1(x_n, v) + \beta w_1(x_n, Tx_n) + \gamma w_1(v, Tv) \\ &\quad + L w_1(v, Tx_n)) \end{aligned}$$

which implies

$$\begin{aligned} 2\tau + F(w_1(x_{n+1}, Tv)) &\leq 2\tau + F(H_w(Tx_n, Tv)) \\ &\leq F(\alpha w_1(x_n, v) + \beta w_1(x_n, Tx_n) + \gamma w_1(v, Tv) \\ &\quad + L w_1(v, Tx_n)) \\ &\leq F(\alpha w_1(x_n, v) + \beta w_1(x_n, x_{n+1}) + \gamma w_1(v, Tv) \\ &\quad + L w_1(v, x_{n+1})) \end{aligned}$$

Since F is strictly increasing, we have

$$\begin{aligned} w_1(x_{n+1}, Tv) &< \alpha w_1(x_n, v) + \beta w_1(x_n, x_{n+1}) + \gamma w_1(v, Tv) \\ &\quad + L w_1(v, x_{n+1}). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the previous inequality, as $\gamma < 1$ we have $w_1(v, Tv) \leq \gamma w_1(v, Tv) < w_1(v, Tv)$, which implies $w_1(v, Tv) = 0$. Since Tv is w-closed, we obtain that $v \in Tv$, that is, v is a fixed point of T. □

Remark 16. *If we consider $T : X \longrightarrow CB(T)$ in Theorem 15 i.e., we are relaxing compactness of co-domain of mapping T but then we have to assume T be continuous. In this case, we can write proof as Theorem 15 upto Cauchy. Further, by the completeness of D, we have $v \in D$ such that $x_n \to v$. Since T is continuous, we have $\lim_{n \longrightarrow \infty} H_w(Tx_n, T_v) = 0$ and as $x_{n+1} \in Tx_n$ with $x_{n+1} \to v$ then by Lemma 6 we obtain $v \in Tv$. Hence v is fixed point of T.*

4. Application to Integral Equations

Integral equations arise in many scientific and engineering problems. A large class of initial and boundary value problem can be converted to Volterra or Fredholm integral equation (see for instant [14]).

In this section we consider the following integral equation:

$$u(t) = \beta A\big(u(t)\big) + \gamma B\big(u(t)\big) + g(t),\ t \in [0, T],\ T > 0 \tag{4.1}$$

where

$$A\big(u(t)\big) = \int_0^t K_1\big(t, s, u(s)\big) ds,\ \ B\big(u(t)\big) = \int_0^t K_2\big(t, s, u(s)\big) ds \quad \text{and} \quad \beta, \gamma \geq 0$$

Let $C(I, \mathbb{R})$ be the space of all continuous functions on I, where $I = [0, T]$ with the norm $\|u\| = \sup_{t \in I} |u(t)|$ and the metric $w_\lambda(u, v) := \frac{1}{\lambda}\|u - v\| = \frac{1}{\lambda} d(u, v)$ for all $u, v \in C(I, \mathbb{R})$. For $r > 0$

and $u \in C(I,\mathbb{R})$ we denote by $B_\lambda(u,r) = \{v \in C(I,\mathbb{R}) : w_\lambda(u,v) \le r\}$ the closed ball concerned at u and of radius r.

Theorem 17. *Let $r > 0$ be a fixed real number and the following conditions are satisfied:*

(i) $K : I \times I \times \mathbb{R} \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *are continuous;*
(ii) *there exists* $u_0 \in C(I,\mathbb{R})$ *such that* $\beta A(u_0(t)) + \gamma B(u_0(t)) + g(t) \in B(u_0, r)$;
(iii) *if* $v \in B_\lambda(u,r)$, $\lambda > 0$, *then*

$$|K_i(t,s,u(s)) - K_i(t,s,v(s))| \le L_i(t,s,u(s),v(s)) \frac{|u(s)-v(s)|}{\left(1+\tau\sqrt{\frac{|u(s)-v(s)|}{\lambda}}\right)^2},\ i = 1,2$$

for all $t,s \in I, u,v \in \mathbb{R}$ *and for some continuous functions* $L_1, L_2 : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$.

such that $L_i(t,s,u(s),v(s))(\beta+\gamma)T \le 1,\ i = 1,2$ *for all* $s,t \in I$, *then the integral Equation (4.1) admit a solution.*

Proof. Note that $(C(I,\mathbb{R}), w_\lambda)$ is a complete modular metric space. Define $T : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ by

$$T(u(t)) = \beta A(u(t)) + \gamma B(u(t)) + g(t), t \in I$$

Since $v \in B_\lambda(u,r)$, then by the definition of T and (iii) we have

$$\begin{aligned}
w_\lambda(Tu,Tv) &= \frac{1}{\lambda}\sup_{t\in I} |\beta A(u(t)) + \gamma B(u(t)) - \beta A(v(t)) - \gamma B(v(t))| \\
&= \frac{1}{\lambda}\sup_{t\in I} \bigg|\beta \int_0^t [K_1(t,s,u(s)) - K_1(t,s,v(s))]ds \\
&\quad + \gamma \int_0^t [K_2(t,s,u(s)) - K_2(t,s,v(s))]ds\bigg| \\
&\le \frac{1}{\lambda}\sup_{t\in I} \bigg\{\beta \int_0^t |K_1(t,s,u(s)) - K_1(t,s,v(s))|ds \\
&\quad + \gamma \int_0^t |K_2(t,s,u(s)) - K_2(t,s,v(s))|ds\bigg\} \\
&\le \frac{1}{\lambda}\sup_{t\in I} \bigg\{\beta \int_0^t |L_1(t,s,u(s),v(s)) \frac{|u(s)-v(s)|}{\left(1+\tau\sqrt{\frac{|u(s)-v(s)|}{\lambda}}\right)^2} ds \\
&\quad + \gamma \int_0^t |L_2(t,s,u(s),v(s)) \frac{|u(s)-v(s)|}{\left(1+\tau\sqrt{\frac{|u(s)-v(s)|}{\lambda}}\right)^2} ds\bigg\} \\
&\le \frac{|u(s)-v(s)|}{\lambda} \frac{1}{\left(1+\tau\sqrt{\frac{|u(s)-v(s)|}{\lambda}}\right)^2} \sup_{t\in I}\left\{\beta\int_0^t \frac{1}{(\beta+\gamma)T}ds + \gamma\int_0^t \frac{1}{(\beta+\gamma)T}ds\right\} \\
&\le \frac{w_\lambda(u,v)}{(1+\tau\sqrt{w_\lambda(u,v)})^2}\sup_{t\in I}\left\{\frac{t}{T}\right\}
\end{aligned}$$

This implies

$$w_\lambda(Tu,Tv) \le \frac{w_\lambda(u,v)}{(1+\tau\sqrt{w_\lambda(u,v)})^2}$$

□

Now, we observe that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(\alpha) = -\frac{1}{\sqrt{\alpha}}, \alpha > 0$ is in $\mathcal{F}$ and so we deduce that the mapping T satisfies all condition of Theorem 13 with $\mathcal{M}(u, v) = w_\lambda(u, v)$ for $\lambda = 1$. Hence there exists a solution of the integral Equation (4.1).

Remark 18. *Our above Theorem 4.1 is an abstract application of F- contraction mapping which can not be covered by Banach contraction principle.*

Acknowledgments: The authors thank Editor-in-Chief and Referee(s) for their valuable comments and suggestions, which were very useful to improve the paper significantly. The first author thanks for the support of Petchra Pra Jom Klao Doctoral Scholarship Academic. This work was completed while the second author (Dhananjay Gopal) was visiting Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok, Thailand, during 15 October–8 November 2015. He thanks Professor Poom Kumam and the University for their hospitality and support.

Author Contributions: All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Banach, S. Sur les operations dans les ensembles abstraits et leur applications aux équations intégrales. *Fund. Math.* **1922**, *3*, 133–181.
2. Wardowski, D. Fixed points of new type of contractive mappings in complete metric space. *Fixed Point Theory Appl.* **2012**, *94*, doi:10.1186/1687-1812-2012-94.
3. Sgroi, M.; Vetro, C. Multi-valued F-contractions and the solution of certain functional and integral equations. *Filomat* **2013**, *27*, 1259–1268.
4. Chistyakov, V.V. Modular metric spaces I basic concepts. *Nonlinear Anal.* **2010**, *72*, 1–14.
5. Abdou, A.A.N.; Khamsi, M.A. Fixed points of multivalued contraction mappings in modular metric spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 249.
6. Abdo, A.A.N.; Khamsi, M.A. Fixed point results of pointwise contractions in modular metric spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 163.
7. Mongkolkeha, C.; Sintunavarat, W.; Kumam, P. Fixed point theorems for contraction mappings in modular metric spaces. *Fixed Point Theory Appl.* **2011**, *2011*, 93.
8. Alfuraidan, M.R. Fixed points of multivalued contraction mappings in modular metric spaces. *Fixed Point Theory Appl.* **2015**, *2015*, 46.
9. Chaipunya, P.; Mongkolkeha, C.; Sintunavarat, W.; Kumam, P. Fixed-point theorems for multivalued mappings in modular metric spaces. *Abstr. Appl. Anal.* **2012**, *2012*, 503504.
10. Kadelburg, Z.; Radenovic, S. Remarks on some recent M. Borcut's results in partially ordered metric spaces. *Int. J. Nonlinear Anal. Appl.* **2015**, *6*, 96–104.
11. Nadler, S.B., Jr. Multi-valued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–488.
12. Padcharoen, A.; Kumam, P.; Gopal, D.; Chaipunya, P. Fixed points and periodic point results for α-type F-contractions in modular metric spaces. *Fixed Point Theory Appl.* **2016**, *39*, doi:10.1186/s13663-016-0525-4.
13. Acar, O.; Durmaz, G.; Minak, G. Generalized multivalued F-contractions on complete metric spaces. *Bull. Iranian Math. Soc.* **2014**, *40*, 1469–1478.
14. Moradi, S.; Mohammadi Anjedani, M.; Analoei, E. On existence and uniqueness of solutions of a nonlinear Volterra-Fredholm integral equation. *Int. J. Nonlinear Anal. Appl.* **2015**, *6*, 62–68.