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Some Determinantal Expressions and Recurrence Relations of the Bernoulli Polynomials

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Abstract: In the paper, the authors recall some known determinantal expressions in terms of the Hessenberg determinants for the Bernoulli numbers and polynomials, find alternative determinantal expressions in terms of the Hessenberg determinants for the Bernoulli numbers and polynomials, and present several new recurrence relations for the Bernoulli numbers and polynomials.

Keywords: determinantal expression; recurrence relation; Bernoulli number; Bernoulli polynomial; Hessenberg determinant

MSC: 11B68; 11B83; 11C20; 15A15; 26A06; 26C05; 41A58

1. Introduction and Main Results

It is general knowledge that the Bernoulli numbers and polynomials B_k and $B_k(t)$ can be generated by

$$\frac{z}{e^z-1}=\sum_{k=0}^{\infty}B_k\frac{z^k}{k!}=1-\frac{z}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{z^{2k}}{(2k)!} \tag{1}$$

and

$$\frac{ze^{tz}}{e^z-1}=\sum_{k=0}^{\infty}B_k(t)\frac{z^k}{k!}$$

for $|z|<2\pi$ respectively. It is clear that $B_k(0)=B_k$. Because the function $\frac{x}{e^x-1}-1+\frac{x}{2}$ is even in $x\in\mathbb{R}$, all the Bernoulli numbers B_{2k+1} for $k\in\mathbb{N}$ equal 0. In addition to $B_0=1$ and $B_1=-\frac{1}{2}$, the first few Bernoulli numbers B_{2k} are

$$B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\quad B_6=\frac{1}{42},\quad B_8=-\frac{1}{30},\quad B_{10}=\frac{5}{66},\quad B_{12}=-\frac{691}{2730}.$$

The first five Bernoulli polynomials $B_k(t)$ for $0\le k\le 4$ are

$$1,\quad t-\frac{1}{2},\quad t^2-t+\frac{1}{6},\quad t^3-\frac{3}{2}t^2+\frac{1}{2}t,\quad t^4-2t^3+t^2-\frac{1}{30}.$$

It is well known that a matrix $H=(h_{ij})_{n\times n}$ is called a lower (respectively upper) Hessenberg matrix if $h_{ij}=0$ for all pairs (i,j) such that $i+1<j$ (respectively $j+1<i$). Correspondingly, we can define a Hessenberg determinant.

In [1] (p. 40), it was mentioned that

$$B_k = \frac{(-1)^k}{(k+1)!}\begin{vmatrix} \binom{2}{0} & \binom{2}{1} & 0 & 0 & \cdots & 0 & 0 \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 & \cdots & 0 & 0 \\ \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \cdots & 0 & 0 \\ \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \binom{k-2}{3} & \cdots & 0 & 0 \\ \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \binom{k-1}{3} & \cdots & \binom{k-1}{k-2} & 0 \\ \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \binom{k}{3} & \cdots & \binom{k}{k-2} & \binom{k}{k-1} \\ \binom{k+1}{0} & \binom{k+1}{1} & \binom{k+1}{2} & \binom{k+1}{3} & \cdots & \binom{k+1}{k-2} & \binom{k+1}{k-1} \end{vmatrix} \tag{2}$$

and

$$B_k(x) = \frac{(-1)^k}{(k-1)!}\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ x & \frac{1}{2} & \binom{1}{0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ x^2 & \frac{1}{3} & \binom{2}{0} & \binom{2}{1} & 0 & 0 & \cdots & 0 & 0 \\ x^3 & \frac{1}{4} & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 & \cdots & 0 & 0 \\ x^4 & \frac{1}{5} & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x^{k-2} & \frac{1}{k-1} & \binom{k-2}{0} & \binom{k-2}{1} & \binom{k-2}{2} & \binom{k-2}{3} & \cdots & \binom{k-2}{k-3} & 0 \\ x^{k-1} & \frac{1}{k} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \binom{k-1}{3} & \cdots & \binom{k-1}{k-3} & \binom{k-1}{k-2} \\ x^k & \frac{1}{k+1} & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \binom{k}{3} & \cdots & \binom{k}{k-3} & \binom{k}{k-2} \end{vmatrix} \tag{3}$$

for $k \in \mathbb{N}$. The determinant in (2) is a sub-determinant of the determinant in (3). It was pointed out in [1] (p. 40) that these two determinantal expressions were recorded in [2] and can be traced back to the book [3].

The Bernoulli polynomials $B_k(t)$ were represented in [1] in terms of the Hessenberg determinants as

$$B_k(t) = (-1)^k k!\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{t}{1!} & \frac{1}{2!} & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{t^2}{2!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{t^3}{3!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{t^{k-2}}{(k-2)!} & \frac{1}{(k-1)!} & \frac{1}{(k-2)!} & \frac{1}{(k-3)!} & \frac{1}{(k-4)!} & \frac{1}{(k-5)!} & \cdots & 1 & 0 \\ \frac{t^{k-1}}{(k-1)!} & \frac{1}{k!} & \frac{1}{(k-1)!} & \frac{1}{(k-2)!} & \frac{1}{(k-3)!} & \frac{1}{(k-4)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{t^k}{k!} & \frac{1}{(k+1)!} & \frac{1}{k!} & \frac{1}{(k-1)!} & \frac{1}{(k-2)!} & \frac{1}{(k-3)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} \tag{4}$$

$$
= \frac{(-1)^k}{\prod_{m=1}^{k-1} m!}
\begin{vmatrix}
1 & \frac{0!}{1!} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
t & \frac{1!}{2!} & \frac{1!}{1!} & 0 & 0 & 0 & \cdots & 0 & 0 \\
t^2 & \frac{2!}{3!} & \frac{2!}{2!} & \frac{2!}{1!} & 0 & 0 & \cdots & 0 & 0 \\
t^3 & \frac{3!}{4!} & \frac{3!}{3!} & \frac{3!}{2!} & \frac{3!}{1!} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
t^{k-2} & \frac{(k-2)!}{(k-1)!} & \frac{(k-2)!}{(k-2)!} & \frac{(k-2)!}{(k-3)!} & \frac{(k-2)!}{(k-4)!} & \frac{(k-2)!}{(k-5)!} & \cdots & \frac{(k-2)!}{1!} & 0 \\
t^{k-1} & \frac{(k-1)!}{k!} & \frac{(k-1)!}{(k-1)!} & \frac{(k-1)!}{(k-2)!} & \frac{(k-1)!}{(k-3)!} & \frac{(k-1)!}{(k-4)!} & \cdots & \frac{(k-1)!}{2!} & \frac{(k-1)!}{1!} \\
t^k & \frac{k!}{(k+1)!} & \frac{k!}{k!} & \frac{k!}{(k-1)!} & \frac{k!}{(k-2)!} & \frac{k!}{(k-3)!} & \cdots & \frac{k!}{3!} & \frac{k!}{2!}
\end{vmatrix}. \tag{5}
$$

In [4] (Section 21.5) and [5] (p. 1), the determinantal expression

$$
B_k = k! \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\frac{1}{2!} & 1 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{1}{(k-1)!} & \frac{1}{(k-2)!} & \frac{1}{(k-3)!} & \cdots & 1 & 0 & 0 \\
\frac{1}{k!} & \frac{1}{(k-1)!} & \frac{1}{(k-2)!} & \cdots & \frac{1}{2!} & 1 & 0 \\
\frac{1}{(k+1)!} & \frac{1}{k!} & \frac{1}{(k-1)!} & \cdots & \frac{1}{3!} & \frac{1}{2!} & 0
\end{vmatrix}
$$

for the Bernoulli numbers B_k for $k \ge 0$ was listed. This expression can also be deduced from taking the limit $t \to 0$ in (4).

Let $\{a_m\}_{0 \le m \le \infty}$ be a sequence of complex numbers and let $\{D_k(a_m)\}_{0 \le k \le \infty}$ be a sequence of the Hessenberg determinants such that $D_0(a_m) = 1$ and

$$
D_k(a_m) = \begin{vmatrix}
a_1 & a_0 & 0 & \cdots & 0 \\
a_2 & a_1 & a_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_0 \\
a_k & a_{k-1} & a_{k-2} & \cdots & a_1
\end{vmatrix}, \quad k \in \mathbb{N}.
$$

In [6], the determinantal expressions

$$
B_{2k} = (-1)^k \frac{(2k)!}{2} \left\{ \sum_{\ell=0}^{k} \frac{(-1)^\ell}{(2\ell)!} D_{k-\ell}\left(\frac{1}{(2m+1)!}\right) + D_k\left(\frac{1}{(2m+1)!}\right) \right\} \tag{6}
$$

and

$$
B_{2k} = (-1)^{k+1} \frac{(2k)!}{2(2^{2k-1}-1)} D_k\left(\frac{1}{(2m+1)!}\right) \tag{7}
$$

for $k \in \mathbb{N}$ were established.

In [7] (Theorem 1.1), it was shown that, if

$$
A(z) = \sum_{m=0}^{\infty} a_n z^m \quad \text{and} \quad B(z) = \sum_{m=0}^{\infty} b_n z^m
$$

are the ordinary generating functions of $\{a_m\}_{0 \le m \le \infty}$ and $\{b_m\}_{0 \le m \le \infty}$ such that $A(x)B(x) = 1$, then $a_0 \ne 0$ and

$$b_k = (-1)^k \frac{D_k(a_m)}{a_0^{k+1}}.$$

As applications of [7] (Theorem 1.1), among other things, some properties of $D_k(a_m)$ were discovered and applied to give an elegant proof of (6) and (7). In particular, the Hessenberg determinantal expressions

$$B_k = (-1)^k k! D_k\left(\frac{1}{(m+1)!}\right),$$

which recovers [8] (Equation (4)), and

$$B_k = k! D_k\left(\frac{(-1)^m}{(m+1)!}\right)$$

were derived.

In [9] (Theorem 1.2), the Bernoulli polynomials $B_k(t)$ for $k \in \mathbb{N}$ were expressed as a Hessenberg determinant by

$$B_k(t) = (-1)^k \left|\frac{1}{\ell+1}\binom{\ell+1}{m}\left[(1-t)^{\ell-m+1} - (-t)^{\ell-m+1}\right]\right|_{1\le\ell\le k, 0\le m\le k-1} \tag{8}$$

under the conventions that $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \ge 0$. Consequently, the Bernoulli numbers B_k for $k \in \mathbb{N}$ were be represented as

$$B_k = (-1)^k \left|\frac{1}{\ell+1}\binom{\ell+1}{m}\right|_{1\le\ell\le k, 0\le m\le k-1}. \tag{9}$$

The first aim of this paper is to find alternative determinantal expressions in terms of the Hessenberg determinants for the Bernoulli polynomials $B_k(t)$ and the Bernoulli numbers B_k. The second aim is to derive several recurrence relations for the Bernoulli polynomials $B_k(t)$ and the Bernoulli numbers B_k.

Our main results can be formulated as a theorem and a corollary below.

Theorem 1. *For $k \ge 0$, the Bernoulli polynomials $B_k(t)$ can be expressed in terms of a Hessenberg determinant as*

$$B_k(t) = (-1)^k \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ t & \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 & 0 \\ t^2 & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 & 0 & 0 \\ t^3 & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \frac{1}{2}\binom{3}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ t^{k-3} & \frac{1}{k-2}\binom{k-3}{0} & \frac{1}{k-3}\binom{k-3}{1} & \frac{1}{k-4}\binom{k-3}{2} & \cdots & 1 & 0 & 0 \\ t^{k-2} & \frac{1}{k-1}\binom{k-2}{0} & \frac{1}{k-2}\binom{k-2}{1} & \frac{1}{k-3}\binom{k-2}{2} & \cdots & \frac{1}{2}\binom{k-2}{k-3} & 1 & 0 \\ t^{k-1} & \frac{1}{k}\binom{k-1}{0} & \frac{1}{k-1}\binom{k-1}{1} & \frac{1}{k-2}\binom{k-1}{2} & \cdots & \frac{1}{3}\binom{k-1}{k-3} & \frac{1}{2}\binom{k-1}{k-2} & 1 \\ t^k & \frac{1}{k+1}\binom{k}{0} & \frac{1}{k}\binom{k}{1} & \frac{1}{k-1}\binom{k}{2} & \cdots & \frac{1}{4}\binom{k}{k-3} & \frac{1}{3}\binom{k}{k-2} & \frac{1}{2}\binom{k}{k-1} \end{vmatrix} \tag{10}$$

and, consequently, the Bernoulli numbers B_k can be expressed as

$$B_k = (-1)^k \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \frac{1}{2}\binom{3}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \frac{1}{k-2}\binom{k-3}{0} & \frac{1}{k-3}\binom{k-3}{1} & \frac{1}{k-4}\binom{k-3}{2} & \cdots & 1 & 0 & 0 \\ 0 & \frac{1}{k-1}\binom{k-2}{0} & \frac{1}{k-2}\binom{k-2}{1} & \frac{1}{k-3}\binom{k-2}{2} & \cdots & \frac{1}{2}\binom{k-2}{k-3} & 1 & 0 \\ 0 & \frac{1}{k}\binom{k-1}{0} & \frac{1}{k-1}\binom{k-1}{1} & \frac{1}{k-2}\binom{k-1}{2} & \cdots & \frac{1}{3}\binom{k-1}{k-3} & \frac{1}{2}\binom{k-1}{k-2} & 1 \\ 0 & \frac{1}{k+1}\binom{k}{0} & \frac{1}{k}\binom{k}{1} & \frac{1}{k-1}\binom{k}{2} & \cdots & \frac{1}{4}\binom{k}{k-3} & \frac{1}{3}\binom{k}{k-2} & \frac{1}{2}\binom{k}{k-1} \end{vmatrix}. \tag{11}$$

Corollary 1. *For $k \ge 1$, the Bernoulli polynomials $B_k(t)$ satisfy the recurrence relations*

$$B_k(t) = t^k - \frac{1}{k+1}\sum_{\ell=0}^{k-1}\binom{k+1}{\ell}B_\ell(t) \tag{12}$$

and

$$B_k(t) = t^k - \frac{k}{2}B_{k-1}(t) - \sum_{\ell=1}^{k-1}\frac{1}{k-\ell+2}\binom{k}{\ell-1}B_{\ell-1}(t). \tag{13}$$

Consequently, the Bernoulli numbers B_k satisfy the recurrence relations

$$B_k = -\frac{1}{k+1}\sum_{\ell=0}^{k-1}\binom{k+1}{\ell}B_\ell \tag{14}$$

and

$$B_k = -\frac{k}{2}B_{k-1} - \sum_{\ell=1}^{k-1}\frac{1}{k-\ell+2}\binom{k}{\ell-1}B_{\ell-1}. \tag{15}$$

2. Lemmas

In order to obtain our aims and to prove our main results, we need the following lemmas.

Lemma 1 ([10] (p. 40, Exercise 5)). *Let $u(t)$ and $v(t) \ne 0$ be two differentiable functions, let $U_{(n+1)\times 1}(t)$ be an $(n+1)\times 1$ matrix whose elements are $u_{k,1}(t) = u^{(k-1)}(t)$ for $1 \le k \le n+1$, let $V_{(n+1)\times n}(t)$ be an $(n+1)\times n$ matrix whose elements are*

$$v_{i,j}(t) = \begin{cases} \binom{i-1}{j-1}v^{(i-j)}(t), & i-j \ge 0 \\ 0, & i-j < 0 \end{cases}$$

for $1 \le i \le n+1$ and $1 \le j \le n$, and let $|W_{(n+1)\times(n+1)}(t)|$ denote the determinant of the $(n+1)\times(n+1)$ matrix

$$W_{(n+1)\times(n+1)}(t) = \begin{pmatrix} U_{(n+1)\times 1}(t) & V_{(n+1)\times n}(t) \end{pmatrix}.$$

Then the nth derivative of the ratio $\frac{u(t)}{v(t)}$ *can be computed by*

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left[\frac{u(t)}{v(t)}\right] = (-1)^n \frac{\left|W_{(n+1)\times(n+1)}(t)\right|}{v^{n+1}(t)}. \tag{16}$$

Lemma 2 ([11] (p. 222, Theorem) and [12] (Remark 3))**.** *Let* $M_0 = 1$ *and*

$$M_n = \begin{vmatrix} m_{1,1} & m_{1,2} & 0 & \dots & 0 & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & \dots & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \dots & m_{n-2,n-1} & 0 \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \dots & m_{n-1,n-1} & m_{n-1,n} \\ m_{n,1} & m_{n,2} & m_{n,3} & \dots & m_{n,n-1} & m_{n,n} \end{vmatrix}$$

for $n \in \mathbb{N}$*. Then the sequence* M_n *for* $n \geq 0$ *satisfies* $M_1 = m_{1,1}$ *and*

$$M_n = m_{n,n} M_{n-1} + \sum_{r=1}^{n-1} (-1)^{n-r} m_{n,r} \left(\prod_{j=r}^{n-1} m_{j,j+1}\right) M_{r-1}, \quad n \geq 2. \tag{17}$$

3. Proofs of Theorem 1 and Corollary 1

We are now in a position to prove our main results.

Proof of Theorem 1. We can write the generating function of the Bernoulli polynomials $B_k(t)$ as

$$\frac{ze^{tz}}{e^z - 1} = \frac{e^{tz}}{(e^z - 1)/z} = \frac{e^{tz}}{\int_1^e s^{z-1}\,\mathrm{d}\,s}.$$

Therefore, with the help of Equation (16) applied to $u(t) = e^{tz}$ and $v(t) = \int_1^e s^{z-1}\,\mathrm{d}\,s$, we obtain the kth derivative

$$\left(\frac{e^{tz}}{\int_1^e s^{z-1}\,\mathrm{d}\,s}\right)^{(k)} = \frac{(-1)^k}{\left(\int_1^e s^{z-1}\,\mathrm{d}\,s\right)^{k+1}} \begin{vmatrix} A_{k+1,1}(z,t) & C_{k+1,k}(z,t) \end{vmatrix}_{(k+1)\times(k+1)},$$

where

$$A_{k+1,1}(z,t) = (a_{i,1}(z,t))_{1\leq i\leq k+1} \quad \text{and} \quad C_{k+1,k}(z,t) = (c_{i,j}(z,t))_{1\leq i\leq k+1, 1\leq j\leq k}$$

are matrices whose elements are $a_{i,1}(z,t) = t^{i-1}e^{tz} \to t^{i-1}$ and

$$c_{i,j}(z,t) = \begin{cases} 0, & i < j \\ \binom{i-1}{j-1}\int_1^e (\ln s)^{i-j} s^{z-1}\,\mathrm{d}\,s, & i \geq j \end{cases} \to \begin{cases} 0, & i < j \\ \frac{1}{i-j+1}\binom{i-1}{j-1}, & i \geq j \end{cases}$$

as $z \to 0$, respectively. This implies by definition of the generating function that

$$B_k(t) = \lim_{z\to 0}\left(\frac{ze^{tz}}{e^z - 1}\right)^{(k)} = \lim_{z\to 0}\left(\frac{e^{tz}}{\int_1^e s^{z-1}\,\mathrm{d}\,s}\right)^{(k)}$$

$$
= (-1)^k \begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
t & \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 & 0 \\
t^2 & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 & 0 & 0 \\
t^3 & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \frac{1}{2}\binom{3}{2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
t^{k-3} & \frac{1}{k-2}\binom{k-3}{0} & \frac{1}{k-3}\binom{k-3}{1} & \frac{1}{k-4}\binom{k-3}{2} & \cdots & 1 & 0 & 0 \\
t^{k-2} & \frac{1}{k-1}\binom{k-2}{0} & \frac{1}{k-2}\binom{k-2}{1} & \frac{1}{k-3}\binom{k-2}{2} & \cdots & \frac{1}{2}\binom{k-2}{k-3} & 1 & 0 \\
t^{k-1} & \frac{1}{k}\binom{k-1}{0} & \frac{1}{k-1}\binom{k-1}{1} & \frac{1}{k-2}\binom{k-1}{2} & \cdots & \frac{1}{3}\binom{k-1}{k-3} & \frac{1}{2}\binom{k-1}{k-2} & 1 \\
t^{k} & \frac{1}{k+1}\binom{k}{0} & \frac{1}{k}\binom{k}{1} & \frac{1}{k-1}\binom{k}{2} & \cdots & \frac{1}{4}\binom{k}{k-3} & \frac{1}{3}\binom{k}{k-2} & \frac{1}{2}\binom{k}{k-1}
\end{vmatrix}.
$$

The determinantal Equation (10) is thus proved.

Letting $t \to 0$ in (10) leads immediately to (11). The proof of Theorem 1 is complete. □

Proof of Corollary 1. Expanding the determinant in (10) by the last column consecutively and employing (10) inductively reveal

$$
(-1)^k B_k(t) = \frac{1}{2}\binom{k}{k-1} \begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
t & \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 \\
t^2 & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 & 0 \\
t^3 & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \frac{1}{2}\binom{3}{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
t^{k-3} & \frac{1}{k-2}\binom{k-3}{0} & \frac{1}{k-3}\binom{k-3}{1} & \frac{1}{k-4}\binom{k-3}{2} & \cdots & 1 & 0 \\
t^{k-2} & \frac{1}{k-1}\binom{k-2}{0} & \frac{1}{k-2}\binom{k-2}{1} & \frac{1}{k-3}\binom{k-2}{2} & \cdots & \frac{1}{2}\binom{k-2}{k-3} & 1 \\
t^{k-1} & \frac{1}{k}\binom{k-1}{0} & \frac{1}{k-1}\binom{k-1}{1} & \frac{1}{k-2}\binom{k-1}{2} & \cdots & \frac{1}{3}\binom{k-1}{k-3} & \frac{1}{2}\binom{k-1}{k-2}
\end{vmatrix}
$$

$$
- \begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
t & \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 \\
t^2 & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 & 0 \\
t^3 & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \frac{1}{2}\binom{3}{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
t^{k-3} & \frac{1}{k-2}\binom{k-3}{0} & \frac{1}{k-3}\binom{k-3}{1} & \frac{1}{k-4}\binom{k-3}{2} & \cdots & 1 & 0 \\
t^{k-2} & \frac{1}{k-1}\binom{k-2}{0} & \frac{1}{k-2}\binom{k-2}{1} & \frac{1}{k-3}\binom{k-2}{2} & \cdots & \frac{1}{2}\binom{k-2}{k-3} & 1 \\
t^{k} & \frac{1}{k+1}\binom{k}{0} & \frac{1}{k}\binom{k}{1} & \frac{1}{k-1}\binom{k}{2} & \cdots & \frac{1}{4}\binom{k}{k-3} & \frac{1}{3}\binom{k}{k-2}
\end{vmatrix}
$$

$$
= \frac{1}{2}\binom{k}{k-1} \begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
t & \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 & 0 \\
t^2 & \frac{1}{3}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 & 0 \\
t^3 & \frac{1}{4}\binom{3}{0} & \frac{1}{3}\binom{3}{1} & \frac{1}{2}\binom{3}{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
t^{k-3} & \frac{1}{k-2}\binom{k-3}{0} & \frac{1}{k-3}\binom{k-3}{1} & \frac{1}{k-4}\binom{k-3}{2} & \cdots & 1 & 0 \\
t^{k-2} & \frac{1}{k-1}\binom{k-2}{0} & \frac{1}{k-2}\binom{k-2}{1} & \frac{1}{k-3}\binom{k-2}{2} & \cdots & \frac{1}{2}\binom{k-2}{k-3} & 1 \\
t^{k-1} & \frac{1}{k}\binom{k-1}{0} & \frac{1}{k-1}\binom{k-1}{1} & \frac{1}{k-2}\binom{k-1}{2} & \cdots & \frac{1}{3}\binom{k-1}{k-3} & \frac{1}{2}\binom{k-1}{k-2}
\end{vmatrix}
$$

$$
\begin{aligned}
&-\frac{1}{3}\binom{k}{k-2}\begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & \\
t & \frac{1}{2}\tbinom{1}{0} & 1 & 0 & \cdots & 0 & 0\\
t^2 & \frac{1}{3}\tbinom{2}{0} & \frac{1}{2}\tbinom{2}{1} & 1 & \cdots & 0 & 0\\
t^3 & \frac{1}{4}\tbinom{3}{0} & \frac{1}{3}\tbinom{3}{1} & \frac{1}{2}\tbinom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
t^{k-3} & \frac{1}{k-2}\tbinom{k-3}{0} & \frac{1}{k-3}\tbinom{k-3}{1} & \frac{1}{k-4}\tbinom{k-3}{2} & \cdots & \frac{1}{2}\tbinom{k-3}{k-4} & 1\\
t^{k-2} & \frac{1}{k-1}\tbinom{k-2}{0} & \frac{1}{k-2}\tbinom{k-2}{1} & \frac{1}{k-3}\tbinom{k-2}{2} & \cdots & \frac{1}{3}\tbinom{k-2}{k-4} & \frac{1}{2}\tbinom{k-2}{k-3}
\end{vmatrix}\\
&+\frac{1}{4}\binom{k}{k-3}\begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0\\
t & \frac{1}{2}\tbinom{1}{0} & 1 & 0 & \cdots & 0 & 0\\
t^2 & \frac{1}{3}\tbinom{2}{0} & \frac{1}{2}\tbinom{2}{1} & 1 & \cdots & 0 & 0\\
t^3 & \frac{1}{4}\tbinom{3}{0} & \frac{1}{3}\tbinom{3}{1} & \frac{1}{2}\tbinom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
t^{k-4} & \frac{1}{k-3}\tbinom{k-4}{0} & \frac{1}{k-4}\tbinom{k-4}{1} & \frac{1}{k-5}\tbinom{k-4}{2} & \cdots & \frac{1}{2}\tbinom{k-4}{k-5} & 1\\
t^{k-3} & \frac{1}{k-2}\tbinom{k-3}{0} & \frac{1}{k-3}\tbinom{k-3}{1} & \frac{1}{k-4}\tbinom{k-3}{2} & \cdots & \frac{1}{3}\tbinom{k-3}{k-5} & \frac{1}{2}\tbinom{k-3}{k-4}
\end{vmatrix}\\
&-\begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0\\
t & \frac{1}{2}\tbinom{1}{0} & 1 & 0 & \cdots & 0 & 0\\
t^2 & \frac{1}{3}\tbinom{2}{0} & \frac{1}{2}\tbinom{2}{1} & 1 & \cdots & 0 & 0\\
t^3 & \frac{1}{4}\tbinom{3}{0} & \frac{1}{3}\tbinom{3}{1} & \frac{1}{2}\tbinom{3}{2} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
t^{k-4} & \frac{1}{k-3}\tbinom{k-4}{0} & \frac{1}{k-4}\tbinom{k-4}{1} & \frac{1}{k-5}\tbinom{k-4}{2} & \cdots & \frac{1}{2}\tbinom{k-4}{k-5} & 1\\
t^{k} & \frac{1}{k+1}\tbinom{k}{0} & \frac{1}{k}\tbinom{k}{1} & \frac{1}{k-1}\tbinom{k}{2} & \cdots & \frac{1}{4}\tbinom{k}{k-3} & \frac{1}{5}\tbinom{k}{k-4}
\end{vmatrix}\\
&=\sum_{\ell=1}^{m}(-1)^{\ell+1}\frac{1}{\ell+1}\binom{k}{k-\ell}(-1)^{k-\ell}B_{k-\ell}(t)+(-1)^{m+2}\\
&\times\begin{vmatrix}
1 & 1 & 0 & \cdots & 0 & 0\\
t & \frac{1}{2}\tbinom{1}{0} & 1 & \cdots & 0 & 0\\
t^2 & \frac{1}{3}\tbinom{2}{0} & \frac{1}{2}\tbinom{2}{1} & \cdots & 0 & 0\\
t^3 & \frac{1}{4}\tbinom{3}{0} & \frac{1}{3}\tbinom{3}{1} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
t^{k-m-2} & \frac{1}{k-m-1}\tbinom{k-m-2}{0} & \frac{1}{k-m-2}\tbinom{k-m-2}{1} & \cdots & \frac{1}{2}\tbinom{k-m-1}{k-m-2} & 1\\
t^{k} & \frac{1}{k+1}\tbinom{k}{0} & \frac{1}{k}\tbinom{k}{1} & \cdots & \frac{1}{m+3}\tbinom{k}{k-m-2} & \frac{1}{m+2}\tbinom{k}{k-m-1}
\end{vmatrix}\\
&=\sum_{\ell=1}^{k-1}(-1)^{\ell+1}\frac{1}{\ell+1}\binom{k}{k-\ell}(-1)^{k-\ell}B_{k-\ell}(t)+(-1)^{k+1}\begin{vmatrix}1 & 1\\ t^k & \frac{1}{k+1}\tbinom{k}{0}\end{vmatrix}\\
&=\sum_{\ell=1}^{k-1}(-1)^{\ell+1}\frac{1}{\ell+1}\binom{k}{k-\ell}(-1)^{k-\ell}B_{k-\ell}(t)+(-1)^{k+1}\left[\frac{1}{k+1}\binom{k}{0}-t^k\right]\\
&=\sum_{\ell=1}^{k}(-1)^{\ell+1}\frac{1}{\ell+1}\binom{k}{k-\ell}(-1)^{k-\ell}B_{k-\ell}(t)+(-1)^{k+2}t^k\\
&=(-1)^{k+1}\left[\sum_{\ell=1}^{k}\frac{1}{\ell+1}\binom{k}{k-\ell}B_{k-\ell}(t)-t^k\right].
\end{aligned}
$$

Accordingly, it follows that

$$B_k(t) = -\left[\sum_{\ell=1}^{k} \frac{1}{\ell+1}\binom{k}{k-\ell} B_{k-\ell}(t) - t^k\right] = -\sum_{\ell=0}^{k-1} \frac{1}{k-\ell+1}\binom{k}{\ell} B_\ell(t) + t^k.$$

Further considering the identity

$$\frac{1}{i-j+1}\binom{i-1}{j-1} = \frac{1}{i}\binom{i}{j-1}, \tag{18}$$

the relation (12) is thus proved.

Taking $t = 0$ in (12) results in the relation (14).

A straightforward application of the recurrence Equation (17) to the determinantal Equations (10) and (11) leads to (13) and (15), respectively. The proof of Corollary 1 is complete. □

4. Remarks

Finally, we list several remarks below.

Remark 1. *The determinantal Equation* (11) *can also be derived from differentiating the generating function*

$$\frac{z}{e^z - 1} = \frac{1}{\int_1^e s^{z-1}\,\mathrm{d}\,s}$$

in (1) *by virtue of Formula* (16) *and taking the limit* $z \to 0$.

Remark 2. *The recurrence relations* (12) *and* (14) *recover the well-known identities*

$$\sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell(t) = (k+1)t^k \quad \text{and} \quad \sum_{\ell=0}^{k} \binom{k+1}{\ell} B_\ell = 0.$$

in [13] (p. 4, Equations (1.10) and (1.11)).

Remark 3. *Dividing the jth column in* (3) *by* $j-2$ *for* $3 \le j \le n+1$ *immediately gives* (10). *Similarly, the determinantal Equation* (11) *can be derived from* (2). *This means that the determinantal Equations* (10) *and* (11) *are equivalent to* (3) *and* (2) *respectively. In this sense, Theorem 1 recovers the determinantal Equations* (2) *and* (3) *by an alternative method.*

Remark 4. *Formulas* (2) *to* (5) *are reformulations of corresponding ones in [1]. By this, some typos appearing in [1] are corrected.*

Remark 5. *By virtue of* (18)*, the determinantal Equation* (11) *can be rewritten as* (9)*, while the determinantal Equation* (10) *can be reformulated as*

$$B_k(t) = \frac{(-1)^k}{(k+1)!}\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{t}{2} & \binom{2}{0} & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ \frac{t^2}{3} & \binom{3}{0} & \binom{3}{1} & \frac{1}{3} & \cdots & 0 & 0 & 0 \\ \frac{t^3}{4} & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{t^{k-2}}{k-1} & \binom{k-1}{0} & \binom{k-1}{1} & \binom{k-1}{2} & \cdots & \binom{k-1}{k-3} & \frac{1}{k-1} & 0 \\ \frac{t^{k-1}}{k} & \binom{k}{0} & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-3} & \binom{k}{k-2} & \frac{1}{k} \\ \frac{t^k}{k+1} & \binom{k+1}{0} & \binom{k+1}{1} & \binom{k+1}{2} & \cdots & \binom{k+1}{k-3} & \binom{k+1}{k-2} & \binom{k+1}{k-1} \end{vmatrix} \tag{19}$$

for $k \geq 0$.

Remark 6. *Applying similar arguments in the proof of Theorem 1 to discuss the determinantal Equations (2) to (8), one can find or recover more recurrence relations for the Bernoulli numbers* B_k *and the Bernoulli polynomials* $B_k(t)$.

Because the last rows in the determinantal Equations (2)–(5), (8), (9), and (19) are different, when applying (17) to them, we can acquire different recurrence relations for the Bernoulli numbers B_n *and the Bernoulli polynomials* $B_n(t)$.

Remark 7. *In [14] (Section 4.1), the determinantal Equation (10) was derived by a different method.*

In [14] (Section 4.3), our main results in [15] (Theorems 1 and 2) were obtained by a different method.

We believe that Lemma 1 and Formula (17) can be applied to reformulate the paper [14]. In other words, this paper provides an alternative method to express some mathematical quantities in terms of the Hessenberg determinants or tridiagonal determinants.

Remark 8. *Formula (16) in Lemma 1 has been applied in the papers [9,12,15–26] to express the Apostol-Bernoulli polynomials, the Cauchy product of central Delannoy numbers, the Bernoulli polynomials, the Schröder numbers, the (generalized) Fibonacci polynomials, the Catalan numbers, derangement numbers, and the Euler numbers and polynomials in terms of the Hessenberg and tridiagonal determinants. This implies that Formula (16) in Lemma 1 is effectual to express some mathematical quantities in terms of the Hessenberg and tridiagonal determinants.*

5. Conclusions

Conclusively speaking, we recall some known determinantal expressions in terms of the Hessenberg determinants for the Bernoulli numbers and polynomials, find alternative determinantal expressions in terms of the Hessenberg determinants for the Bernoulli numbers and polynomials, and present several new recurrence relations for the Bernoulli numbers and polynomials.

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