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Results on Coincidence and Common Fixed Points for $(\psi, \varphi)_g$-Generalized Weakly Contractive Mappings in Ordered Metric Spaces

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Abstract: Inspired by a metrical-fixed point theorem from Choudhury et al. (*Nonlinear Anal.* **2011**, *74*, 2116–2126), we prove some order-theoretic results which generalize several core results of the existing literature, especially the two main results of Harjani and Sadarangani (*Nonlinear Anal.* **2009**, *71*, 3403–3410 and **2010**, *72*, 1188–1197). We demonstrate the realized improvement obtained in our results by using a suitable example. As an application, we also prove a result for mappings satisfying integral type $(\psi, \varphi)_g$-generalized weakly contractive conditions.

Keywords: fixed point; partially ordered metric space; weakly contractive; $(\psi, \varphi)_g$-generalized weekly contractive

MSC: 47H10; 54H25

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1. Introduction and Preliminaries

Banach contraction principle is a pivotal result of metric-fixed point theory. In subsequent years, this classical result has been generalized and improved in numerous ways and by now there exists extensive literature on this theme. In 1997, Alber and Guerre-Delabriere [1] introduced the notion of weak contraction and utilized the same to prove the existence and uniqueness of a fixed point of a self-mapping, satisfying a weak contraction condition on Hilbert spaces. In 2001, Rhoades [2] showed that this result remains true for complete metric spaces too. In recent years, the idea of weak contraction has been exploited by several researchers (e.g., [3–13]).

On the other hand, in 2004, Ran and Reurings [14] proved an order-theoretic analogue of Banach contraction principle which marks the beginning of a vigorous research activity. This noted-paper of Ran and Reurings is well followed by two very useful articles from Nieto and Rodríguez-López [15,16]. Presently, proving an order-theoretic analogue of metric-fixed point results is an area of active research and by now there exists considerable literature on this topic (e.g., [17–27]). Our work in this paper is on similar lines wherein our results are proved using $(\psi, \varphi)_g$-generalized weakly contractive mappings.

To present our main results, the following definitions, basic results and relevant historical overviews are needed.

We denote by $\mathbb{N}_0$ the set of natural numbers including zero, i.e., $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As usual, I_X stands for the identity mapping defined on X. For brevity, we write fx instead of $f(x)$.

Definition 1. *[28] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if it is continuous, increasing and satisfies $\psi(t) = 0$ if and only if $t = 0$. We denote the set of all altering distance functions by Ψ.*

Definition 2. *[7] A self-mapping f on a metric space (X, d) is said to be (ψ, φ)-weakly contractive mapping if for all $x, y \in X$,*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \tag{1}$$

where $\psi, \varphi \in \Psi$.

Remark 1. *In Definition 2, if we set $\psi := I_{[0,\infty)}$, then f is known as φ-weakly contractive mapping (see [1]).*

Definition 3. *[29] A self-mapping f on a metric space (X, d) is said to be (ψ, φ)-generalized weakly contractive mapping if for all $x, y \in X$,*

$$\psi(d(fx, fy)) \leq \psi(M_f(x, y)) - \varphi(max\{d(x, y), d(y, fy)\}), \tag{2}$$

where $M_f(x, y) = max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\})$, $\psi \in \Psi$ and φ:$[0, \infty) \to [0, \infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

Definition 4. *[27] A triple $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is an ordered set. Moreover, two elements $x, y \in X$ are said to be comparable if either $x \preceq y$ or $y \preceq x$. For brevity, we denote it by $x \prec\succ y$.*

Remark 2. *With a view to emphasize the order-theoretic analogue of Definition 2 (resp. Definition 3), it can be pointed out that the inequality (1) (resp. (2)) is required to hold merely for comparable elements, i.e., for all $x, y \in X$ such that $x \preceq y$ (rather than for every pair of elements in X).*

Definition 5. *[21] Let (f, g) be a pair of self-mappings on an ordered set $(X, \preceq)$. Then the mapping*

(i) *f is said to be g-increasing if $gx \preceq gy \Rightarrow fx \preceq fy$, for all $x, y \in X$,*
(ii) *f is said to be g-decreasing if $gx \preceq gy \Rightarrow fx \succeq fy$, for all $x, y \in X$,*
(iii) *f is said to be g-monotone if f is either g-increasing or g-decreasing.*

Definition 6. *[30] Let (f, g) be a pair of self-mappings on a metric space (X, d) and $x \in X$. We say that f is g-continuous at x if $gx_n \xrightarrow{d} gx \Rightarrow fx_n \xrightarrow{d} fx$, for any sequence $\{x_n\} \subset X$. Moreover, f is called g-continuous if it is g-continuous at every point of X.*

Let $\{x_n\}$ be a sequence in an ordered metric space $(X, d, \preceq)$. If $\{x_n\}$ is an increasing (resp. decreasing, monotone) and converges to x, we denote it by $x_n \uparrow x$ (resp. $x_n \downarrow x$, $x_n \uparrow\downarrow x$).

Definition 7. *[20] Let (f, g) be a pair of self-mappings on an ordered metric space $(X, d, \preceq)$ and $x \in X$. Then f is called $(g, \overline{O})$-continuous (resp. $(g, \underline{O})$-continuous, (g, O)-continuous) at $x \in X$ if $fx_n \xrightarrow{d} fx$, for every sequence $\{x_n\} \subset X$ with $gx_n \uparrow gx$ (resp. $gx_n \downarrow gx$, $gx_n \uparrow\downarrow gx$). Moreover, f is called (g, O)-continuous (resp. $(g, \overline{O})$-continuous, $(g, \underline{O})$-continuous) if it is (g, O)-continuous (resp. $(g, \overline{O})$-continuous, $(g, \underline{O})$-continuous) at every point of X.*

On setting $g := I_X$, Definition 7 reduces to the usual definition of $\overline{O}$-continuity (resp. $\underline{O}$-continuity, O-continuity) of self-mapping f on X.

Remark 3. *In an ordered metric space, g-continuity $\Rightarrow$ (g, O)-continuity $\Rightarrow$ $(g, \overline{O})$-continuity (as well as $(g, \underline{O})$-continuity).*

Definition 8. *Let (f, g) be a pair of self-mappings on an ordered metric space $(X, d, \preceq)$. Then the pair (f, g) is said to be*

- [31] *compatible if* $\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0$, *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n$.
- [20] $\overline{\mathrm{O}}$*-compatible (resp.* $\underline{\mathrm{O}}$*-compatible,* O*-compatible) if* $\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0$, *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\{gx_n\}$ *and* $\{fx_n\}$ *are increasing (resp. decreasing, monotone) sequences with* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$.
- [32] *weakly compatible if* $g(fx) = f(gx)$, *for every coincidence point* $x \in X$ *of* f *and* g.

Remark 4. *In an ordered metric space, compatibility* $\Rightarrow$ O*-compatibility* $\Rightarrow$ $\overline{\mathrm{O}}$*-compatibility (as well as* $\underline{\mathrm{O}}$*-compatibility)* $\Rightarrow$ *weak compatibility.*

Definition 9. [20] *An ordered metric space* $(X, d, \preceq)$ *is called* $\overline{\mathrm{O}}$*-complete (resp.* $\underline{\mathrm{O}}$*-complete,* O*-complete) if every increasing (resp. decreasing, monotone) Cauchy sequence in* X *converges to a point of* X.

Remark 5. *In an ordered metric space, completeness* $\Rightarrow$ O*-completeness* $\Rightarrow$ $\overline{\mathrm{O}}$*-completeness (as well as* $\underline{\mathrm{O}}$*-completeness).*

Definition 10. [20] *Let* (f, g) *be a pair of self-mappings on an ordered metric space* $(X, d, \preceq)$. *Then*

(i) $(X, d, \preceq)$ *is said to have g-ICU-property (Increasing-Convergence-Upper-Bound) if g-image of every increasing convergent sequence* $\{x_n\}$ *in* X *is bounded above by g-image of its limit, i.e.,*

$$x_n \uparrow x \Rightarrow g(x_n) \preceq g(x) \ \ \forall\, n \in \mathbb{N}_0,$$

(ii) $(X, d, \preceq)$ *is said to have g-DCL-property (Decreasing-convergence-Lower-Bound) if g-image of every decreasing convergent sequence* $\{x_n\}$ *in* X *is bounded below by g-image of its limit, i.e.,*

$$x_n \downarrow x \Rightarrow g(x_n) \succeq g(x) \ \ \forall\, n \in \mathbb{N}_0\,,$$

(iii) $(X, d, \preceq)$ *is said to have g-MCB-property (Monotone-Convergence-Boundedness) if it has both g-ICU as well as g-DCL-property.*

On setting $g := I_X$, Definition 10(i) (resp. 10(ii), 10(iii)) reduces to the definition of the *ICU*-property (resp. *DCL*-property, *MCB*-property).

Definition 11. [24] *Let* D *be a subset of an ordered set* $(X, \preceq)$ *and* g *a self-mapping on* X. *We say that* D *is g-directed if for every pair of elements* $x, y \in D$, *there is* $z \in X$ *such that* $x \prec\succ gz$ *and* $y \prec\succ gz$.

Notice that, on setting $g := I_X$ in Definition 11, D is said to be directed due to [24].
The following three lemmas are needed to prove our results:

Lemma 1. [33] *Let* (f, g) *be a pair of self-mappings defined on an ordered set* $(X, \preceq)$. *If* f *is g-monotone and* $gx = gy$, *then* $fx = fy$.

Lemma 2. [33] *Let* (f, g) *be a pair of weakly compatible self-mappings defined on non-empty set* X. *Then every point of coincidence of the pair* (f, g) *is also a coincidence point.*

Proof. Let x be a point of coincidence of f and g such that $fx = gx = x^*$ for some $x^* \in X$. On using the weak compatibility of f and g, we have

$$gx^* = g(fx) = f(gx) = fx^*,$$

which implies that x^* is a coincidence point of f and g. □

The following lemma was proved as a part of the proof of Theorem 2.1 of [23].

Lemma 3. *[23] Let $(X, d, \preceq)$ be an ordered metric space and $\{x_n\}$ a sequence in X such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that*

(i) $n_k > m_k \geq k$,
(ii) $d(x_{m_k}, x_{n_k}) \geq \epsilon$,
(iii) $d(x_{m_k}, x_{n_k-1}) < \epsilon$,
(iv) *the sequences* $d(x_{m_k}, x_{n_k})$, $d(x_{m_k+1}, x_{n_k})$, $d(x_{m_k}, x_{n_k+1})$, $d(x_{m_k+1}, x_{n_k+1})$ *tend to* ϵ *when* $k \to \infty$.

Alber and Guerre-Delabriere [1] proved that every φ-weakly contractive mapping defined on a Hilbert space possesses a unique fixed point. Thereafter, Rhoads [2] proved that this result is also true for complete metric spaces.

Theorem 1. *[2] (Theorem 1) Let (X, d) be a complete metric space. If the mapping $f : X \to X$ is a φ-weakly contractive mapping, then f has a unique fixed point.*

It is worth noting that, Alber and Guerre-Delabriere [1] assumed that the altering distance function φ satisfies an extra condition (which is $\lim_{t\to\infty} \varphi(t) = \infty$), but Rhoades [2] obtained the above result without using this condition.

Thereafter, Dutta and Choudhury [7] proved a generalization of Theorem 1 as follows:

Theorem 2. *[7] (Theorem 2.1) Let (X, d) be a complete metric space and $f : X \to X$ a (ψ, φ)-weakly contractive mapping. Then f has a unique fixed point.*

Choudhury et al. [29] proved a generalization of the above two theorems as follows:

Theorem 3. *[29] (Theorem 3.1) Let (X, d) be a complete metric space and $f : X \to X$ a (ψ, φ)-generalized weakly contractive mapping on X. Then f has a unique fixed point.*

On the other hand, in the setting of ordered metric spaces, Harjani and Sadarangani [22] proved an order-theoretic analogue of Theorem 1 as follows:

Theorem 4. *[22] (Theorems 2 and 3) Let $(X, d, \preceq)$ be a complete ordered metric space and f an increasing self-mapping on X. Suppose that the following conditions hold:*

(i) *f is a φ-weakly contractive mapping with $\lim_{t\to\infty} \varphi(t) = \infty$,*
(ii) *either f is a continuous mapping or $(X, d, \preceq)$ enjoys ICU-property.*

Then f has a fixed point provided there exists $x_0 \in X$ such that $x_0 \preceq f x_0$.

Subsequently, Harjani and Sadarangani [23] proved the following result which is an order-theoretic analogue of Theorem 2 as well as a generalization of Theorem 4.

Theorem 5. *[23] (Theorems 2.1 and 2.2) Let $(X, d, \preceq)$ be a complete ordered metric space and f an increasing self-mapping on X. Suppose that the following conditions hold:*

(i) *f is a (ψ, φ)-weakly contractive mapping,*
(ii) *either f is a continuous mapping or $(X, d, \preceq)$ enjoys ICU-property.*

Then f has a fixed point provided there exists $x_0 \in X$ such that $x_0 \preceq f x_0$.

Here, it can be pointed out that Harjani and Sadarangani [22,23] proposed the following sufficient condition for the uniqueness of the fixed point in Theorems 4 and 5:

$$X \text{ is directed .} \tag{3}$$

The aim of this article is to prove an order-theoretic analogue of Theorem 3 so as to improve and generalize Theorems 4 and 5. The improvement realized in our results is three-fold which we describe as under:

(a) relatively weaker notions of the continuity and completeness are employed,
(b) the (ψ, φ)-weak contractive condition is replaced by a $(\psi, \varphi)_g$-generalized weak contractive condition (defined later) involving a pair of self mappings,
(c) a weaker uniqueness condition is utilized.

We demonstrate the genuineness of our results by a suitable example. As an application, we prove a result for mappings satisfying integral type $(\psi, \varphi)_g$-generalized weak contractive condition.

2. Results on Coincidence Point

In the sequel, we use the following definition:

Definition 12. *Let (f, g) be a pair of self-mappings on an ordered metric space $(X, d, \preceq)$. Then f is said to be a $(\psi, \varphi)_g$-generalized weakly contractive mapping if for all $x, y \in X$ such that $gx \preceq gy$, we have*

$$\psi(d(fx, fy)) \leq \psi(M_{f,g}(x,y)) - \varphi(max\{d(gx,gy), d(gy,fy)\}), \tag{4}$$

where $M_{f,g}(x,y) = max\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{1}{2}[d(gx,fy) + d(gy,fx)]\}$, $\psi \in \Psi$ and $\varphi : [0,\infty) \to [0,\infty)$ is a lower-semi continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

Observe that, on setting $g := I_X$, Definition 12 remains relatively weaker than the order-theoretic analogue of Definition 3 as the class of lower-semi continuous functions is larger than the class of continuous functions.

Now, we prove our main result as follows:

Theorem 6. *Let $(X, d, \preceq)$ be an ordered metric space and Y an $\overline{O}$-complete subspace of X. Let (f, g) be a pair of self-mappings on X such that the mapping f is g-increasing. Suppose the following conditions hold:*

(i) f is a $(\psi, \varphi)_g$-generalized weakly contractive mapping,
(ii) (a) $f(X) \subseteq Y \subseteq g(X)$ and
(b) either f is $(g, \overline{O})$-continuous or f and g are continuous or $(Y, d, \preceq)$ has ICU-property.

Then the pair (f, g) has a coincidence point provided there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$.

Proof. Choose $x_0 \in X$ such that $gx_0 \preceq fx_0$. As the mapping f is g-increasing and $f(X) \subseteq g(X)$, we can define increasing mapping sequences $\{gx_n\}$ and $\{fx_n\}$ in X such that for all $n \in \mathbb{N}_0$

$$gx_{n+1} = fx_n. \tag{5}$$

Observe that, $\{gx_n\}$ and $\{fx_n\}$ are in Y. Moreover, if $d(gx_n, gx_{n+1}) = 0$ for some $n \in \mathbb{N}_0$, then x_n is the required coincidence point and we are done. Henceforth, we assume that $d(gx_n, gx_{n+1}) > 0$ for all $n \in \mathbb{N}_0$.

We assert that $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0$. On setting $x = x_n, y = x_{n+1}$ in (4), we get

$$\begin{aligned}\psi(d(gx_{n+1}, gx_{n+2})) &= \psi(d(fx_n, fx_{n+1})) \\ &\leq \psi(M_{f,g}(x_n, x_{n+1})) - \varphi(max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})\}) \end{aligned} \tag{6}$$

for all $n \in \mathbb{N}_0$, where

$$\begin{aligned} M_{f,g}(x_n, x_{n+1}) &= max\Big\{d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1}), \frac{d(gx_n, fx_{n+1}) + d(gx_{n+1}, fx_n)}{2}\Big\} \\ &= max\Big\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), \frac{d(gx_n, gx_{n+2})}{2}\Big\}\end{aligned}$$

By the triangular inequality, $max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})\} \geq \frac{1}{2}d(gx_n, gx_{n+2})$. If possible, assume $M_{f,g}(x_n, x_{n+1}) = d(gx_{n+1}, gx_{n+2})$, then $d(gx_n, gx_{n+1})) \leq d(gx_{n+1}, gx_{n+2})$ so that (6) reduces to

$$\begin{aligned}\psi(d(gx_{n+1}, gx_{n+2})) &\leq \psi(d(gx_{n+1}, gx_{n+2}) - \varphi(d(gx_{n+1}, gx_{n+2})) \\ &< \psi(d(gx_{n+1}, gx_{n+2})),\end{aligned}$$

a contradiction. Thus, $M_{f,g}(x_n, x_{n+1}) = d(gx_n, gx_{n+1})$ and (6) becomes

$$\begin{aligned}\psi(d(gx_{n+1}, gx_{n+2})) &\leq \psi(d(gx_n, gx_{n+1})) - \varphi(d(gx_n, gx_{n+1})) \\ &< \psi(d(gx_n, gx_{n+1})).\end{aligned}$$

As ψ is an increasing function, $\{d(gx_n, gx_{n+1})\}$ is a decreasing sequence of positive real numbers so that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \alpha \geq 0.$$

On taking the limit superior as $n \to \infty$ in inequality (6), we obtain

$$\lim_{n\to\infty} \sup \psi(d(gx_{n+1}, gx_{n+2})) \leq \lim_{n\to\infty} \sup \psi(d(gx_n, gx_{n+1})) - \lim_{n\to\infty} \inf \varphi(d(gx_n, gx_{n+1}))$$

which implies that $\psi(\alpha) \leq \psi(\alpha) - \varphi(\alpha)$, a contradiction. Therefore, $\alpha = 0$, i.e., $\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0$.

Now, we assert that $\{gx_n\}$ is a Cauchy sequence in Y. For if it is not Cauchy, owing to Lemma 3, there exist $\epsilon > 0$ and two subsequences $\{gx_{n_k}\}$ and $\{gx_{m_k}\}$ of $\{gx_n\}$ such that $n_k > m_k \geq k$, $d(gx_{m_k}, gx_{n_k}) \geq \epsilon$, $d(gx_{n_k-1}, gx_{m_k}) < \epsilon$ and

$$\begin{aligned}\lim_{k\to\infty} d(gx_{m_k}, gx_{n_k}) &= \lim_{k\to\infty} d(gx_{m_k+1}, gx_{n_k}) \\ &= \lim_{k\to\infty} d(gx_{m_k}, gx_{n_k+1}) \\ &= \lim_{k\to\infty} d(gx_{m_k+1}, gx_{n_k+1}) \\ &= \epsilon.\end{aligned}$$

Since $n_k > m_k$, on putting $x = x_{n_k}$ and $y = x_{m_k}$ in (4), we have (for all $k \in \mathbb{N}$)

$$\begin{aligned}\psi(d(gx_{n_k+1}, gx_{m_k+1})) &= \psi(d(fx_{n_k}, fx_{m_k})) \\ &\leq \psi(M_{f,g}(x_{n_k}, x_{m_k})) - \varphi(max\{d(gx_{n_k}, gx_{m_k}), d(gx_{m_k}, gx_{m_k+1})\})\end{aligned} \tag{7}$$

where

$$M_{f,g}(x_{n_k}, x_{m_k})) = max\Big\{d(gx_{n_k}, gx_{m_k}), d((gx_{n_k}, gx_{n_k+1}), d(gx_{m_k}, gx_{m_k+1}), \frac{1}{2}[d(gx_{n_k}, gx_{m_k+1}) + d(gx_{m_k}, gx_{n_k+1})]\Big\}.$$

Taking limit superior as $n \to \infty$ in (7), we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon),$$

a contradiction. Thus, $\{gx_n\}$ is a Cauchy sequence in Y. Therefore, there exists some $x \in Y$ such that

$$gx_n \uparrow x. \tag{8}$$

Due to the condition (ii)a, there exists some $z \in X$ such that $x = gz$, so that

$$gx_n \uparrow gz. \tag{9}$$

Now, using the condition (ii)b, we show that z is a coincidence point of the pair (f, g). Firstly, assume that f is $(g, \overline{\text{O}})$-continuous. In view of (9), we have $fx_n \to fz$ which (in view of (5)) by the uniqueness of the limit implies $gz = fz$.

Secondly, let f and g be continuous mappings. Then, the proof can be outlined on the lines of the proof of Theorem 1 in [20].

Lastly, assume that $(Y, d, \preceq)$ enjoys *ICU*-property. Then, $gx_n \preceq gz \;\; \forall\, n \;\in \mathbb{N}$, and on setting $x = x_n, y = z$ in (4), we have (for all $n \in \mathbb{N}_0$)

$$\begin{aligned}\psi(d(gx_{n+1}, fz)) &= \psi(d(fx_n, fz)) \\ &\leq \psi(M_{f,g}(x_n, z)) - \varphi(max\{d(gx_n, gz), d(gz, fz)\})\end{aligned} \tag{10}$$

where

$$M_{f,g}(x_n, z) = max\{d(gx_n, gz), d(gx_n, gx_{n+1}), d(gz, fz), \frac{1}{2}[d(gx_n, fz) + d(gz, gx_{n+1})]\}.$$

On using (5), (9) and taking limit superior in (10) as $n \to \infty$, we have

$$\psi(d(gz, fz)) \leq \psi(d(gz, fz)) - \varphi(d(gz, fz)),$$

a contradiction unless $gz = fz$. This concludes the proof. □

Theorem 7. *Theorem 6 remains true if assumptions embodied in the condition (ii) are replaced by the following (besides retaining the rest of the hypotheses).*

($\tilde{ii}$) *(a)* $f(X) \subseteq Y \cap g(X)$,
(b) *g is $\overline{\text{O}}$-continuous,*
(c) *(f, g) is $\overline{\text{O}}$-compatible pair and*
(d) *either f is $\overline{\text{O}}$-continuous or $(Y, d, \preceq)$ has g-ICU-property.*

Proof. The proof runs on the lines of the proof of Theorem 6 except wherever we used conditions in (ii), which can be altered as follows: Owing to (5) and (8), we have

$$fx_n \uparrow x \text{ and } gx_n \uparrow x, \tag{11}$$

where $x \in Y$. In view of the condition $(\tilde{ii})$b, we have

$$\lim_{n\to\infty} g(fx_n) = gx = \lim_{n\to\infty} g(gx_n).$$

Also, in view of the condition $(\tilde{ii})$c, we have

$$\lim_{n\to\infty} d(g(fx_n), f(gx_n)) = 0,$$

so that,

$$\lim_{n\to\infty} f(gx_n) = gx.$$

Now, on using the condition $(\tilde{ii})$d, we show that x is a coincidence point of f and g. Let f be $\overline{\text{O}}$-continuous. Then, from (11), we have

$$\lim_{n\to\infty} f(gx_n) = f(\lim_{n\to\infty} gx_n) = fx.$$

Combining last two equations, we get $fx = gx$ and hence we are done.

Alternately, let $(Y, d, \preceq)$ enjoy g-*ICU*-property. By (11), we have $g(gx_n) \preceq gx$ for all $n \in \mathbb{N}_0$. On putting $x = gx_n, y = x$ in (4), we get

$$\psi(d(fgx_n, fx)) \leq \psi(M_{f,g}(gx_n, x)) - \varphi(max\{d(ggx_n, gx), d(gx, fy)) \tag{12}$$

for all $n \in \mathbb{N}_0$, where,

$$M_{f,g}(gx_n, x)) = max\Big\{d(ggx_n, gx), d(ggx_n, fgx_n), d(gx, fx), \frac{1}{2}[d(ggx_n, fx) + d(gx, fgx_n)]\Big\}.$$

On taking the limit of (12) as $n \to \infty$, we arrive at a contradiction unless $gx = fx$. This concludes the proof. □

Remark 6. *Observe that the condition $(\tilde{ii})$a utilized in Theorem 7 is relatively weaker than the condition (ii)a of Theorem 6.*

On setting $g := I_X$ in Theorems 6 and 7, we deduce the following:

Corollary 1. *Let $(X, d, \preceq)$ be an ordered metric space, Y an $\overline{\text{O}}$-complete subspace of X and f an increasing self-mapping on X such that $f(X) \subseteq Y$. Suppose the following conditions hold:*

(i) f is a (ψ, φ)-generalized weakly contractive mapping,
(ii) either f is $\overline{\text{O}}$-continuous or $(Y, d, \preceq)$ has ICU-property.

Then, f has a fixed point provided there exists $x_0 \in X$ such that $x_0 \preceq fx_0$.

Remark 7.

(a) If $M_f(x, y) = d(x, y)$, then Corollary 1 reduces to a sharpened version of Theorem 5, as the increasing condition on the altering distance function φ is found unnecessary and a weaker notion of the continuity of φ is utilized.
(b) If $M_f(x, y) = d(x, y)$ and $\psi := I_{[0,\infty))}$ in Corollary 1, we get Theorem 4 without the assumption $\lim_{t\to\infty} \varphi(t) = \infty$.
(c) The completeness in Theorems 4 and 5 is merely required on any subspace rather than the whole space X such that this subspace contains $f(X)$. Further, these results can be obtained utilizing a relatively weaker notion of the continuity and completeness.

Example 1. *Consider $X = (-1, 0]$ endowed with the usual metric d. Then, $(X, d, \preceq)$ is an $\overline{\text{O}}$-complete ordered metric space wherein the partial order '$\preceq$' is defined by: $x \preceq y$ iff $x \le y$ for $x, y \in (-1, 0)$ and $0 \preceq 0$. Define $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = 3t$ and $\varphi := I_{[0,\infty)}$. Consider f and g two self-mappings on X defined by: $f(x) = \frac{1}{3}x$ and $g(x) = \frac{2}{3}x$. Then, the left hand side of the inequality (4) is*

$$\psi(d(fx, fy)) = |x - y| = \begin{cases} x - y, & \text{for } y \le x \\ y - x, & \text{for } y \ge x. \end{cases}$$

To compute the right hand side of the inequality, we have

$$\begin{aligned} \psi\left(M_{f,g}(x,y)\right) &= \psi\left(max\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{1}{2}[d(gx,fy) + d(gy,fx)]\}\right) \\ &= \psi\left(max\{\frac{2}{3}|x-y|, \frac{1}{3}|x|, \frac{1}{3}|y|, \frac{1}{6}\left(|x-2y| + |y-2x|\right)\right)\} \\ &= \begin{cases} 2(x-y), & \text{for } -1 < y \le 2x \\ -y, & \text{for } 2x \le y \le x \\ -x, & \text{for } x \le y \le \frac{1}{2}x \\ 2(y-x), & \text{for } \frac{1}{2}x \le y \le 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \varphi\left(max\{d(gx,gy), d(gy,fy)\}\right) &= max\left\{\frac{2}{3}|x-y|, \frac{1}{3}|y|\right\} \\ &= \begin{cases} \frac{2}{3}(x-y), & \text{for } -1 < y \le 2x \\ -\frac{1}{3}y, & \text{for } 2x \le y \le \frac{2}{3}x \\ \frac{2}{3}(y-x), & \text{for } \frac{2}{3}x \le y \le 0. \end{cases} \end{aligned}$$

Thus, the right hand side of (4) is

$$\psi(M_{f,g}(x,y)) - \varphi(max\{d(gx,gy), d(gy,fy)\}) = \begin{cases} \frac{4}{3}(x-y), & \text{for } -1 < y \le 2x \\ -\frac{2}{3}y, & \text{for } 2x \le y \le x \\ -x + \frac{1}{3}y, & \text{for } x \le y \le \frac{2}{3}x \\ -\frac{1}{3}(x+2y), & \text{for } \frac{2}{3}x \le y \le \frac{1}{2}x \\ \frac{4}{3}(y-x), & \text{for } \frac{x}{2} < y \le 0. \end{cases}$$

By a routine calculation, we can see that inequality (4) is satisfied, that is, f is a $(\psi, \varphi)_g$-generalized weakly contractive mapping and the pair (f, g) has a coincidence point (namely $x = 0$) supporting Theorems 6 and 7.

On setting $g =: I_X$ in Example 1, we create a situation wherein neither Theorem 4 nor Theorem 5 can be used, as the whole space is not complete while our Corollary 1 works. This substantiates the genuineness of our results proved in this paper.

Definition 13. *Let (f, g) be a pair of self-mappings on an ordered metric space $(X, d, \preceq)$. Then, f is said to be a lean $(\psi, \varphi)_g$-generalized weakly contractive mapping if for all $x, y \in X$ such that $gx \preceq gy$, we have*

$$\psi(d(fx, fy)) \le \psi(m_{f,g}(x,y)) - \varphi(max\{d(gx,gy), d(gy,fy)\}), \tag{13}$$

where $m_{f,g}(x,y) = max\{d(gx,gy), \frac{1}{2}[d(gx,fx) + d(gy,fy)], \frac{1}{2}[d(gx,fy) + d(gy,fx)]\})$, $\psi \in \Psi$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

As $m_{f,g}(x,y) \le M_{f,g}(x,y)$, Definition 12 is weaker than Definition 13.

Corollary 2. *Theorem 6 remains true if the condition (i) is replaced by the following condition (besides retaining the rest of the hypothesis).*

$(\H{i})$ f *is a lean* $(\psi, \varphi)_g$*-generalized weakly contractive mapping.*

Corollary 3. *Theorem 7 remains true if the condition (i) is replaced by the condition* $(\H{i})$ *(besides retaining the rest of the hypothesis).*

3. Results on Common Fixed Points

Theorem 8. *In addition to the hypotheses of Corollary 2, if* $f(X)$ *is* g*-directed, then the pair* (f, g) *has a unique point of coincidence.*

Proof. Let $x, y, \overline{x}, \overline{y} \in X$ be such that

$$gx = fx = \overline{x} \text{ and } gy = fy = \overline{y}.$$

We assert that $\overline{x} = \overline{y}$. By the hypothesis, there exists $z \in X$ such that gz is comparable to both fx and fy. For $fx \prec\succ gz$, we may assume $fx \preceq gz$ (other case is similar).

Set $z_0 = z$. Since $f(X) \subseteq g(X)$ and f is a g-increasing mapping, one can define a sequence $\{z_n\} \subset X$ such that

$$gz_{n+1} = fz_n \text{ and } gx \preceq gz_n \text{ for all } n \in \mathbb{N}.$$

We assert that

$$\lim_{n\to\infty} d(gx, gz_n) = 0. \tag{14}$$

To establish the assertion, we distinguish two cases:

Firstly, if $d(gx, gz_m) = 0$ for some $m \in \mathbb{N}$. Then by Lemma 1, $d(fx, fz_m) = 0$, that is, $d(gx, gz_{m+1}) = 0$. On using induction on m, $d(gx, gz_n) = 0$ for all $n \geq m$ establishing the assertion in this case.

Secondly, if $d(gx, gz_n) > 0$ for all $n \in \mathbb{N}_0$, then on setting $x = x$ and $y = z_n$ in (13), we get

$$\begin{aligned} \psi(d(gx, gz_{n+1})) &= \psi(d(fx, fz_n)) \\ &\leq \psi(m_f(x, z_n)) - \varphi(max\{d(gx, gz_n), d(gz_n, fz_n)\}) \end{aligned} \tag{15}$$

for all $n \in \mathbb{N}_0$, where

$$m_f(x, z_n) = max\Big\{d(gx, gz_n), \frac{1}{2}[d(gx, fx) + d(gz_n, gz_{n+1})], \frac{1}{2}[d(gx, gz_{n+1}) + d(gz_n, gx)]\Big\}$$

Obviously, $\frac{1}{2}[d(gz_n, gz_{n+1})] \leq \frac{1}{2}[d(gx, gz_{n+1}) + d(gz_n, gx)]$. Assume that $d(gx, gz_{n+1}) > d(gx, gz_n)$. Then $m_f(x, z_n) = \frac{1}{2}[d(gx, gz_{n+1}) + d(gz_n, gx)]$. Therefore, from (15), we have

$$\psi(d(gx, gz_{n+1})) < \psi\Big(\frac{1}{2}[d(gx, gz_{n+1}) + d(gz_n, gx)]\Big).$$

As ψ is increasing, we have $d(gx, gz_{n+1}) \leq d(gz_n, gx)$, a contradiction to our assumption. Hence, $d(gx, gz_{n+1}) \leq d(gx, gz_n)$ so that $m_f(x, z_n) = d(gx, gz_n)$ and (15) reduces to

$$\psi(d(gx, gz_{n+1})) \leq \psi(d(gx, gz_n)) \text{ for all } n \in \mathbb{N}_0$$

Now, $\{d(gx, gz_n)\}$ is a decreasing sequence of strictly positive real numbers which must posses a limit $r \geq 0$. Letting $n \to \infty$ in (15), we get $\psi(r) \leq \psi(r) - \varphi(2r)$ which is a contradiction unless $r = 0$. Thus, in all, our assertion is established.

Similarly, when $fy \prec\succ gz$, one can show that

$$\lim_{n\to\infty} d(gy, gz_n) = 0 \tag{16}$$

On using triangular inequality, (14) and (16), we have

$$d(\overline{x}, \overline{y}) = d(gx, gy) \leq d(gx, gz_n) + d(gz_n, gy) \to 0 \quad as \quad n \to \infty,$$

which shows that the pair (f, g) has a unique point of coincidence. □

Theorem 9. *In addition to the hypotheses of Theorem 8, if the pair (f, g) is weakly compatible, then the pair has a unique common fixed point.*

Proof. Let $x \in X$ be an arbitrary coincidence point of the pair (f, g). Due to Theorem 8, there exists a unique point of coincidence $w \in X$ (say) such that $fx = gx = w$. By Lemma 2, w itself is a coincidence point, i.e., $fw = gw$. Now, again, Theorem 8 ensures that $fw = gw = w$, i.e., w is a unique common fixed point of f and g. □

Theorem 10. *In addition to the hypotheses of Corollary 3, if $f(X)$ is g-directed, then the pair (f, g) has a unique common fixed point.*

Proof. On the lines of the proof of Theorem 8, one can show that the pair (f, g) has a unique point of coincidence. In view of the hypothesis (condition 1c of Theorem 7), (f, g) is an $\overline{\text{O}}$-compatible pair and hence is a weakly compatible pair (by Remark 4). Now, the proof can be completed on the lines of the proof of Theorem 9. □

Remark 8. *On setting $g := I_X$, the uniqueness condition utilized in Theorem 8 (also in Theorem 10) remains slightly weaker than the condition (3).*

Remark 9. *One can obtain dual type results corresponding to all results in Sections 2 and 3 by replacing "$\overline{\text{O}}$-analogues" with "$\underline{\text{O}}$-analogues" and "ICU-property" with "DCL-property" provided the existence of $x_0 \in X$ such that $gx_0 \preceq fx_0$ is replaced by the existence of $x_0 \in X$ such that $gx_0 \succeq fx_0$.*

Remark 10. *One can obtain companied type results corresponding to all results in Sections 2 and 3 by replacing "$\overline{\text{O}}$-analogues" with "O-analogues" and "ICU-property" with "MCU-property" provided the existence of $x_0 \in X$ such that $gx_0 \preceq fx_0$ is replaced by the existence of $x_0 \in X$ such that $gx_0 \prec\succ fx_0$.*

Remark 11. *By using Zermelo's well-ordering Theorem, the set X can be well ordered and the contraction conditions in all above results of Sections 2 and 3 are valid for each $x, y \in X$. Therefore, each of Theorems 9 and 10 covers Theorems 1, 2, 3 and Theorem 2.1 of [4]*

As an application of Theorem 6 (resp. Theorem 7), we have the following result on coincidence point for mappings satisfying integral type $(\psi, \varphi)_g$-weakly contraction in ordered metric space.

Let Λ be the set of functions $\omega : [0, \infty) \to [0, \infty)$ satisfying the following:

(a) ω is a Lebesgue-integrable mapping on each compact subset of $[0, \infty)$;
(b) $\int_0^{\epsilon} \omega(t)dt > 0$ for all $\epsilon > 0$.

Theorem 11. *Let $(X, d, \preceq)$ be an ordered metric space and Y an $\overline{\text{O}}$-complete subspace of X. Let (f, g) be a pair of self-mappings on X such that f is g-increasing. Suppose that for every $x, y \in X$ with $x \preceq y$ and $\omega \in \Lambda$, we have*

$$\int_0^{\psi(d(fx,fy))} \omega(t)dt \le \int_0^{\psi(M_{f,g}(x,y))} \omega(t)dt - \int_0^{\varphi(max\{d(gx,gy),d(gy,fy)\})} \omega(t)dt, \tag{17}$$

where ψ and φ are as in Definition 12. If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$ and the condition (ii) of Theorem 6 (resp. condition ($\tilde{ii}$) of Theorem 7) is satisfied, then the pair (f, g) has a coincidence point.

Proof. Define $\Gamma : [0, \infty) \to [0, \infty)$ by $\Gamma(x) = \int_0^x \omega(t)dt$, then (17) can be written as

$$\Gamma(\psi(d(fx, fy))) \le \Gamma(\psi(M_{f,g}(x, y))) - \Gamma(\varphi(max\{d(gx, gy), d(gy, fy)\})).$$

Since $\Gamma \circ \psi : [0, \infty) \to [0, \infty)$ is an altering distance function and $\Gamma \circ \varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $(\Gamma \circ \varphi)(t) = 0$ if and only if $t = 0$. The desired result follows from Theorem 6 (resp. Theorem 7). □

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