Article

Weyl and Marchaud Derivatives: A Forgotten History †

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† Dedicated to Sandro Salsa for his 68th birthday.

Abstract: In this paper, we recall the contribution given by Hermann Weyl and André Marchaud to the notion of fractional derivative. In addition, we discuss some relationships between the fractional Laplace operator and Marchaud derivative in the perspective to generalize these objects to different fields of the mathematics.

1. Introduction

Exactly one century ago, while we are writing, in 1917, a paper by Hermann Weyl, *Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung*, appeared, [1]. It dealt with the definition of a fractional derivative in a weaker sense with respect to the approach classically known at that time with the name of Riemann–Liouville derivative.

Ten years later, in 1927, the thesis of a misunderstood French mathematician, Adré Paul Weyl, was published, who discussed at the age of forty his PhD work entitled *Sur les dérivées et sur les différences des fonctions de variables réelles*, [2].

In [3], the names Weyl and Marchaud appear associated with the notion of fractional derivative more than two hundred times. Nevertheless, in my opinion, the name Marchaud is not so popular even among the mathematicians dealing with fractional calculus, in particular among scientists coming from Western countries. Due to the huge quantity of papers dealing with fractional subjects, my previous statement could appear debatable. In any case, this opinion can be tested just consulting, for instance a database. We tried, for instance, with the American Mathematical Society database MathShiNet. In fact, inserting the keyword "Marchaud" anywhere, we obtain around two hundred files. Among these two hundred files, improving the request by also searching the word Marchaud in the titles of the papers, we find around fifty files. In addition, by reading these titles, covering for example the last twenty years, we realize at a first glance that the frequency of mathematicians from Eastern countries is prevalent. Indeed, on the contrary of what we stated about Western mathematicians, Marchaud's name is recurrent in fractional calculus literature and among mathematicians coming from Eastern Europe, let us recall one more time the number of citations that appear in [3].

Concerning Hermann Weyl, of course, we are considering a very popular mathematician for many other mathematical reasons. Nevertheless, we have to say that also in this case Weyl's name is not usually associated with the fractional calculus even if the specialists in the field are aware of the importance of his contribution in fractional calculus. It could be interesting to understand whether many of them know why a fractional derivative is entitled to him, but this is another story.

For different reasons, the authors of the two cited papers will not publish any more results explicitly amenable to the fractional derivative. Thus, accepting the previous interpretation, they

appear as isolated points in the *mare magnum* of the fractional calculus, where the more popular names are nowadays others.

In Marchaud's doctoral thesis, see p. 47, Section 27 Formula (23) in [2], or the definition (23) in the published paper [4] at p. 383, he defined the following fractional differentiation for sufficiently regular real functions $f : (0,1) \rightarrow \mathbb{R}$ extended with 0 for $x \leq 0$, whenever $\alpha \in (0,1)$:

$$\mathbf{D}^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt.$$

This definition can be easily given for a function defined in all of $\mathbb{R}$ and for every $\alpha \in (0,1)$ distinguishing two types of derivative, see [3], respectively from the right and from the left:

$$\mathbf{D}^\alpha_+ f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt$$

and

$$\mathbf{D}^\alpha_- f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(x) - f(x+t)}{t^{1+\alpha}} dt.$$

The construction of these operators will be briefly described in the next section following the original motivation contained in Marchaud's thesis.

The problem of giving a coherent definition of derivative of a function for all positive real numbers has a long history—for instance, see [3,5–8] for some detailed information. In any case, Abel's contribution for solving the tautochrone problem, [9], and the work by Liouville [10] and Riemann [11] in application to geometry are fundamental and well known at the beginning of the fractional calculus. Many other authors have written papers that contributed to improving the knowledge of this subject. Nevertheless, I think that a very special role has to be recognized to Hermann Weyl because, probably following the path traced by Riemann, as Weyl himself writes in [1], he introduced, maybe first, the nonlocal operator that is known as Marchaud derivative, for people who know it, in a significative, even if particular, case. We shall dedicate Section 5 to this aspect.

Our interest to this subject comes out after the celebrated contribution given by the paper [12]. Indeed, the authors developed an idea that was already contained in [13]. In any case, in [12], the authors dedicated their interest to a different type of nonlocal operator with respect to the fractional derivative: the fractional Laplace operator. In particular, in [12], a different perspective in the interpretation of the nonlocal operators was introduced using a method based on an *extension* approach (see also [13]). We do not want to bore the reader too much with this subject. However, some words are in order. Heuristically, following the extension approach, idea it is possible to deduce the properties of a nonlocal operator from the ones of a local operator. In [12], the authors were concerned with the fractional Laplace operator, while the local operator obtained after the extension construction was a degenerate elliptic operator in divergence form. This approach can be developed considering the solution of ad hoc Dirichlet problem formulated in an unbounded set, where an auxiliary variable has been added, and then taking the limit of a weighted normal derivative of the solution of the Dirichlet problem, when this auxiliary variable vanishes. The scientific follow up of [12] produced an enormous amount of papers. Moreover, in [14], such an idea was generalized considering an abstract approach in a very powerful way. Following this stream of ideas, in [15], an intrinsic characterization of the fractional sub-Laplace operators in Carnot groups was obtained. Roughly speaking, the operators considered in this last case are sums of squares of smooth vector fields satisfying the Hörmander condition in a non-commutative structure.

The approach described in [12,14], and then in [15], was also extended to the case of fractional operators in [16] and independently also, as very often it happens when the time is ripe, in [17]. Indeed, with this aim, commenting for instance [17], we faced the problem of defining the Marchaud derivative via an extension approach in order to obtain a Harnack inequality for solutions of homogeneous fractional equations. As a consequence of this research, we realized in particular

that Marchaud derivative and Weyl derivative have been, in a sense, perhaps a little put aside in the last time, especially considering the great development and the large popularity that research about nonlocal operators has recently had. This last remark is essentially based on the popularity of other fractional derivatives, for instance the Riemann–Liouville derivative or the Caputo derivative (see [7,18,19]) for a modern approach to these operators. Indeed, see also [20] for a recent example of application involving Caputo derivative.

On the other hand, by reading the monumental opera [3], it is possible to verify, as we pointed out at the beginning of this introduction, that Weyl and Marchaud names are cited many times. Thus, the curiosity of explaining this situation was strong. Why do only few people associate Weyl and Marchaud names to the fractional subject? More precisely, why do only few people utilize these fractional derivatives for applications, simply preferring other definitions, even if it appears natural to use Weyl and Marchaud operators? We do not have any conclusive reply. In any case, it is quite difficult to understand what the true motivations of this apparent amnesia are. Of course, the specialists of the fractional calculus know Marchaud and Weyl derivatives in reading in particular [3] where the right tribute to both of these mathematicians has been given. Perhaps only recently a new awakened interest about these definitions has spread out. For this reason we think, hopefully, that the contribution of this paper might be useful in consolidating this new trend.

Anyhow, a partial reply to the previous questions, partially related with beginning of these events, can be found in the social contour that strongly influenced the life of the two mathematicians. The period during which this research was developed was very uproarious for Europe. Weyl's paper [1] dated back to 1917, Marchaud's thesis [2] was published in 1927 and both the lives of these two people were, for different reasons, affected by the two world wars events (see e.g., [21] for some biography details about Weyl's life and [22,23] for some information about Marchaud).

In this paper, we want to analyze the definition of fractional derivative given by Weyl and Marchaud, concentrating on those aspects that, in perspective, seem to be more flexible for generalizing to other situations the notion of nonlocal operator (see e.g., [24] for facing the case of the semigroup approach in its abstract generality and then for recalling the contribution given in [14], for fractional Laplace operators, and then recalling [25] for several generalizations in an abstract approach, including in principle: Riemannian manifolds, Lie groups, infinite dimensions and non-symmetric operators, see also [15] for the particular case of Carnot groups). We would also like to point out some recent research establishing few relationships between Marchaud derivative and some nonlocal operators in non-commutative structures (see [26]).

With this aim, we summarize the plan of the paper. After this introduction, the reader can find in Section 2 some basic biographic information about Weyl and Marchaud. In Section 3, we discuss briefly how Marchaud came to define his derivative followed by reviewing a part of his Ph.D. thesis. In Section 4, we recall the basic idea already developed earlier by Grünwald and Letnikov that is at the base of the fractional derivative given by Weyl and Marchaud. In Section 5, we recall the seminal Weyl's paper [1] discussing some details about the relationship between his contribution and Marchaud's derivative. In Section 6, we comment the basic ideas of the respective definitions. We face the modern general setting of Marchaud derivative in Section 7 also making some remarks about its properties with respect to partial differential equations. In Section 8, we continue our work by recalling the definition of fractional Laplace operator, while, in Section 9, we deal with the definition of Marchaud derivative via an extension approach and eventually, in Section 10, we conclude our effort making evident the relationship between the fractional Laplace operator and the Marchaud derivative.

In order to outline a few aspects of Weyl's and Marchaud's biographies, we list below only some key facts of the period during which the results about fractional derivative were written. For further curiosities or remarks, we suggest consulting [21–23].

Closing this introduction, we remark that, from a historical point of view, it would be interesting to deepen our knowledge of these two characters of the mathematical world, especially considering the

influence and the role of the respective mathematical schools compared to the other mathematicians of their time and their scientific legacy.

2. Short Historical Placement

In this section, we introduce some information about the lives of Weyl and Marchaud, mainly regarding the period of publication of their papers on fractional derivative without pretending to consider this parallel description exhaustive.

Hermann Weyl was born in Germany in 1885. André Paul Marchaud was born in France in 1887.

In 1913, Weyl was professor at the ETH (Swiss Federal Institute of Technology) in Zürich where he interacted also with Einstein. In 1915, Weyl was called up for military service in Germany, but, since 1916, he was exempted from military duties for health reasons. Later on, he came back to Germany as a successor of Hilbert in Göttingen, but, in 1933, he left to go to the *Institute for Advanced Study* in Princeton, escaping from the Nazi regime, where he continued his brilliant career (see e.g., [21] until he died in Zürich in 1955).

In 1913, Marchaud had not gotten his PhD thesis yet, probably because of his health problems. He was professor in a lyceum when he was mobilized by the French army in 1914. In the same year, Marchaud was taken prisoner. He stayed in an *Oflag* (a prison camp for officers only) from 1914 to 1918, at the beginning in Germany and then, thanks to the help of the Red Cross who intervened because he was ill, since 1917 in Switzerland. Marchaud discussed his PhD thesis later on, only in 1927. He continued his career mainly serving as *Rector* (provost) of French universities, even during the Nazi occupation of France in the Second World War, until 1957 (see [22]), when he retired. He died in Paris in 1973.

3. The Marchaud Approach

As we have already announced, in this section, we represent the Marchaud approach following the main steps of a part of his PhD thesis.

The Liouville–Riemann integral of order $\alpha > 0$ of a function $f : [a, b] \to \mathbb{R}$ is defined as:

$$I_a^{(\alpha)} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^{x-a} t^{\alpha-1} f(x-t) dt.$$

In this case, the derivative of order $\alpha < n$ where n is a positive integer is defined by

$$\mathcal{D}_a^\alpha f(x) = D^n I_a^{n-\alpha} f(x).$$

In fact, this definition is well posed because it is independent to n. In particular, it is coherent since the following fundamental identity holds for all the functions in $L^1([a,b])$ that are bounded, for every $\alpha, \alpha' > 0$:

$$I_a^{(\alpha)}[I_a^{(\alpha')} f(x)] = I_a^{(\alpha+\alpha')} f(x).$$

The point of starting by Liouville, as Marchaud observed in [4], is that, if $a = -\infty$, then

$$\mathcal{D}_{-\infty}^\alpha e^{kx} = k^\alpha e^{kx}.$$

Moreover, for $\beta \geq 0$, it results that

$$I_a^\beta(1) = \frac{(x-a)^\beta}{\Gamma(\beta+1)},$$

and in particular $I_a^0(1) = 1$, so that, for every $\beta \geq 0$, and for every $\alpha > 0$, we get

$$\mathcal{D}_a^{\alpha} I_a^{\beta}(1) = D^{(n)} I_a^{n-\alpha} I_a^{\beta}(1) = D^{(n)} I_a^{n-\alpha+\beta} 1 = D^{(n)} \frac{(x-a)^{n-\alpha+\beta}}{\Gamma(n-\alpha+\beta+1)} = \frac{(x-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}.$$

Hence, for $\beta = 0$,

$$\mathcal{D}_a^{\alpha}(1) = \mathcal{D}_a^{\alpha} I_a^0(1) = \frac{(x-a)^{-\alpha}}{\Gamma(-\alpha+1)}.$$

As a consequence, the fractional derivative of order α for α non-integer is in general infinite for $x = a$.

Trying to define the fractional derivative as the fractional integral of negative order α, we obtain a divergent integral. In fact, formally, we should obtain

$$I_a^{(-\alpha)} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^{x-a} t^{-\alpha-1} f(x-t) dt.$$

Then, Marchaud argues in this way. Taking the integral of order α and assuming to consider the function extended with 0 from $-\infty$ to a, we get that

$$I_{-\infty}^{(\alpha)} f(x) \Gamma(-\alpha) = \int_0^{\infty} t^{\alpha-1} f(x-t) dt,$$

that is, making clear the definition of Γ,

$$I_{-\infty}^{(\alpha)} f(x) \int_0^{\infty} t^{\alpha-1} e^{-t} dt = \int_0^{\infty} t^{\alpha-1} f(x-t) dt.$$

The same formula holds for every positive integer k, so that performing a change of variable like $t = ks$ in both the integrals we get:

$$I_{-\infty}^{(\alpha)} f(x) k \int_0^{\infty} (ks)^{\alpha-1} e^{-ks} ds = \int_0^{\infty} (ks)^{\alpha-1} f(x-ks) k ds,$$

which implies

$$I_{-\infty}^{(\alpha)} f(x) \int_0^{\infty} s^{\alpha-1} e^{-ks} ds = \int_0^{\infty} s^{\alpha-1} f(x-ks) ds.$$

Then, taking a linear combination od order $p+1$ for a finite sequence of integer positive decreasing number $\{k_i\}_{0 \le i \le p}$, we obtain summing terms by terms

$$I_{-\infty}^{(\alpha)} f(x) \int_0^{\infty} s^{\alpha-1} \psi(s) ds = \int_0^{\infty} s^{\alpha-1} \varphi(x,s) ds,$$

where

$$\psi(s) = \sum_{i=0}^{p} C_i e^{-k_i s}, \quad \varphi(s) = \sum_{i=0}^{p} C_i f(x-k_i s),$$

and $\{C_i\}_{1 \le 0 \le p} \subset \mathbb{R}$. At this point, Marchaud asks that passing to negative exponent $-\alpha$ the following relation makes sense

$$I_{-\infty}^{(-\alpha)} f(x) \int_0^{\infty} s^{-\alpha-1} \psi(s) ds = \int_0^{\infty} s^{-\alpha-1} \varphi(x,s) ds,$$

calling the function $I_{-\infty}^{(-\alpha)} f(x)$ the fractional derivative of order α, that is $\mathcal{D}^{\alpha} f(x)$ is implicitly defined by

$$\mathcal{D}^{\alpha} f(x) \int_0^{\infty} s^{-\alpha-1} \psi(s) ds = \int_0^{\infty} s^{-\alpha-1} \varphi(x,s) ds,$$

supposing that it is possible to choose ψ in such a way $\gamma(\alpha) := \int_0^{\infty} s^{-\alpha-1} \psi(s) ds$ does not vanish, and, as a consequence, obtains the expression of φ. Discussing this problem, Marchaud finds that if it is possible to find ψ and φ with previous properties, then

$$\gamma(\alpha)\mathcal{D}^{\alpha}f(x) = \int_0^{\infty} s^{-\alpha-1}\varphi_{\alpha}(x,s)ds,$$

where

$$\varphi_{\alpha}(x,s) = \sum_{i=0}^{p} C_{k_i}f(x - k_is),$$

with a possible choice for ψ that it is given by

$$\psi(t) = e^{-t}(1 - e^{-t})^p = \sum_{j=0}^{p}(-1)^j \binom{p}{j}e^{(-1-j)t}$$

and

$$\varphi_{\alpha}(x,s) = \sum_{i=1}^{p}(-1)^{j-1}\binom{p}{j}f(x - js).$$

After a detailed computation, Marchaud concludes that the existence of the fractional derivative $\mathcal{D}^{\alpha}f(x)$ continuous for continuous functions defined in (a,b) is equivalent to the uniform convergence of the following integral

$$\int_{\epsilon}^{\infty} s^{-1-\alpha}\varphi(x,s)ds$$

in every interval $(a',b) \subset (a,b)$ as $\epsilon \to 0^+$, and it is independent from the choice of the positive numbers $\{k_i\}_{1 \le i \le p}$. At this point Marchaud defines the fractional derivative of order $\alpha < p$ of a function defined in all of $\mathbb{R}$ implicitly:

$$\mathbf{D}^{\alpha}f(x)\int_0^{\infty} s^{-1-\alpha}e^{-s}(1 - e^{-s})^p ds = \int_0^{\infty} s^{-\alpha-1}\sum_{j=1}^{p}(-1)^{j-1}\binom{p}{j}f(x - js)ds,$$

or taking $\psi(t) = (1 - e^{-t})^p - (1 - e^{-2t})$, it is possible to obtain

$$\mathbf{D}^{\alpha}f(x)\int_0^{\infty} s^{-1-\alpha}\left((1 - e^{-s})^p - (1 - e^{-2s})^p\right)ds = \int_0^{\infty} s^{-\alpha-1}(\Delta_{-s}^p f(x) - \Delta_{-2s}^p f(x))ds,$$

where

$$\Delta_{-s}^p f(x) = \sum_{j=0}^{p}(-1)^j\binom{p}{j}f(x - (p-j)s).$$

Hence, separating the integral, remarking that

$$\int_0^{\infty} s^{-1-\alpha}(1 - e^{-2s})^p ds = 2^{\alpha}\int_0^{\infty} t^{-1-\alpha}(1 - e^{-t})^p dt,$$

and

$$\int_0^{\infty} s^{-\alpha-1}\Delta_{-2s}^p f(x)ds = 2^{\alpha}\int_0^{\infty} t^{-\alpha-1}\Delta_{-t}^p f(x)dt,$$

we obtain

$$(1 - 2^{\alpha})\mathbf{D}^{\alpha}f(x)\int_0^{\infty} s^{-1-\alpha}(1 - e^{-s})^p ds = (1 - 2^{\alpha})\int_0^{\infty} s^{-\alpha-1}\Delta_{-s}^p f(x)ds.$$

As a consequence, we also obtain this representation

$$\mathbf{D}^{\alpha}f(x)\int_0^{\infty} s^{-1-\alpha}(1 - e^{-s})^p ds = \int_0^{\infty} s^{-\alpha-1}\Delta_{-s}^p f(x)ds. \tag{1}$$

4. Grünwald–Letnikov Derivative

It is impossible to deal with fractional Marchaud derivative without recalling the contribution of Grünwald and Letnikov (see [27,28]). In fact, for giving a different perspective of the Marchaud derivative, we have to introduce the Grünwald–Letnikov derivative (Indeed, from this point of view, after we had completed this manuscript, Francesco Mainardi pointed out the survey paper [29] dedicated to Marchaud and Grünwald–Letnikov derivatives). To do this, we need some new notation.

We recall that the binomial coefficients can be defined for every $\alpha \in \mathbb{C}$ and $n \in \mathbb{N} \cup \{0\}$ as:

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \frac{(-1)^n(-\alpha)_n}{n!}, \quad n \in \mathbb{N}. \tag{2}$$

It is also true that

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha+n-1)}$$

for $\alpha \in \mathbb{C} \setminus -\mathbb{N}$ and $n \in \mathbb{N}$.

We introduce now the following notation concerning the difference of fractional order $\alpha \in \mathbb{R}$ for a function f as follows. Let us denote

$$(\Delta_h^\alpha f)(x) = \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} f(x-kh).$$

We are now in position to define the Grünwald–Letnikov fractional derivative (see [3,27,28]). Let $\alpha \in (0,1)$ be fixed and let $f : \mathbb{R} \to \mathbb{R}$ be a given function. The Grünwald–Letnikov derivative of order α of f is defined, separating the two cases, respectively as:

$$f_+^{\alpha)}(x) = \lim_{h \to 0^+} \frac{(\Delta_h^\alpha f)(x)}{h^\alpha}$$

and

$$f_-^{\alpha)}(x) = \lim_{h \to 0^+} \frac{(\Delta_{-h}^\alpha)f(x)}{h^\alpha},$$

whenever the limit exists.

In order to understand better the reason of this definition, we introduce the following definition.

Definition 1. *We define a non-centered difference of increment h on $f : \mathbb{R} \to \mathbb{R}$, as*

$$(I - \tau^{-t})f(x) = f(x) - f(x-t).$$

Then, we obtain for every $m \in \mathbb{N}$ so that

$$(I - \tau^{-t})^m = \sum_{k=0}^m (-1)^k \binom{m}{k} (\tau^{-t})^k$$

and

$$(I - \tau^{-t})^m f(x) = \sum_{k=0}^m (-1)^k \binom{m}{k} (\tau^{-t})^k f(x) = \sum_{k=0}^m (-1)^k \binom{m}{k} f(x - kt).$$

On the other hand, taking the Taylor expansion of the function $t \to (1+t)^\alpha$ in the center $t_0 = 0$ and $\alpha \in (0,1)$, we get

$$(1+t)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} t^k,$$

where

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \frac{(-1)^n(-\alpha)_n}{n!}, \quad n \in \mathbb{N}.$$

Thus, we can extend our definition to the fractional case, and it is possible to define for $\alpha \in (0,1)$

$$(I - \tau^{-t})^\alpha f(x) = \sum_{k=0}^{+\infty}(-1)^k \binom{\alpha}{k}(\tau^{-t})^k f(x) = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} f(x - kt).$$

In this way, we still maintain the semigroup property for the $\Delta_h^\alpha = (I - \tau^{-h})^\alpha$, because for every $\alpha_1, \alpha_2 \in \mathbb{R}$

$$\Delta_h^{\alpha_1}\Delta_h^{\alpha_2} = (I - \tau^{-h})^{\alpha_1}(I - \tau^{-h})^{\alpha_2} = (I - \tau^{-h})^{\alpha_1+\alpha_2}$$

and $(I - \tau^{-h})^0 = I$. Here, we simply discuss the case of $\Delta_{+,h}^\alpha$ for $\alpha \in (0,1)$, but the results may be generalized to different exponents.

Moreover, the following result holds, see e.g., [7].

Theorem 1. *Let $\alpha, \beta > 0$. Then, for every bounded function:*

$$\Delta_h^\alpha \Delta_h^\beta f = \Delta_h^{\alpha+\beta} f.$$

In addition, considering one more time [7], and recalling also the contribution given in [30], we have that:

Theorem 2. *Let $\alpha > 0$. Then, for every $f \in L^1(\mathbb{R})$*

$$\mathcal{F}(\Delta_h^\alpha f)(x) = (1 - e^{ixh})^\alpha \mathcal{F}(f)(x).$$

In particular, it is true that the Grünwald–Letnikov derivative of order $\alpha \in (0,1)$ coincides with the Marchaud derivative of the same order. Indeed, in consideration of the two previous trivial properties, the following result holds.

Theorem 3. *Let $f \in L^p(\mathbb{R})$, $p \geq 1$. Then, for every $q \geq 1$, there exist*

$$f_\pm^{(\alpha)}(x) = \lim_{h \to 0, \, in \, L^q} \frac{\Delta_{\pm h}^\alpha f(x)}{h^\alpha}$$

and

$$\mathbf{D}_\pm^\alpha f(x) = \lim_{\epsilon \to 0, \, in \, L^q} C(\alpha) \int_\epsilon^{+\infty} \frac{f(x) - f(x \mp h)}{h^{1+\alpha}} dh.$$

Moreover,

$$f_\pm^{(\alpha)}(x) = \mathbf{D}_\pm^\alpha f(x),$$

independently from p and q.

The proof is quite long and can be found in [3], Theorem 20.4. Moreover, about this topic, we recall the very recent contribution [31]. By the way, this last paper can be considered also as a further signal of the renascent interest for a Marchaud derivative. In fact, in that manuscript, it has been recently proved the coincidence of the Marchaud derivative and the Grünwald–Letnikov derivative for functions in Hölder spaces with explicit rates of convergence. Previous results encode many facts. The first concerns the commutativity of the Grünwald–Letnikov derivative as well as the Marchaud derivative, namely $(f^\alpha))^\beta) = (f^\beta))^\alpha) = f^{\alpha+\beta}$ and $\mathbf{D}^\alpha\mathbf{D}^\beta = \mathbf{D}^\beta\mathbf{D}^\alpha = \mathbf{D}^{\alpha+\beta}$.

5. Weyl Derivative

Hermann Weyl's name is associated with many important scientific results in physics and mathematics. In particular, concerning fractional derivative, Weyl made an important contribution that is strictly linked to the Marchaud derivative. By the truth Weyl introduced in its paper [1], in p. 302, exactly the definition of the fractional derivative that Marchaud gave in [4]. The paper written by Weyl appeared in 1917, while the Marchaud thesis was published in 1927 (see [2,4]). It is not clear if the two definitions were discovered independently. The cited Weyl paper, whose title is *Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung* (remarks on the notion of the differential quotient of a broken order), concerns the notion of a fractional derivative. In the introduction of his paper, Weyl recognizes at first the efforts made by Bernanrd Riemann for obtaining a notion of a derivative for every positive real number. In particular, Weyl cited the contents of the unpublished Riemann notes reported in the XIX paper of the published Riemann opera post. About this fact, Weyl recalls, as the editor of that volume, remarks that Riemann surely did not think that those computations would have been published, at least in that form. In any case, Weyl faces the problem of starting from those notes and having in mind that he wants to obtain a definition that works for periodic functions. In order to avoid the problem of introducing some privileged points, as very often happens in literature concerning fractional derivatives, he assumes that periodic functions have to have a zero mean. We do not enter into the details here (see Section); however, Weyl uses the properties of Fourier series and, on p. 302 [1], the following relationship

$$g(x) = \beta \int_0^\infty \frac{f(x) - f(x - \xi)}{\xi^{1+\beta}} d\xi \tag{3}$$

for functions f Hölder having modulus of continuity with exponent α and $\alpha > \beta$ and knowing that g denotes the fractional derivative of order β. The same argument was reported in [32] on p. 226 (see Formula (3) in IX, 9.81). Nevertheless, the previous formula, apparently, disappeared in the final version of the book [33] published later on, probably because the author was mainly interested in the periodic properties of the functions, but we do not have any proof of this statement.

Anyhow, also in [3], the definition of the Weyl derivative can be found. Starting from Fourier expansion of a periodic function, Weyl defines the kernel

$$\psi_\pm^\alpha(t) = \sum_{k=-\infty, k\neq 0}^{+\infty} \frac{e^{ikt}}{(\pm ik)^\alpha} = 2 \sum_{k=1}^\infty \frac{\cos(kt \mp \alpha \frac{\pi}{2})}{k^\alpha}.$$

Thus, the so-called Marchaud–Weyl derivative is defined as

$$\mathbf{D}_\pm^{(\alpha)} f(x) = \frac{1}{2\pi} \int_0^{2\pi} (f(x) - f(x-t)) \frac{d}{dt} \psi_\pm^{1-\alpha}(t) dt. \tag{4}$$

If the function f is 2π periodic, then it results that

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt = \frac{1}{2\pi} \int_0^{2\pi} (f(x) - f(x-t)) \frac{d}{dt} \psi_\pm^{1-\alpha}(t) dt.$$

The previous relationship has to be correctly expressed in the following sense (see Lemma 19.4 in [3]).

Proposition 1 (Lemma 19.4, [3]). *For every $f \in L^p(0, 2\pi)$, $1 \geq p < +\infty$, the following limits converge for almost every $x \in (0, 2\pi)$ simultaneously:*

$$\lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_\epsilon^{2\pi} (f(x) - f(x-t)) \frac{d}{dt} \psi_\pm^{1-\alpha}(t) dt,$$

$$\lim_{\epsilon \to 0^+} \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt,$$

and

$$\mathbf{D}_+^{(\alpha)} f(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_\epsilon^{2\pi} (f(x) - f(x-t)) \frac{d}{dt} \psi_\pm^{1-\alpha}(t) dt$$

$$= \lim_{\epsilon \to 0^+} \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt = \mathbf{D}_+^\alpha f(x).$$

By the way, concerning the parallel situation for the fractional Laplace operator on the torus, we point out [34], where similar results to the Proposition 1 have been proved.

We do not enter in the details concerning the question whether the definition of this type of fractional derivative has been invented by Weyl or Marchaud, or maybe by Riemann himself indeed as Weyl seems to suggest in the introduction of his paper [1]. Nevertheless the Formula (3) appeared in [1], as already written, ten years before the Marchaud thesis. In any case, Marchaud correctly cites the Weyl's paper [1]. More precisely Marchaud at p. 50 of his thesis acknowledges to Weyl to have obtained the result in the case of dimension $n = 1$, by referring to the representation (6). In addition Marchaud also admits that Weyl's approach was more powerful with respect to the one established by Montel, see [35]. Montel approach used polynomial approximation, as Marchaud stated. On the contrary, Marchaud remarked, that Weyl's approach is more direct. In [3], XXXIII, the authors faced indirectly that question in the note dedicated to the historical outline of the subject. There, they explained that Formula (6) appeared earlier in [1] by accident. Nevertheless, they concluded that Weyl did not develop his idea, as on the contrary Marchaud did in [2,4]. It would be interesting to know if any interaction between Weyl and Marchaud happened. In any case the Weyl's paper [1] is not one of the most cited among all the important results obtained by Weyl during his fruitful career.

6. Basic Ideas

If we compare the Marchaud derivative with respect to the Riemann–Liouville one, we immediately realize that, in the latter one, the classical derivative operator appears, while, in the first one, it does not. This is one of the key points that Marchaud's definition makes evident. That is, Marchaud derivative avoids applying the classical derivative after an integration in order to define the fractional operator. In a sense, this approach recalls the one that has characterized the Sobolev's approach (see, for instance [36], and, in in a sense, it could be considered as precursive of the notion of a weak solution to a PDE). In fact, roughly speaking, we recall that Sobolev's approach is based on the integration of both sides of an equation. In this way, we reduced looking for functions that satisfy the obtained integral equation.

In this order of ideas, in the Riemann–Liouville definition, the classical derivation still appears. On the contrary, in the Marchaud derivative, we simply recognize a singular integral where the reminiscence of the derivative is given by the kernel that multiplies the difference between the values of the function in two points. On the other hand, Marchaud's definition includes the Riemann–Liouville's one when the initial point is $-\infty$ and the functions are sufficiently smooth. We come back to this aspect later on in the section. From a philosophical point of view, the Marchaud derivative seems to make evident its non-local character. On the contrary, in the initial historical approach described by the Riemann–Liouville derivative, the classical derivative operator, which is a local object, still remains.

For instance, by considering a function defined in all of $\mathbb{R}$ and having a minimal smoothness, we, in principle, can modify its definition locally, for example simply changing its derivative in a *small* set of points. Nevertheless, the remaining part of the function is not affected by this modification. On the contrary, the Marchaud derivative, but also the Riemann–Liouville one, even if in a spurious way, determines a quantity that heavily depends on the modified function. This fact is evident thanks to the presence of the integral operator. Summarizing, by modifying the given function even only in a small

set, the value of the fractional derivative will change, in general, in all the points where this fractional derivative will be evaluated.

Now, we comment separately on the Marchaud derivative and Weyl derivative.

6.1. Marchaud Derivative

The Marchaud derivative acts like an operator that associates to a function a new function that in general does not maintain local properties like the differential (of the function) do far away to the set where the function has been modified. Nevertheless, this operator, the fractional one, in a sense, still contains the classical derivative. Indeed, the classical derivative materializes as a particular (let say like an exception) case who realizes when the order of the fractional derivative goes to an integer. This focusing phenomenon is particularly interesting.

In order to clarify this remark, let us consider the Definition (1), in the case $p = 2$ and $\alpha = 1$. Then, we obtain:

$$\mathbf{D}^\alpha f(x) \int_0^\infty s^{-2}(1 - e^{-s})^2 ds = \int_0^\infty s^{-\alpha-1} \left(f(x) - f(x-s) + f(x-2s) \right) ds, \tag{5}$$

and since

$$\int_0^\infty s^{-2}(1 - e^{-s})^2 ds = 2\log 2,$$

we get:

$$\mathbf{D}f(x) = \frac{1}{2\log 2} \int_0^\infty \frac{f(x) - 2f(x-s) + f(x-2s)}{s^2} ds.$$

It is worth saying that here we have the value of the classical derivative in a point represented via an integral! Let us say: from the global to the local. How to explain this fact? We remark that, if f is a C^2 function with compact support or even $f \in \mathcal{S}(\mathbb{R})$, then

$$\mathcal{F}\left(\int_0^\infty \frac{f(x) - 2f(x-s) + f(x-2s)}{s^2} ds \right) = \mathcal{F}f(\xi) \int_0^\infty \frac{(1 - e^{-is\xi})^2}{s^2} ds$$
$$= 2\log 2 (i\xi) \mathcal{F}f(\xi) = 2\log 2 \mathcal{F}(f').$$

This implies, recalling that the Fourier transform is invertible on Schwartz space $\mathcal{S}(\mathbb{R})$, that, for every $f \in \mathcal{S}(\mathbb{R})$, Formula (5) truly gives a representation of the derivative of a function in a point. We shall come back in Section 7 on this fact. On the other hand, the relationship (5) is correctly defined in a larger space of functions with respect to $\mathcal{S}(\mathbb{R})$.

In the case $p = 1$, $\alpha < 1$

$$\mathbf{D}^\alpha f(x) \int_0^\infty s^{-1-\alpha}(1 - e^{-s}) ds = \int_0^\infty s^{-\alpha-1} \left(f(x) - f(x-s) \right) ds,$$

but

$$\int_0^\infty s^{-1-\alpha}(1 - e^{-s}) ds = [-\alpha^{-1}s^{-\alpha}(1 - e^{-s})]_{s=0}^{s=\infty} + \alpha^{-1} \int_0^\infty s^{-\alpha}e^{-s} ds = \alpha^{-1}\Gamma(1-\alpha).$$

As a consequence,

$$\mathbf{D}^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x-s)}{s^{1+\alpha}} ds \tag{6}$$

and, even in this case, the easier case among Marchaud derivatives concerning the function f for $\alpha \in (0,1)$, we can read the non-locality of this definition and, in addition, for sufficiently smooth functions,

$$\lim_{\alpha \to 1^-} \mathbf{D}^\alpha f(x) = Df(x).$$

In this case, an important role is played by the normalizing constant $\frac{\alpha}{\Gamma(1-\alpha)}$ that multiplies the integral in the definition of Marchaud derivative.

The fact that, for sufficiently "good" functions, the fractional derivative $\mathbf{D}^\alpha f$ coincides with the Riemann–Liouville derivative

$$\mathcal{D}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} \frac{f(t)}{(x-t)^\alpha}$$

can be checked straightforwardly. Moreover, the definition given by Marchaud can be applied even for functions that may grow at infinity less than α. On the contrary, the definition of the Liouville derivative is less flexible since it does not admit (see p. XXXIII [3]) being applied to constant functions.

Let us check that the Marchaud derivative $\mathbf{D}_+^\alpha f$ coincides with the Riemann–Louville derivative from the right. In fact, since

$$\mathcal{D}_+^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^{+\infty} \frac{f(x-t)}{t^\alpha} dt,$$

and supposing that $f \in C^1(\mathbb{R})$ and $f = o(|x|^{\alpha-1-\epsilon})$, $x \to +\infty$ for $\epsilon > 0$, then by Lebesgue dominated convergence theorem first and then integrating by parts, we get:

$$
\begin{aligned}
\mathcal{D}_+^\alpha f(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f'(x-t)}{t^\alpha} dt = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} f'(x-t) \left(\int_t^\infty \tau^{-\alpha-1} d\tau \right) dt \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\epsilon \to 0^+} \left\{ [-f(x-t) \left(\int_t^{+\infty} \tau^{-\alpha-1} d\tau \right)]_{t=\epsilon}^{+\infty} - \int_\epsilon^{+\infty} \frac{f(x-t)}{t^{1+\alpha}} dt \right\} \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\epsilon \to 0^+} \left\{ f(x-\epsilon) \left(\int_\epsilon^{+\infty} \tau^{-\alpha-1} d\tau \right) - \int_\epsilon^{+\infty} \frac{f(x-t)}{t^{1+\alpha}} dt \right\} \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\epsilon \to 0^+} \left\{ \int_\epsilon^{+\infty} \frac{f(x)-f(x-t)}{t^{1+\alpha}} dt + (f(x-\epsilon)-f(x)) \int_\epsilon^{+\infty} \tau^{-\alpha-1} d\tau \right\} = \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{f(x)-f(x-t)}{t^{1+\alpha}} dt = \mathbf{D}_+^\alpha f(x)
\end{aligned}
$$

(7)

because there exists $\eta \in]x-\epsilon, x[$ such that, as $\epsilon \to 0$:

$$\left| \alpha(f(x-\epsilon)-f(x)) \int_\epsilon^{+\infty} \tau^{-\alpha-1} d\tau \right| = |\alpha f'(x-\eta)| \leq \sup_{\tau \in [x-\epsilon, x]} |f'(\tau)| \epsilon^{1-\alpha} \to 0.$$

Thus, from this point of view, the Marchaud derivative is a sort of weaker version of the Riemann–Liouville derivative.

For example, constants satisfy $\mathbf{D}_+^\alpha f(x) = 0$ in the Marchaud sense, even if we can not consider, in all of $\mathbb{R}$, the Riemann–Liouville derivative of a constant. In fact, the parallel integral is divergent. This is, of course, absolutely unpleasant! Indeed, both Marchaud and Weyl were motivated also from this fact in order for looking for a different type of definition of fractional derivative.

We also think that the Marchaud derivative as some further properties that have to be better understood in its application. In order to focus on one of these aspects, we remark (see also [26]), that the sum of the two Marchaud derivatives ($\mathbf{D}_+^\alpha f$ and $\mathbf{D}_-^\alpha f$) gives, in a sense, the Riesz derivative in one dimension, namely the fractional Laplace operator in dimension 1. More precisely:

$$\mathbf{D}_+^\alpha f(x) + \mathbf{D}_-^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \frac{2f(x)-f(x-t)-f(x+t)}{t^{1+\alpha}} dt$$

or

$$\mathbf{D}^\alpha_+ f(x) + \mathbf{D}^\alpha_- f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \left(\int_0^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt + \int_0^{+\infty} \frac{f(x) - f(x+t)}{t^{1+\alpha}} dt \right),$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \left(\int_0^{+\infty} \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt + \int_{-\infty}^0 \frac{f(x) - f(x-t)}{|t|^{1+\alpha}} dt \right),$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{+\infty} \frac{f(x) - f(x-t)}{|t|^{1+\alpha}} dt = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{+\infty} \frac{f(x) - f(\xi)}{|x - \xi|^{1+\alpha}} d\xi,$$

$$= \frac{\alpha}{c(1, \frac{\alpha}{2}) \Gamma(1-\alpha)} \left(-\frac{d^2}{dx^2} \right)^{\frac{\alpha}{2}} f(x),$$

where $c(1, \frac{\alpha}{2})$ is the normalizing constant associated with the fractional Laplace operator $\left(-\frac{d^2}{dx^2} \right)^{\frac{\alpha}{2}}$, and whose value we will recall later on in this paper. This fact was implicitly remarked in [3] and it seems that it can be connected with the different type of variable considered. In case of $\mathbf{D}^\alpha_+ f$ and $\mathbf{D}^\alpha_- f$, the only one variable in $\mathbb{R}$ has a privileged direction in the two definitions of fractional derivatives. For instance, we can think of it as the time variable. On the contrary, considering the fractional Laplace operator

$$\left(-\frac{d^2}{dx^2} \right)^{\frac{\alpha}{2}} f$$

in $\mathbb{R}$ (the same also in $\mathbb{R}^n$), there is not any privileged direction. Namely, the space (in this case $\mathbb{R}$) is homogeneous so that the previous connection is particularly interesting.

6.2. Weyl Derivative

As far as Weyl's approach is concerned, the relationship between spectral theory and fractional derivative is explicit. Indeed, supposing that working with a 2π-periodic function as having a zero average, it is well known that the associated Fourier series is

$$\sum_{k=-\infty}^{+\infty} c_k e^{ikx},$$

where, of course, $\{c_k\}_{k\in\mathbb{Z}}$ denotes the sequence of Fourier coefficients.

Then, by computing formally the derivative of this series, we obtain

$$\sum_{k=-\infty}^{+\infty} c_k (ik) e^{ikx}.$$

It is obvious that defining a new function for a fixed $\alpha < 1$, as

$$\sum_{k=-\infty}^{+\infty} \frac{c_k}{(ik)^\alpha} e^{ikx},$$

we formally obtain taking then a derivative we obtain

$$D \left(\sum_{k=-\infty}^{+\infty} \frac{c_k}{(ik)^\alpha} e^{ikx} \right) = \sum_{k=-\infty}^{+\infty} \frac{c_k}{(ik)^{\alpha-1}} e^{ikx}. \tag{8}$$

In this way, Weyl defines the parallel fractional integral so that it is natural to define the fractional derivative of f as

$$\sum_{k=-\infty,}^{+\infty} c_k (ik)^\alpha e^{ikx}.$$

On the other hand, we recall that, given two periodic functions f, g, the new function

$$\frac{1}{2\pi} \int_0^{2\pi} g(t) f(x - t) dt$$

is represented by the Fourier series

$$\sum_{k=-\infty}^{\infty} g_k c_k e^{ikx},$$

where $\{g_k\}_{k \in \mathbb{Z}}$ and $\{c_k\}_{k \in \mathbb{Z}}$ are the respective Fourier coefficients.

As a consequence, considering

$$\sum_{k=-\infty}^{+\infty} \frac{c_k}{(ik)^\alpha} e^{ikx},$$

as representing the Fourier series of an integral like the following one:

$$\frac{1}{2\pi} \int_0^{2\pi} g(t) f(x - t) dt,$$

we deduce that previous integral has to be written in the following form:

$$\frac{1}{2\pi} \int_0^{2\pi} f(x - t) \left(\sum_{k=-\infty, k \neq 0}^{+\infty} \frac{e^{ikt}}{(ik)^\alpha} \right) dt.$$

Since it can prove that (see [3]) that

$$\sum_{k=-\infty, k \neq 0}^{+\infty} \frac{e^{ikt}}{(ik)^\alpha} = 2 \sum_{k=1}^{\infty} \frac{\cos(kt - \alpha \frac{\pi}{2})}{k^\alpha}.$$

Then, denoting the kernel

$$\psi_+^\alpha(t) := \sum_{k=-\infty, k \neq 0}^{+\infty} \frac{e^{ikt}}{(ik)^\alpha},$$

Weyl obtains the fractional integral

$$I_+^{(\alpha)} f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x - t) \psi_+^\alpha(t) dt.$$

At this point, by recalling (8), Weyl defines the fractional derivative as

$$\mathcal{D}_+^{(\alpha)}(x) = D \left(I_+^{(1-\alpha)} f \right)(x).$$

This definition corresponds to the Weyl-Riemann-Lioville version of this derivative, see [3] for the details. Then taking formally the derivative Weyl obtains the Weyl-Marchaud derivative, see also (4), discussed in Section 5:

$$\mathbf{D}_+^{(\alpha)} f(x) = \frac{1}{2\pi} \int_0^{2\pi} (f(x) - f(x - t)) \frac{d}{dt} \psi_+^{1-\alpha}(t) dt.$$

Of course, the case concerning $\mathbf{D}_-^{(\alpha)} f$ is analogous to the one just described for $\mathbf{D}_+^{(\alpha)} f$.

7. General Setting of Marchaud Derivative and Some Further Remarks

The definition of Marchaud derivative, as it is known since [3], can be extended to all $\alpha > 0$ in the following way, see [3,37]. Let $l \in \mathbb{N}$, $l \geq 1$ and $\alpha < l$. We define for every $f \in \mathcal{S}(\mathbb{R})$

$$\mathbf{D}_{\pm}^{\alpha} f(x) = \frac{1}{\chi(\alpha, l)} \int_0^{+\infty} \frac{\Delta_{\pm \tau}^l f(x)}{\tau^{1+\alpha}} d\tau,$$

where

$$\chi(\alpha, l) = \Gamma(-\alpha) A_l(\alpha) = \int_0^{+\infty} \frac{(1 - e^{-t})^l}{t^{1+\alpha}} dt,$$

$$A_l(\alpha) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} k^{\alpha}, \tag{9}$$

and

$$\Delta_{\pm \tau}^l f(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x \mp k\tau).$$

Of course, this definition can be generalized to the case of functions with several variables having a nice behavior both at infinity and locally, simply considering for every $\xi \in \mathbb{R}^n$, $\xi \neq 0$ and for every $\alpha > 0$ and $l \in \mathbb{N}$, $l \geq 1$ such that $\alpha < l$ and defining:

$$\mathcal{D}_{\pm,\xi}^{\alpha)} f(x) = \frac{1}{\chi(\alpha, l)} \int_0^{+\infty} \frac{\Delta_{\pm \tau, \xi}^l f(x)}{\tau^{1+\alpha}} d\tau,$$

where

$$\Delta_{\pm \tau, \xi}^l f(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x \mp k\tau\xi).$$

It is worth saying that $\lim_{\alpha \to l^-} \mathcal{D}_{\pm,\xi}^{\alpha)} f(x) = \pm D_{\xi}^l f(x)$, in the local (classical sense) and $\lim_{\alpha \to (l-1)^+} \mathcal{D}_{\pm,\xi}^{\alpha)} f(x) = \pm D_{\xi}^{l-1} f(x)$, where $\mathcal{D}_{\xi}^{0)} = I$, whenever f is sufficiently smooth (for example in $\mathcal{S}(\mathbb{R}^n)$).

In this way, it is possible to consider interesting representation of local operators. For example, denoting by e_i the vector of the canonic base of $\mathbb{R}^n$, for every $i = 1, \ldots, n$, we get

$$\frac{\partial f(x)}{\partial x_i} = \frac{1}{\chi(1, 2)} \int_0^{+\infty} \frac{\Delta_{\tau, e_i}^2 f(x)}{\tau^2} d\tau \tag{10}$$

and

$$\frac{\partial^2 f(x)}{\partial x_i^2} = \frac{1}{\chi(2, 3)} \int_0^{+\infty} \frac{\Delta_{\tau, e_i}^3 f(x)}{\tau^3} d\tau. \tag{11}$$

As a consequence for every $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial f(x)}{\partial x_i} = \Delta f(x) = \frac{1}{\chi(1, 2)} \int_0^{+\infty} \frac{\sum_{i=1}^{n} \left(\frac{\partial f(x)}{\partial x_i} + \sum_{k=1}^{2} (-1)^{k+1} k \binom{2}{k} \tau \frac{\partial f(x - k\tau\xi)}{\partial x_i} \right)}{\tau^2} d\tau \tag{12}$$

and

$$\Delta f(x) = \sum_{i=1}^{n} \frac{\partial^2 f(x)}{\partial x_i^2} = \frac{1}{\chi(2, 3)} \int_0^{+\infty} \frac{\sum_{i=1}^{n} \Delta_{\tau, e_i}^3 f(x)}{\tau^3} d\tau, \tag{13}$$

that is

$$\Delta f(x) = \frac{1}{\chi(2,3)} \int_0^{+\infty} \frac{\sum_{k=0}^3 (-1)^k \binom{3}{k} \sum_{i=1}^n f(x - k\tau e_i)}{\tau^3} d\tau. \tag{14}$$

From the Liouville Theorem, it is well known that there exists a unique function $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\Delta f(x) = 0$ in $\mathbb{R}^n$ that is $f = 0$. Thus, the unique function $f \in \mathcal{S}(\mathbb{R}^n)$ that satisfies

$$\frac{1}{\chi(2,3)} \int_0^{+\infty} \frac{\sum_{i=1}^n \Delta_{\tau,e_i}^3 f(x)}{\tau^3} d\tau = 0$$

has to be $f = 0$.

About the properties of the Marchaud derivative, we like to remind readers that, for every function $f \in \mathcal{S}(\mathbb{R})$,

$$\mathbf{D}f(x) = \frac{1}{2\log 2} \int_0^\infty \frac{f(x) - 2f(x - s) + f(x - 2s)}{s^2} ds.$$

On the other hand, for every $f, g \in \mathcal{S}(\mathbb{R})$, $fg \in \mathcal{S}(\mathbb{R})$, so that

$$\mathbf{D}(fg)(x) = \frac{1}{2\log 2} \int_0^\infty \frac{f(x)g(x) - 2f(x - s)g(x - s) + f(x - 2s)g(x - 2s)}{s^2} ds.$$

On the other hand, we know that $\mathbf{D}(fg)(x) = \mathbf{D}(f)(x)g(x) + \mathbf{D}(g)(x)f(x)$. Then, as a by-product, we obtain the following formula for every $f, g \in \mathcal{S}(\mathbb{R})$

$$\int_0^\infty \frac{f(x)g(x) - 2f(x - s)g(x - s) + f(x - 2s)g(x - 2s)}{s^2} ds$$
$$= \int_0^\infty \frac{f(x) - 2f(x - s) + f(x - 2s)}{s^2} ds\, g(x) + \int_0^\infty \frac{g(x) - 2g(x - s) + g(x - 2s)}{s^2} ds\, f(x).$$

Nevertheless, for instance, for every $\alpha \in (0,1)$ and for every $f, g \in \mathcal{S}(\mathbb{R})$, we get that $fg \in \mathcal{S}(\mathbb{R})$ and

$$\mathbf{D}^\alpha(fg)(x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{f(x)g(x) - f(x - t)g(x - t)}{t^{1+\alpha}} dt$$
$$= \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{f(x) - f(x - t)}{t^{1+\alpha}} dt\, g(x) + \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{f(x - t)(g(x) - g(x - t))}{t^{1+\alpha}} dt$$
$$= \mathbf{D}^\alpha f(x)g(x) + \mathbf{D}^\alpha g(x)f(x) - \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{(f(x) - f(x - t))(g(x) - g(x - t))}{t^{1+\alpha}} dt.$$

This remark implies that the usual differential rule for the product of two functions does not hold. Nevertheless,

$$\frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{(f(x) - f(x - t))(g(x) - g(x - t))}{t^{1+\alpha}} dt \to 0$$

whenever $\alpha \to 1^-$. In fact,

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{(f(x) - f(x-t))(g(x) - g(x-t))}{t^{1+\alpha}} dt$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\eta \frac{(f(x) - f(x-t))(g(x) - g(x-t))}{t^{1+\alpha}} dt$$

$$+ \frac{\alpha}{\Gamma(1-\alpha)} \int_\eta^\infty \frac{(f(x) - f(x-t))(g(x) - g(x-t))}{t^{1+\alpha}} dt$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} Df(x) Dg(x) \int_0^\eta t^{1-\alpha} dt + o(\eta^{2-\alpha})$$

$$+ \frac{\alpha}{\Gamma(1-\alpha)} \int_\eta^\infty \frac{(f(x) - f(x-t))(g(x) - g(x-t))}{t^{1+\alpha}} dt \to 0,$$

whenever $\alpha \to 1^-$, because $\frac{\alpha}{\Gamma(1-\alpha)} \to 0$, and there exists a positive constant such that

$$\left| \int_\eta^\infty \frac{(f(x) - f(x-t))(g(x) - g(x-t))}{t^{1+\alpha}} dt \right| \leq M$$

uniformly for every $\alpha \in (0, 1)$ and for every fixed $\eta > 0$.

In this way, we obtain one more time the classical rule for the usual derivative of order one because $\mathbf{D}^\alpha(fg)(x) \to \mathbf{D}(fg)(x)$ and $\mathbf{D}^\alpha f(x)g(x) + \mathbf{D}^\alpha g(x)f(x) \to \mathbf{D}f(x)g(x) + \mathbf{D}g(x)f(x)$ if $\alpha \to 1^-$.

This behavior is heuristically clear thinking to the fractional operator as a nonlocal object. That is, the fractional derivative in a point *measures* something that depends on all the values of the function before that point. Thus, it is in a sense expected that, for this type of operator, a term depending on the interplay of the quantity associated with the fractional derivative of the functions acting, has to appear. In the special case of $\alpha \to 1^-$, this third term appears with value 0 thanks to the locality of the quantity expressed by the classical derivative of order one. The Marchaud derivative of order α rescales with the law λ^α. In fact, we have that, for every $f \in \mathcal{S}(\mathbb{R})$, the function $x \to f(\lambda x) = f_\lambda(x)$ has the following behavior with respect to the Marchaud fractional derivative:

$$\mathbf{D}^\alpha f_\lambda(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(\lambda x) - f(\lambda(x-t))}{t^{1+\alpha}} dt$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(\lambda x) - f(\lambda x - \lambda t))}{t^{1+\alpha}} dt = \lambda^\alpha \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(\lambda x) - f(\lambda x - \tau)}{\tau^{1+\alpha}} d\tau$$

$$= \lambda^\alpha \mathbf{D}^\alpha f(\lambda x).$$

Remark also that, with respect to a different representation of the Marchaud fractional derivative, letting us see the case $\alpha < 2$, we get the same rescaling law:

$$\mathbf{D}^\alpha f_\lambda(x) = \frac{1}{\chi(\alpha, 2)} \int_0^\infty \frac{f(\lambda x) - 2f(\lambda(x-t)) + f(\lambda(x - 2t))}{t^{1+\alpha}} dt$$

$$= \frac{1}{\chi(\alpha, 2)} \int_0^\infty \frac{f(\lambda x) - 2f(\lambda x - \lambda t)) + f(\lambda x - 2\lambda t))}{t^{1+\alpha}} dt$$

$$= \lambda^\alpha \frac{1}{\chi(\alpha, 2)} \int_0^\infty \frac{f(\lambda x) - 2f(\lambda x - \tau) + f(\lambda x - 2\tau)}{\tau^{1+\alpha}} d\tau$$

$$= \lambda^\alpha \mathbf{D}^\alpha f(\lambda x).$$

The definition of the Marchaud derivative makes sense for a larger class of functions, with respect to the set $\mathcal{S}(\mathbb{R})$. For instance, all the constants have Marchaud derivative zero. The exponential function $x \to e^{\lambda x}$ does not belong to $\mathcal{S}(\mathbb{R})$. Nevertheless, for $\lambda \geq 0$ (here, we are using the Marchaud derivative

$\mathbf{D}^{\alpha}_{+}$, but, for the sake of simplicity, we omit writing the sign $+$), the function $e^{\lambda x}$ has Marchaud derivative and

$$\mathbf{D}^{\alpha}e^{\lambda x} = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{e^{\lambda x}-e^{\lambda(x-t)}}{t^{1+\alpha}}dt = e^{\lambda x}\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{1-e^{-\lambda t}}{t^{1+\alpha}}dt$$

$$= \lambda^{\alpha}e^{\lambda x}\frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{1-e^{-\tau}}{\tau^{1+\alpha}}d\tau = \lambda^{\alpha}e^{\lambda x}$$

because, integrating by parts, we obtain:

$$\int_0^{\infty}\frac{1-e^{-\tau}}{\tau^{1+\alpha}}d\tau = \frac{1}{\alpha}\int_0^{\infty}\frac{e^{-\tau}}{\tau^{\alpha}}d\tau = \frac{\Gamma(1-\alpha)}{\alpha}.$$

As a consequence, $e^{\lambda x}$ is solution of the fractional differential equation

$$\mathbf{D}^{\alpha}f(x) = \lambda^{\alpha}f(x).$$

8. Fractional Laplace Operator

The fractional Laplace operator can be represented in several ways. We should have to cite the contribution of many authors. We recall, for instance, [38–43]. Using the Fourier transform, for every $s \in (0,1)$ and for every $u \in \mathcal{S}(\mathbb{R}^n)$, the fractional Laplace operator is usually defined as

$$(-\Delta)^s u = \mathcal{F}^{-1}(\|\xi\|^{2s}\mathcal{F})u.$$

As a consequence, for every $u \in L^2(\mathbb{R}^n)$ if $\|\xi\|^{2s}\mathcal{F}u \in L^2(\mathbb{R}^n)$, then the fractional Laplace operator is defined by $\mathcal{F}^{-1}(\|\xi\|^{2s}\mathcal{F})u$.

On the other hand, for every $u \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0,1)$, we can define the operator

$$\mathcal{L}_s u(x) = c(\alpha,n)\int_{\mathbb{R}^n}\frac{f(x)-f(y)}{\|x-y\|^{n+2s}}dy := \lim_{\epsilon\to 0}c(s,n)\int_{\mathbb{R}^n\setminus B_{\epsilon}(x)}\frac{f(x)-f(y)}{\|x-y\|^{n+2s}}dy,$$

where $c(\alpha,n)$ is a normalizing constant, then $\mathcal{L}_s = (-\Delta)^s$ and

$$c(s,n) = \left(\int_{\mathbb{R}^n}\frac{1-\cos(\xi_1)}{\|\xi\|^{n+2s}}d\xi\right)^{-1}.$$

In addition, see [44], if $n > 1$, we get:

$$\lim_{s\to 1^-}\frac{\omega_{n-1}c(s,n)}{4ns(1-s)} = 1$$

and

$$\lim_{s\to 0^+}\frac{\omega_{n-1}c(s,n)}{2s(1-s)} = 1.$$

In addition in Lemma 5, [14], the previous constant has been surprisingly computed in a precise way so that it results as:

$$c_{s,n} = \frac{4^s\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(-\sigma)}.$$

We recall that in [3] a different expression of the fractional Laplace operator has been given, introducing a different constant of normalization and considering a more general situation. In fact, for every $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha > 0, l \in \mathbb{N}, n \geq 1, \alpha < l$, we may define the following operator:

$$(-S)^{\frac{\alpha}{2}}f(x) = \frac{\sin(\alpha\frac{\pi}{2})}{\beta_n(\alpha)A_l(\alpha)}\int_{\mathbb{R}^n}\frac{\Delta_y^l f(x)}{\|y\|^{n+\alpha}}dy,$$

where $A_l(\alpha)$ is defined in (9),

$$\beta_n(\alpha) = \frac{\pi^{1+\frac{n}{2}}}{2^\alpha \Gamma(1+\frac{\alpha}{2})\Gamma(\frac{n+\alpha}{2})}$$

and

$$\Delta_y^l f(x) = \sum_{k=0}^{l}(-1)^k \binom{l}{k} f(x-ky)$$

denote the non-centered differences. Then, in [3], see Lemma 25.3, it is possible to find the proof that $(-S)^{\alpha/2} = (-\Delta)^{\alpha/2}$ in $\mathcal{S}(\mathbb{R}^n)$.

Another way of introducing the fractional Laplace operator can be done considering if $U : \mathbb{R}^n \times]0, +\infty[\to \mathbb{R}$ solution of the following nonlocal problem,

$$\begin{cases} \text{div}_{(x,y)}(y^{1-2s}\nabla U(x,y)) = 0, & \text{in} \quad \mathbb{R}^n \times]0, +\infty[, \\ U(\cdot, 0) = u, \quad x \in \mathbb{R}^n. \end{cases}$$

Then, defining

$$\mathcal{N}_s u := \lim_{y \to 0} y^{1-2s} \frac{\partial U(\cdot, y)}{\partial y},$$

it results, possibly up to a multiplicative factor depending only on s and n to $\mathcal{N}_s$, that $(-\Delta)^s = (-S)^s = \mathcal{L}_s = \mathcal{N}_s$ for every $u \in \mathcal{S}(\mathbb{R}^n)$. Among the application of this extension approach, we have the application to Carnot groups (see [14,15]).

In the next Section 9, we shall discuss the relationship of the Marchaud derivative with respect to the previous representation of the fractional Laplace operator. We recall, however, that, for the sake of completeness, the fractional Laplace operator may be represented also defining the operator

$$\mathcal{A}_s = -\frac{s}{\Gamma(1-s)} \int_0^{+\infty} (e^{t\Delta} - Id) \frac{dt}{t^{1+s}},$$

where $e^{t\Delta}$ denotes the heat semigroup generated by the Laplace operator Δ and it is also well known that defining the operator

$$\mathcal{B}_s = c(s,n) \int_0^{+\infty} \lambda^s dE(\lambda),$$

where $\{E(\lambda)\}_{\lambda \in [0,\infty[}$ is, as usual, the family of spectral projectors of the Laplace operator, we can conclude that, at least in $\mathcal{S}(\mathbb{R}^n)$, $(-\Delta)^s = (-S)^s = \mathcal{L}_s = \mathcal{N}_s = \mathcal{A}_s = \mathcal{B}_s$. We conclude this section recalling [24], where the semigroup method has been introduced and [45]. In [14], this approach has been developed and then generalized in [25] to a very large class of operators. The fractional Laplace operator in its representation via an extension has been applied in [46] for facing the regularity of the thin obstacle (see also [47]). In particular, we point out that this approach opened the way to a large number of papers in which this idea applied to many other problems. Other applications of the fractional calculus to the geometric measure theory can be found, for instance, in [48–50] and also coming to a very recently result [51,52], where the definition of nonlocal (fractional) perimeters is discussed. For further insights to the properties of the fractional Laplace operator, in addition to [3,38,44,45,53,54], we point out also the very recent preprint [55].

9. Extension Approach for Marchaud Derivative

We described here the simplest case given by $s = 1/2$ as follows. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function in $\mathcal{S}(\mathbb{R})$ and U be a solution of the problem

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2}, & (x,t) \in (0,\infty) \times \mathbb{R}, \\ U(0,t) = \varphi(t), & t \in \mathbb{R}. \end{cases} \tag{15}$$

It is worth remarking that this is not the usual Cauchy problem for the heat operator. It is a heat conduction problem.

Without extra assumptions, we can not expect to have a unique solution of problem (15), see [56], Chapter 3.3. Anyhow, if we denote by $T_{1/2}$ the operator that associates with φ the partial derivative $\partial U / \partial x$, whenever U is sufficiently regular, we have that

$$T_{1/2} T_{1/2} \varphi = \frac{d\varphi}{dt}.$$

That is, $T_{1/2}$ acts like an half derivative, indeed

$$\frac{\partial}{\partial x} \frac{\partial U}{\partial x}(x,t) = \frac{\partial U}{\partial t}(x,t) \xrightarrow[x \to 0^+]{} \frac{d\varphi(t)}{dt}.$$

The solution of problem (15) under the reasonable assumptions that φ is bounded and Hölder continuous, is explicitly known (check [56], Chapter 3.3) to be

$$
\begin{aligned}
U(x,t) &= cx \int_{-\infty}^{t} e^{-\frac{x^2}{4(t-\tau)}} (t-\tau)^{-\frac{3}{2}} \varphi(\tau)\, d\tau \\
&= cx \int_{0}^{\infty} e^{-\frac{x^2}{4\tau}} \tau^{-\frac{3}{2}} \varphi(t-\tau)\, d\tau,
\end{aligned}
$$

where the last line is obtained with a change of variables. Using $t = x^2/(4\tau)$ and the integral definition of the Gamma function we have that

$$\int_0^\infty x e^{-\frac{x^2}{4\tau}} \tau^{-\frac{3}{2}}\, d\tau = 2 \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt = 2\Gamma\left(\frac{1}{2}\right).$$

As a consequence,

$$\frac{U(x,t) - U(0,t)}{x} = c \int_0^\infty e^{-\frac{x^2}{4\tau}} \tau^{-\frac{3}{2}} \left(\varphi(t-\tau) - \varphi(t) \right) d\tau,$$

choosing c that takes into account the right normalization. In addition, by passing to the limit, we obtain

$$- \lim_{x \to 0^+} \frac{U(x,t) - U(0,t)}{x} = c \int_0^\infty \frac{\varphi(t) - \varphi(t-\tau)}{\tau^{\frac{3}{2}}}\, d\tau.$$

Hence, with the right choice of the constant, we get exactly $\mathbf{D}^{1/2}\varphi$ i.e., the Marchaud derivative of order $1/2$ of φ.

In [16], and independently also in [17], has been proved the following result.

Theorem 4. *Let $s \in (0,1)$ and $\bar{\gamma} \in (s,1]$ be fixed. Let $\varphi \in C^{\bar{\gamma}}(\mathbb{R})$ be a bounded function and let $U : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a solution of the problem*

$$
\begin{cases}
\dfrac{\partial U}{\partial t}(x,t) = \dfrac{1-2s}{x} \dfrac{\partial U}{\partial x}(x,t) + \dfrac{\partial^2 U}{\partial x^2}(x,t), & (x,t) \in (0,\infty) \times \mathbb{R}, \\
U(0,t) = \varphi(t), & t \in \mathbb{R}, \\
\displaystyle\lim_{x \to +\infty} U(x,t) = 0.
\end{cases}
\tag{16}
$$

Then, U defines the extension operator for φ, such that

$$\mathbf{D}^s \varphi(t) = - \lim_{x \to 0^+} c_s x^{-2s} (U(x,t) - \varphi(t)), \quad \text{where} \quad c_s = 4^s \Gamma(s).$$

An interesting application that follows from this extension procedure is a Harnack inequality for Marchaud stationary functions in an interval $J \subseteq \mathbb{R}$, namely for functions that satisfy $\mathbf{D}^s \varphi = 0$ in J. This fact is not obvious, indeed the set of functions determined by fractional-stationary functions (on an interval) is nontrivial, see e.g., [57].

Theorem 5. *Let $s \in (0, 1)$. There exists a positive constant γ such that, if $\mathbf{D}^s \varphi = 0$ in an interval $J \subseteq \mathbb{R}$ and $\varphi \geq 0$ in $\mathbb{R}$, then*

$$\sup_{[t_0 - \frac{3}{4}\delta, t_0 - \frac{1}{4}\delta]} \varphi \leq \gamma \inf_{[t_0 + \frac{3}{4}\delta, t_0 + \delta]} \varphi$$

for every $t_0 \in \mathbb{R}$ and for every $\delta > 0$ such that $[t_0 - \delta, t_0 + \delta] \subset J$.

The previous result can be deduced from the Harnack inequality proved in [58] for some degenerate parabolic operators (see also [59] for the elliptic setting) that however are local operators. In particular, the constant γ used in Theorem 5 is the same that appears in the parabolic case in [58]. Concerning the Harnack inequality for the Riemann–Liouville fractional derivative, we also point out [60,61]. In concluding this section, we also remark that, as far as in the case of the fractional Laplace case, the result is true if $\varphi \geq 0$ in all of $\mathbb{R}$ (see [62] for a counterexample for the fractional Laplace case).

We end this section remarking that, concerning the numerical computation of the fractional operators, there exist many contributions. Among them, we point out [63] and the recent handbook [64].

10. Relationship between Marchaud Derivative and the Fractional Laplace Operator

In the end, we discuss here some relationships between Marchaud derivative and fractional Laplace operators. An application to this approach can be find in [26] in the first Heisenberg group case. By the way, we would like to point out that recently a major and renewed attention to fractional calculus and operators similar to Marchaud derivative has been testified by the application described in [20].

In order to explain how fractional Laplace and Marchaud derivative are linked, we fix our attention to the case $0 < \alpha < 1$ by considering

$$\mathbf{D}^\alpha_{\pm, \xi} f(x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^{+\infty} \frac{f(x) - f(x \mp t\xi)}{t^{1+\alpha}} dt,$$

where $\xi \in \mathbb{S}^{n-1}$ and $f \in \mathcal{S}(\mathbb{R}^n)$. For clarity, we define a new operator as it follows: for every $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{M}_{\frac{\alpha}{2}} f(x) = \int_{\partial B_1(0)} \mathbf{D}^\alpha_\xi f(x) d\mathcal{H}^{n-1}(\xi).$$

Then, switching the order of integration, we get

$$
\begin{aligned}
\int_{\partial B_1(0)} \mathbf{D}^\alpha_\xi f(x) d\mathcal{H}^{n-1}(\xi) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \left(\int_{\partial B_1(0)} \frac{f(x) - f(x - t\xi)}{t^{1+\alpha}} d\mathcal{H}^{n-1}(\xi) \right) dt \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \left(\int_{\partial B_t(x)} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} d\mathcal{H}^{n-1}(y) \right) \frac{dt}{t^{n-1}} \\
&= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} \left(\int_{\partial B_t(x)} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} d\mathcal{H}^{n-1}(y) \right) dt = \frac{\alpha}{\Gamma(1-\alpha)} \frac{\beta_n(\alpha)}{\sin(\alpha \frac{\pi}{2})} (-\Delta)^{\frac{\alpha}{2}} f(x).
\end{aligned}
\tag{17}
$$

In general, as already remarked in Lemma 26.2, [3], and recalling previous Section 8 for the definition of the constants $\chi(\alpha, l)$, for every $\alpha > 0$, $l \in \mathbb{N}$, $l \geq 0$, $\alpha < l$, denoting

$$\mathcal{D}^{\alpha)}_\xi f(x) = \frac{1}{\chi(\alpha, l)} \int_0^{+\infty} \frac{\Delta^l_\xi f(x)}{\tau^{1+\alpha}} d\tau,$$

we obtain, for every $f \in \mathcal{S}(\mathbb{R}^n)$:

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = -\frac{\Gamma(-\alpha)\sin(\alpha\frac{\pi}{2})}{\beta_n(\alpha)} \int_{\partial B_1(0)} \mathcal{D}_\xi^{(\alpha)} f(x) d\mathcal{H}^{n-1}(\xi). \tag{18}$$

In particular, if $\alpha \in]0,1[$, and $l = 1$

$$\mathbf{D}^{\alpha,+} f(t) = \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \frac{f(t) - f(t-s)}{s^{1+\alpha}} ds,$$

where

$$C_{\alpha,1} = \frac{\Gamma(1-\alpha)}{\alpha}$$

and

$$\mathbf{D}^{\alpha,-} f(t) = \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \frac{f(t) - f(t+s)}{s^{1+\alpha}} ds,$$

where

$$C_{\alpha,1} = \frac{\Gamma(1-\alpha)}{\alpha}.$$

Thus,

$$(\mathbf{D}^{\alpha,+} + \mathbf{D}^{\alpha,-}) f(t) = \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \frac{2f(t) - f(t+\tau) - f(t-\tau)}{\tau^{1+\alpha}} d\tau,$$

and for every $e \in \partial B_1(0)$ and for every $f \in \mathcal{S}(\mathbb{R}^n)$, we have:

$$(\mathbf{D}_e^{\alpha,+} + \mathbf{D}_e^{\alpha,-}) f(x) = \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \frac{2f(x) - f(x+e\tau) - f(x-e\tau)}{\tau^{1+\alpha}} d\tau.$$

As a consequence, integrating on $\partial B_1(0)$, we obtain, as we already remarked in one variable only:

$$\int_{\partial B_1(0)} (\mathbf{D}_e^{\alpha,+} + \mathbf{D}_e^{\alpha,-}) f(x) d\mathcal{H}^{n-1}(e)$$

$$= \frac{1}{C_{\alpha,1}} \int_{\partial B_1(0)} \left(\int_0^{+\infty} \frac{2f(x) - f(x+e\tau) - f(x-e\tau)}{\tau^{1+\alpha}} d\tau \right) d\mathcal{H}^{n-1}(e)$$

$$= \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \left(\int_{\partial B_1(0)} \frac{2f(x) - f(x+e\tau) - f(x-e\tau)}{\tau^{1+\alpha}} d\mathcal{H}^{n-1}(e) \right) d\tau$$

$$= \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \left(\int_{\partial B_\tau(0)} \frac{2f(x) - f(x+\xi) - f(x-\xi)}{\tau^{n+\alpha}} d\mathcal{H}^{n-1}(\xi) \right) d\tau$$

$$= \frac{1}{C_{\alpha,1}} \int_0^{+\infty} \left(\int_{\partial B_\tau(0)} \frac{2f(x) - f(x+\xi) - f(x-\xi)}{|x-\xi|^{n+\alpha}} d\mathcal{H}^{n-1}(\xi) \right) d\tau$$

$$= \frac{1}{C_{\alpha,1}} \int_{\mathbb{R}^n} \frac{2f(x) - f(x+\xi) - f(x-\xi)}{|x-\xi|^{n+\alpha}} d\xi = \frac{2}{C_{\alpha,1} c(\frac{\alpha}{2}, n)} (-\Delta)^{\frac{\alpha}{2}} f(x).$$

Conflicts of Interest: The author declares no conflict of interest.

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