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Article

Expressing Sums of Finite Products of Chebyshev Polynomials of the Second Kind and of Fibonacci Polynomials by Several Orthogonal Polynomials

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Abstract: This paper is concerned with representing sums of the finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials in terms of several classical orthogonal polynomials. Indeed, by explicit computations, each of them is expressed as linear combinations of Hermite, generalized Laguerre, Legendre, Gegenbauer and Jacobi polynomials, which involve the hypergeometric functions ${}_1F_1$ and ${}_2F_1$.

Keywords: chebyshev polynomials of second kind; Fibonacci polynomials; sums of finite products; orthogonal polynomials

1. Introduction and Preliminaries

In this section, we will fix some notations and recall some basic facts about relevant orthogonal polynomials that will be used throughout this paper.

For any nonnegative integer n, the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $< x >_n$ are respectively defined by (see [1])

$$(x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1),\ (x)_0 = 1, \tag{1}$$

$$< x >_n = x(x+1)\cdots(x+n-1),\ (n \geq 1),\ < x >_0 = 1. \tag{2}$$

The two factorial polynomials are related by:

$$(-1)^n (x)_n =< -x >_n,\ (-1)^n < x >_n = (-x)_n. \tag{3}$$

$$\frac{(2n-2s)!}{(n-s)!} = \frac{2^{2n-2s}(-1)^s < \frac{1}{2} >_n}{< \frac{1}{2} - n >_s},\ (n \geq s \geq 0). \tag{4}$$

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{2^{2n} n!},\ (n \geq 0). \tag{5}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+1-n)} = (x)_n, \frac{\Gamma(x+n)}{\Gamma(x)} =< x >_n,\ (n \geq 0). \tag{6}$$

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ \ (Re\,x, Re\,y>0), \tag{7}$$

where $\Gamma(x)$ and $B(x,y)$ are the gamma and beta functions, respectively.

The hypergeometric function is defined by:

$$\begin{aligned} &{}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x) \\ &=\sum_{n=0}^{\infty}\frac{<a_1>_n\cdots<a_p>_n}{<b_1>_n\cdots<b_q>_n}\frac{x^n}{n!},\ (p\le q+1,\ |x|<1). \end{aligned} \tag{8}$$

We are now going to recall some basic facts about Chebyshev polynomials of the second kind $U_n(x)$, Fibonacci polynomials $F_n(x)$, Hermite polynomials $H_n(x)$, generalized (extended) Laguerre polynomials $L_n^{\alpha}(x)$, Legendre polynomials $P_n(x)$, Gegenbauer polynomials $C_n^{(\lambda)}(x)$ and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. All the necessary results on those special polynomials, except Fibonacci polynomials, can be found in [2–7]. Furthermore, the interested reader may refer to [8–11] for full accounts of the fascinating area of orthogonal polynomials.

In terms of generating functions, the above special polynomials are given by:

$$F(t,x)=\frac{1}{1-2xt+t^2}=\sum_{n=0}^{\infty}U_n(x)t^n, \tag{9}$$

$$G(t,x)=\frac{1}{1-xt-t^2}=\sum_{n=0}^{\infty}F_{n+1}(x)t^n, \tag{10}$$

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}, \tag{11}$$

$$(1-t)^{-\alpha-1}\exp(-\tfrac{xt}{1-t})=\sum_{n=0}^{\infty}L_n^{\alpha}(x)t^n,(\alpha>-1), \tag{12}$$

$$(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)t^n, \tag{13}$$

$$\frac{1}{(1-2xt+t^2)^{\lambda}}=\sum_{n=0}^{\infty}C_n^{(\lambda)}(x)t^n,(\lambda>-\frac{1}{2},\lambda\neq 0,|t|<1,|x|\le 1), \tag{14}$$

$$\begin{aligned} &\frac{2^{\alpha+\beta}}{R(1-t+R)^{\alpha}(1+t+R)^{\beta}}=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)t^n, \\ &(R=\sqrt{1-2xt+t^2},\alpha,\beta>-1). \end{aligned} \tag{15}$$

Explicit expressions of special polynomials can be given as in the following.

$$\begin{aligned} U_n(x)&=(n+1){}_2F_1(-n,n+2;\tfrac{3}{2};\tfrac{1-x}{2}) \\ &=\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n-l}{l}(2x)^{n-2l}, \end{aligned} \tag{16}$$

$$F_{n+1}(x)=\sum_{l=0}^{[\frac{n}{2}]}\binom{n-l}{l}x^{n-2l}, \tag{17}$$

$$H_n(x)=n!\sum_{l=0}^{[\frac{n}{2}]}\frac{(-1)^l}{l!(n-2l)!}(2x)^{n-2l}, \tag{18}$$

$$
\begin{aligned}
L_n^{\alpha}(x) &= \frac{<\alpha+1>_n}{n!}\,{}_1F_1(-n,\alpha+1;x) \\
&= \sum_{l=0}^{n} \frac{(-1)^l \binom{n+\alpha}{n-l}}{l!} x^l, \qquad (19) \\
P_n(x) &= {}_2F_1(-n,n+1;1;\tfrac{1-x}{2}) \\
&= \frac{1}{2^n}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n}{l}\binom{2n-2l}{n}x^{n-2l}, \qquad (20) \\
C_n^{(\lambda)}(x) &= \binom{n+2\lambda-1}{n}\,{}_2F_1(-n,n+2\lambda;\lambda+\tfrac{1}{2};\tfrac{1-x}{2}) \\
&= \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)k!(n-2k)!}(2x)^{n-2k}, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{<\alpha+1>_n}{n!}\,{}_2F_1(-n,1+\alpha+\beta+n;\alpha+1;\tfrac{1-x}{2}) \\
&= \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(\tfrac{x-1}{2})^k(\tfrac{x+1}{2})^{n-k}. \qquad (22)
\end{aligned}
$$

Next, we recall Rodrigues-type formulas for Hermite and generalized Laguerre polynomials and Rodrigues' formulas for Legendre, Gegenbauer and Jacobi polynomials.

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, \qquad (23)$$

$$L_n^{\alpha}(x) = \frac{1}{n!}x^{-\alpha}e^x\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}), \qquad (24)$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \qquad (25)$$

$$(1-x^2)^{\lambda-\frac{1}{2}}C_n^{(\lambda)}(x) = \frac{(-2)^n}{n!}\frac{<\lambda>_n}{<n+2\lambda>_n}\frac{d^n}{dx^n}(1-x^2)^{n+\lambda-\frac{1}{2}}, \qquad (26)$$

$$(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}(1-x)^{n+\alpha}(1+x)^{n+\beta}. \qquad (27)$$

The following orthogonalities with respect to various weight functions are enjoyed by Hermite, generalized Laguerre, Legendre, Gegenbauer and Jacobi polynomials. Here, $\delta_{n,m}$ is Kronecker's delta, so that $\delta_{n,m} = 1$, for $n = m$, and $\delta_{n,m} = 0$, for $n \neq m$.

$$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)\,dx = 2^n n!\sqrt{\pi}\delta_{n,m}, \qquad (28)$$

$$\int_{0}^{\infty} x^{\alpha}e^{-x}L_n^{\alpha}(x)L_m^{\alpha}(x)\,dx = \frac{1}{n!}\Gamma(\alpha+n+1)\delta_{n,m}, \qquad (29)$$

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{n,m}, \qquad (30)$$

$$\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}C_n^{(\lambda)}(x)C_m^{(\lambda)}(x)\,dx = \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}\delta_{n,m}, \qquad (31)$$

$$
\begin{aligned}
&\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\,dx \\
&= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}\delta_{n,m}. \qquad (32)
\end{aligned}
$$

For convenience, we put:

$$\gamma_{n,r}(x) = \sum_{i_1+i_2+\cdots+i_{r+1}=n} U_{i_1}(x)U_{i_2}(x)\cdots U_{i_{r+1}}(x),\ (n,r\geq 0), \tag{33}$$

$$\mathcal{E}_{n,r}(x) = \sum_{i_1+i_2+\cdots+i_r=n} F_{i_1+1}(x)F_{i_2+1}(x)\cdots F_{i_r+1}(x),\ (n\geq 0, r\geq 1). \tag{34}$$

We note here that both $\gamma_{n,r}(x)$ and $\mathcal{E}_{n,r}(x)$ have degree n.

The classical linearization problem in general consists of determining the coefficients $c_{nm}(k)$ in the expansion of the product of two polynomials $q_n(x)$ and $r_m(x)$ in terms of an arbitrary polynomial sequence $\{p_k(x)\}_{k\geq 0}$:

$$q_n(x)r_m(x) = \sum_{k=0}^{n+m} c_{nm}(k)p_k(x).$$

Here, we will study the sums of finite products of Chebyshev polynomials of the second kind in (33) and those of Fibonacci polynomials in (34). Then, we would like to express each of $\gamma_{n,r}(x)$ and $\mathcal{E}_{n,r}(x)$ as linear combinations of $H_n(x), L_n^{\alpha}(x), P_n(x), C_n^{(\lambda)}(x)$ and $P_n^{(\alpha,\beta)}(x)$. These will be done by performing explicit computations and exploiting the formulas in Proposition 1. They can be derived from their orthogonalities, Rodrigues' and Rodrigues-like formulas and integration by parts. This may be viewed as a generalization of the above-mentioned linearization problem.

Our main results are as follows:

Theorem 1. *Let n, r be integers with $n\geq 0, r\geq 1$. Then, we have the following.*

$$\sum_{i_1+i_2+\cdots+i_{r+1}=n} U_{i_1}(x)U_{i_2}(x)\cdots U_{i_{r+1}}(x)$$

$$= \frac{(n+r)!}{r!}\sum_{j=0}^{[\frac{n}{2}]}\frac{1}{j!(n-2j)!}\,{}_1F_1(-j;-n-r;-1)H_{n-2j}(x) \tag{35}$$

$$= \frac{2^n\Gamma(\alpha+n+1)}{r!}\sum_{k=0}^{n}\frac{(-1)^k}{\Gamma(\alpha+k+1)} \times \sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-\frac{1}{4})^l(n+r-l)!}{l!(n-k-2l)!(\alpha+n)_{2l}}L_k^{\alpha}(x) \tag{36}$$

$$= \frac{(n+r)!}{r!}\sum_{j=0}^{[\frac{n}{2}]}\frac{(2n-4j+1)}{(n-j+\frac{1}{2})_{n-j}j!}\,{}_2F_1(-j;j-n-\frac{1}{2};-n-r;1)P_{n-2j}(x) \tag{37}$$

$$= \frac{\Gamma(\lambda)(n+r)!}{\Gamma(n+\lambda+1)r!} \times \sum_{j=0}^{[\frac{n}{2}]}\frac{(n+\lambda-2j)(n+\lambda)_j}{j!}\,{}_2F_1(-j;j-n-\lambda;-n-r;1)C_{n-2j}^{(\lambda)}(x) \tag{38}$$

$$= \frac{(-2)^n}{r!}\sum_{k=0}^{n}\frac{(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \times \sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-\frac{1}{4})^l(n+r-l)!}{l!(n-k-2l)!}\,{}_2F_1(k+2l-n,k+\beta+1;2k+\alpha+\beta+2;2) \times P_k^{(\alpha,\beta)}(x). \tag{39}$$

Theorem 2. *Let n, r be integers with $n \geq 0, r \geq 1$. Then, we have the following.*

$$\begin{aligned}
&\sum_{i_1+i_2+\cdots+i_r=n} F_{i_1+1}(x)F_{i_2+1}(x)\cdots F_{i_r+1}(x) \\
&= \frac{(n+r-1)!}{2^n(r-1)!}\sum_{j=0}^{[\frac{n}{2}]}\frac{1}{(n-2j)!j!}{}_1F_1(-j;1-n-r;4)H_{n-2j}(x) \quad (40)\\
&= \frac{\Gamma(\alpha+n+1)}{(r-1)!}\sum_{k=0}^{n}\frac{(-1)^k}{\Gamma(\alpha+k+1)} \\
&\quad\times\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(n+r-l-1)!}{l!(n-k-2l)!(\alpha+n)_{2l}}L_k^{\alpha}(x) \quad (41)\\
&= \frac{(n+r-1)!}{(r-1)!4^n}\sum_{j=0}^{[\frac{n}{2}]}\frac{(2n-4j+1)4^j}{(n-j+\frac{1}{2})_{n-j}j!}{}_2F_1(-j;j-n-\frac{1}{2};1-n-r;-4)P_{n-2j}(x) \quad (42)\\
&= \frac{\Gamma(\lambda)(n+r-1)!}{2^n(r-1)!\Gamma(n+\lambda+1)} \\
&\quad\times\sum_{j=0}^{[\frac{n}{2}]}\frac{(n+\lambda)_j(n+\lambda-2j)}{j!}{}_2F_1(-j;j-n-\lambda;1-n-r;-4)C_{n-2j}^{(\lambda)}(x) \quad (43)\\
&= \frac{(-1)^n}{(r-1)!}\sum_{k=0}^{n}\frac{\Gamma(k+\alpha+\beta+1)(-2)^k}{\Gamma(2k+\alpha+\beta+1)} \\
&\quad\times\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(n+r-1-l)!}{l!(n-k-2l)!}{}_2F_1(k+2l-n,k+\beta+1;2k+\alpha+\beta+2;2) \\
&\quad\times P_k^{(\alpha,\beta)}(x). \quad (44)
\end{aligned}$$

The sums of finite products of Bernoulli, Euler and Genocchi polynomials have been expressed as linear combinations of Bernoulli polynomials in [12–14]. These were done by deriving Fourier series expansions for the functions closely related to those sums of finite products. Further, the same were done for the sums of finite products $\gamma_{n,r}(x)$ and $\mathcal{E}_{n,r}(x)$ in (33) and (34) in [15]. Along the same line as the present paper, sums of finite products of Chebyshev polynomials of the second, third and fourth kinds and of Fibonacci, Legendre and Laguerre polynomials were expressed in terms of all kinds of Chebyshev polynomials in [16–18]. Finally, we let the reader refer to [19,20] for some applications of Chebyshev polynomials and to [21–25] for some similar iteration methods.

2. Proof of Theorem 1

Here, we are going to prove Theorem 1. First, we will state two results that will be needed in showing Theorems 1 and 2.

The results (a), (b), (c), (d) and (e) in Proposition 1 follow respectively from (3.7) of [3], (2.3) of [7] (see also (2.4) of [6]), (2.3) of [4], (2.3) of [2] and (2.7) of [5]. They can be derived from their orthogonalities in (26)–(30), Rodrigues-like and Rodrigues' formulas in (21)–(25) and integration by parts.

Proposition 1. *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then, we have the following.*

$$
\begin{aligned}
(a) \quad & q(x) = \sum_{k=0}^{n} C_{k,1} H_k(x), \text{ where} \\
& C_{k,1} = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} q(x) \frac{d^k}{dx^k} e^{-x^2} dx, \\
(b) \quad & q(x) = \sum_{k=0}^{n} C_{k,2} L_k^{\alpha}(x), \text{ where} \\
& C_{k,2} = \frac{1}{\Gamma(\alpha+k+1)} \int_{0}^{\infty} q(x) \frac{d^k}{dx^k} (e^{-x} x^{k+\alpha}) dx, \\
(c) \quad & q(x) = \sum_{k=0}^{n} C_{k,3} P_k(x), \text{ where} \\
& C_{k,3} = \frac{2k+1}{2^{k+1} k!} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (x^2-1)^k dx, \\
(d) \quad & q(x) = \sum_{k=0}^{n} C_{k,4} C_k^{(\lambda)}(x), \text{ where} \\
& C_{k,4} = \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k \sqrt{\pi} \Gamma(k+\lambda+\frac{1}{2})} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k+\lambda-\frac{1}{2}} dx, \\
(e) \quad & q(x) = \sum_{k=0}^{n} C_{k,5} P_k^{(\alpha,\beta)}(x), \text{ where} \\
& C_{k,5} = \frac{(-1)^k (2k+\alpha+\beta+1) \Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1} \Gamma(\alpha+k+1) \Gamma(\beta+k+1)} \\
& \quad \times \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\alpha} (1+x)^{k+\beta} dx.
\end{aligned}
$$

Proposition 2. *Let m, k be nonnegative integers. Then, we have the following.*

$$
\begin{aligned}
(a) \quad & \int_{-\infty}^{\infty} x^m e^{-x^2} dx \\
& = \begin{cases} 0, & \text{if } m \equiv 1 \pmod{2}, \\ \frac{m! \sqrt{\pi}}{(\frac{m}{2})! 2^m}, & \text{if } m \equiv 0 \pmod{2}, \end{cases} \\
(b) \quad & \int_{-1}^{1} x^m (1-x^2)^k dx \\
& = \begin{cases} 0, & \text{if } m \equiv 1 \pmod{2}, \\ \frac{2^{2k+2} k! m! (k+\frac{m}{2}+1)!}{(\frac{m}{2})! (2k+m+2)!}, & \text{if } m \equiv 0 \pmod{2}, \end{cases} \\
& = 2^{2k+1} k! \sum_{s=0}^{m} \binom{m}{s} 2^s (-1)^{m-s} \frac{(k+s)!}{(2k+s+1)!}, \\
(c) \quad & \int_{-1}^{1} x^m (1-x^2)^{k+\lambda-\frac{1}{2}} dx \\
& = \begin{cases} 0, & \text{if } m \equiv 1 \pmod{2}, \\ \frac{\Gamma(k+\lambda+\frac{1}{2}) \Gamma(\frac{m}{2}+\frac{1}{2})}{\Gamma(k+\lambda+\frac{m}{2}+1)}, & \text{if } m \equiv 0 \pmod{2}, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(d) \quad & \int_{-1}^{1} x^m (1-x)^{k+\alpha}(1+x)^{k+\beta}dx \\
&= 2^{2k+\alpha+\beta+1}\sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}2^s \\
&\quad \times \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+s+1)}{\Gamma(2k+\alpha+\beta+s+2)}.
\end{aligned}
$$

Proof. (a) This is trivial.

(b) The first equality follows from (c) with $\lambda = \frac{1}{2}$ and the second from (d) with $\alpha = \beta = 0$.

$$
\begin{aligned}
(c) \quad & \int_{-1}^{1} x^m (1-x^2)^{k+\lambda-\frac{1}{2}}dx \\
=&(1+(-1)^m)\int_{0}^{1} x^m (1-x^2)^{k+\lambda-\frac{1}{2}}dx \\
=&\frac{1}{2}(1+(-1)^m)\int_{0}^{1} (1-y)^{k+\lambda+\frac{1}{2}-1}y^{\frac{m+1}{2}-1}dy \\
=&\frac{1}{2}(1+(-1)^m)B(k+\lambda+\frac{1}{2},\frac{m+1}{2}).
\end{aligned}
$$

The result now follows from (7).

$$
\begin{aligned}
(d) \quad & \int_{-1}^{1} x^m (1-x)^{k+\alpha}(1+x)^{k+\beta}dx \\
=&2^{2k+\alpha+\beta+1}\int_{0}^{1} (2y-1)^m (1-y)^{k+\alpha}y^{k+\beta}dy \\
=&2^{2k+\alpha+\beta+1}\sum_{s=0}^{m}\binom{m}{s}2^s(-1)^{m-s} \\
&\times \int_{0}^{1}(1-y)^{k+\alpha+1-1}y^{k+\beta+s+1-1}dy \\
=&2^{2k+\alpha+\beta+1}\sum_{s=0}^{m}\binom{m}{s}2^s(-1)^{m-s}B(k+\alpha+1,k+\beta+s+1).
\end{aligned}
$$

The result again follows from (7). Even though the following lemma was shown in [26], we will show it for the sake of completeness. □

Lemma 1. *Let n, r be nonnegative integers. Then, we have the following identity.*

$$
\sum_{i_1+i_2+\cdots+i_{r+1}=n} U_{i_1}(x)U_{i_2}(x)\cdots U_{i_{r+1}}(x) = \frac{1}{2^r r!}U_{n+r}^{(r)}(x), \tag{45}
$$

where the sum runs over all nonnegative integers $i_1, i_2, \cdots i_{r+1}$, with $i_1+i_2+\cdots+i_{r+1} = n$.

Proof. Noting that the degree of $U_n(x)$ has degree n and taking the partial derivative $(\frac{\partial}{\partial x})^r$ on both sides of (9), we have:

$$
\begin{aligned}
r!(2t)^r\Big(1-2xt+t^2\Big)^{-(r+1)} &= r!(2t)^r\Big(\sum_{i=0}^{\infty}U_i(x)\Big)^{r+1} \\
&= \sum_{n=r}^{\infty}U_n^{(r)}(x)t^n,
\end{aligned}
$$

from which our result follows. □

It is immediate to see from (16) that the r-th derivative of $U_n(x)$ is equal to:

$$U_n^{(r)}(x) = \sum_{l=0}^{[\frac{n-r}{2}]} (-1)^l \binom{n-l}{l} (n-2l)_r 2^{n-2l} x^{n-2l-r}. \tag{46}$$

Thus, in particular, we have:

$$U_{n+r}^{(r+k)}(x) = \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r-l}{l} (n+r-2l)_{r+k} 2^{n+r-2l} x^{n-k-2l}. \tag{47}$$

Here, we will show only (35), (37) and (38) in Theorem 1, leaving the proofs for (36) and (39) as an exercise, as they can be proved analogously to those for (41) and (44) in the next section.

With $\gamma_{n,r}(x)$ as in (33), we let:

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,1} H_k(x). \tag{48}$$

Then, from (a) of Proposition 1, (45), (47) and integration by parts k times, we have:

$$\begin{aligned} C_{k,1} &= \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \gamma_{n,r}(x) \frac{d^k}{dx^k} e^{-x^2} dx, \\ &= \frac{(-1)^k}{2^{k+r} k! r! \sqrt{\pi}} \int_{-\infty}^{\infty} U_{n+r}^{(r)}(x) \frac{d^k}{dx^k} e^{-x^2} dx, \\ &= \frac{1}{2^{k+r} k! r! \sqrt{\pi}} \int_{-\infty}^{\infty} U_{n+r}^{(r+k)}(x) e^{-x^2} dx, \\ &= \frac{2^{n-k}}{k! r! \sqrt{\pi}} \sum_{l=0}^{[\frac{n-k}{2}]} (-\frac{1}{4})^l \binom{n+r-l}{l} (n+r-2l)_{r+k} \\ &\quad \times \int_{-\infty}^{\infty} x^{n-k-2l} e^{-x^2} dx. \end{aligned} \tag{49}$$

From (49) and invoking (a) of Proposition 2, we get:

$$\begin{aligned} C_{k,1} &= \frac{2^{n-k}}{k! r! \sqrt{\pi}} \sum_{l=0}^{[\frac{n-k}{2}]} (-\frac{1}{4})^l \binom{n+r-l}{l} (n+r-2l)_{r+k} \\ &\quad \times \begin{cases} 0, & \text{if } k \not\equiv n \pmod 2, \\ \frac{(n-k-2l)! \sqrt{\pi}}{2^{n-k-2l} (\frac{n-k}{2} - l)!}, & \text{if } k \equiv n \pmod 2. \end{cases} \end{aligned} \tag{50}$$

Now, from (48) and (50), and after some simplification, we obtain:

$$\begin{aligned} \gamma_{n,r}(x) &= \frac{1}{r!} \sum_{\substack{0 \le k \le n \\ k \equiv n \pmod 2}} \frac{1}{k!} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-1)^l (n+r-l)!}{l! (\frac{n-k}{2} - l)!} H_k(x) \\ &= \frac{1}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{(n-2j)!} H_{n-2j}(x) \sum_{l=0}^{j} \frac{(-1)^l (n+r-l)!}{l! (j-l)!} \\ &= \frac{(n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{j! (n-2j)!} H_{n-2j}(x) \sum_{l=0}^{j} \frac{(-1)^l < -j >_l}{l! < -n-r >_l} \\ &= \frac{(n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{j! (n-2j)!} {}_1F_1(-j, -n-r; -1) H_{n-2j}(x). \end{aligned} \tag{51}$$

This shows (35) of Theorem 1.

Next, we let:

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,3} P_k(x). \tag{52}$$

Then, from (c) of Proposition 1, (45), (47) and integration by parts k times, we get:

$$\begin{aligned} C_{k,3} &= \frac{2k+1}{2^{k+r+1}k!r!} \int_{-1}^{1} U_{n+r}^{(r)}(x) \frac{d^k}{dx^k}(x^2-1)^k dx \\ &= \frac{(-1)^k(2k+1)}{2^{k+r+1}k!r!} \int_{-1}^{1} U_{n+r}^{(r+k)}(x)(x^2-1)^k dx \\ &= \frac{(2k+1)2^{n-k-1}}{k!r!} \sum_{l=0}^{[\frac{n-k}{2}]} (-\frac{1}{4})^l \binom{n+r-l}{l} (n+r-2l)_{r+k} \\ &\times \int_{-1}^{1} x^{n-k-2l}(1-x^2)^k dx. \end{aligned} \tag{53}$$

From (52) and making use of the first equality of (b) in Proposition 2, we have:

$$\begin{aligned} C_{k,3} &= \frac{(2k+1)2^{n-k-1}}{k!r!} \sum_{l=0}^{[\frac{n-k}{2}]} (-\frac{1}{4})^l \binom{n+r-l}{l} (n+r-2l)_{r+k} \\ &\times \begin{cases} 0, & \text{if } k \not\equiv n \pmod 2, \\ \frac{2^{2k+2}k!(n-k-2l)!(\frac{n+k}{2}-l+1)!}{(\frac{n-k}{2}-l)!(n+k-2l+2)!}, & \text{if } k \equiv n \pmod 2. \end{cases} \end{aligned} \tag{54}$$

From (52), (54), and using (4), we finally obtain:

$$\begin{aligned} \gamma_{n,r}(x) &= \frac{2^{2n+1}}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{2n-4j+1}{2^{2j}} P_{n-2j}(x) \\ &\times \sum_{l=0}^{j} \frac{(-\frac{1}{4})^l (n+r-l)!(n-j+1-l)!}{l!(j-l)!(2n-2j+2-2l)!} \\ &= \frac{(n+r)!}{2r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(2n-4j+1)P_{n-2j}(x)}{(n-j+\frac{1}{2})_{n-j+1} j!} \\ &\times \sum_{l=0}^{j} \frac{< -j >_l < j-n-\frac{1}{2} >_l}{< -n-r >_l \, l!} \\ &= \frac{(n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(2n-4j+1) {}_2F_1(-j, j-n-\frac{1}{2}; -n-r; 1)}{(n-j+\frac{1}{2})_{n-j} j!} P_{n-2j}(x). \end{aligned} \tag{55}$$

This shows (37) of Theorem 1.

Remark 1. *In the step of* (54)*, if we use the second equality of (b) in Proposition 2 instead of the first, we would have the expression:*

$$\begin{aligned} \gamma_{n,r}(x) &= \frac{(-2)^n}{r!} \sum_{k=0}^{n} \frac{(-2)^k k!}{(2k)!} \\ &\times \sum_{j=0}^{[\frac{n-k}{2}]} \frac{(-\frac{1}{4})^l (n+r-l)!}{l!(n-k-2l)!} {}_2F_1(2l+k-n, k+1; 2k+2; 2) P_k(x). \end{aligned} \tag{56}$$

We note here that (56) is (39), with $\alpha = \beta = 0$. This is what we expect, as $P_n(x) = P_n^{(0,0)}(x)$. Finally, we let:

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,4} C_k^{(\lambda)}(x). \tag{57}$$

Then, from (d) of Proposition 1, (45), (47) and integration by parts k times, we obtain:

$$\begin{aligned} C_{k,4} &= \frac{(k+\lambda)\Gamma(\lambda)}{2^{k+r}\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})r!} \times \int_{-1}^{1} U_{n+r}^{(r+k)}(x)(1-x^2)^{k+\lambda-\frac{1}{2}}dx \\ &= \frac{2^{n-k}(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})r!} \sum_{l=0}^{[\frac{n-k}{2}]} (-\frac{1}{4})^l \binom{n+r-l}{l} (n+r-2l)_{r+k} \\ &\times \int_{-1}^{1} x^{n-k-2l}(1-x^2)^{k+\lambda-\frac{1}{2}}dx. \end{aligned} \tag{58}$$

From (58), and exploiting (c) in Proposition 2 and (5), we have:

$$\begin{aligned} C_{k,4} &= \frac{2^{n-k}(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})r!} \sum_{l=0}^{[\frac{n-k}{2}]} (-\frac{1}{4})^l \binom{n+r-l}{l} (n+r-2l)_{r+k} \\ &\times \begin{cases} 0, & \text{if } k \not\equiv n \pmod{2}, \\ \frac{\Gamma(k+\lambda+\frac{1}{2})(n-k-2l)!\sqrt{\pi}}{\Gamma(\frac{n+k}{2}+\lambda-l+1)2^{n-k-2l}(\frac{n-k}{2}-l)!}, & \text{if } k \equiv n \pmod{2}. \end{cases} \end{aligned} \tag{59}$$

Making use of (6), and from (57) and (59), we finally derive:

$$\begin{aligned} \gamma_{n,r}(x) &= \frac{\Gamma(\lambda)}{r!} \sum_{\substack{0\le k\le n \\ k\equiv n \pmod{2}}} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-1)^l(k+\lambda)(n+r-l)!}{l!\Gamma(\frac{n+k}{2}+\lambda-l+1)(\frac{n-k}{2}-l)!} C_k^{(\lambda)}(x) \\ &= \frac{\Gamma(\lambda)(n+r)!}{r!} \sum_{\substack{0\le k\le n \\ k\equiv n \pmod{2}}} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(k+\lambda)}{(\frac{n-k}{2})!\Gamma(\frac{n+k}{2}+\lambda+1)} \\ &\times \frac{(-1)^l(\frac{n-k}{2})_l(\frac{n+k}{2}+\lambda)_l}{l!(n+r)_l} C_k^{(\lambda)}(x) \\ &= \frac{\Gamma(\lambda)(n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \sum_{l=0}^{j} \frac{(n-2j+\lambda)}{j!\Gamma(n-j+\lambda+1)} \\ &\times \frac{(-1)^l(j)_l(n+\lambda-j)_l}{l!(n+r)_l} C_{n-2j}^{(\lambda)}(x) \\ &= \frac{\Gamma(\lambda)(n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \sum_{l=0}^{j} \frac{(n-2j+\lambda)}{j!\Gamma(n-j+\lambda+1)} \\ &\times \frac{<-j>_l<j-n-\lambda>_l}{l!<-n-r>_l} C_{n-2j}^{(\lambda)}(x) \\ &= \frac{\Gamma(\lambda)(n+r)!}{\Gamma(n+\lambda+1)r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(n-2j+\lambda)(n+\lambda)_j}{j!} \\ &\times {}_2F_1(-j, j-n-\lambda; -n-r; 1) C_{n-2j}^{(\lambda)}(x) \end{aligned} \tag{60}$$

This completes the proof for (38) in Theorem 1.

3. Proof of Theorem 2

Here, we will show only (41) and (44) in Theorem 2, leaving the proofs for (40), (42) and (43) as an exercise, as they can be shown similarly to those for (35), (37) and (38).

The following lemma is stated in Equation (9) of [27] and can be derived by differentiating (10).

Lemma 2. *Let n, r be integers with $n \geq 0$, $r \geq 1$. Then, we have the following identity.*

$$\sum_{i_1+i_2+\cdots+i_r=n} F_{i_1+1}(x)F_{i_2+1}(x)\cdots F_{i_r+1}(x) = \frac{1}{(r-1)!}F_{n+r}^{(r-1)}(x), \tag{61}$$

where the sum runs over all nonnegative integers $i_1, i_2, \cdots, i_r$, with $i_1 + i_2 + \cdots + i_r = n$.

From (17), it is easy to show that the r-th derivative of $F_{n+1}(x)$ is given by:

$$F_{n+1}^{(r)}(x) = \sum_{l=0}^{[\frac{n-r}{2}]} \binom{n-l}{l} (n-2l)_r x^{n-r-2l}. \tag{62}$$

Thus, especially, we have:

$$F_{n+r}^{(r+k-1)}(x) = \sum_{l=0}^{[\frac{n-k}{2}]} \binom{n+r-l-1}{l} (n+r-2l-1)_{r+k-1} x^{n-k-2l}. \tag{63}$$

With $\mathcal{E}_{n,r}(x)$ as in (34), we let:

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,2} L_k^{(\alpha)}(x). \tag{64}$$

Then, from (b) of Proposition 1, (61), (63), (6) and integration by parts k times, we have:

$$\begin{aligned} C_{k,2} &= \frac{1}{\Gamma(\alpha+k+1)(r-1)!} \int_0^\infty F_{n+r}^{(r-1)}(x) \frac{d^k}{dx^k}(e^{-x}x^{k+\alpha})dx \\ &= \frac{(-1)^k}{\Gamma(\alpha+k+1)(r-1)!} \int_0^\infty F_{n+r}^{(r+k-1)}(x) e^{-x} x^{k+\alpha} dx \\ &= \frac{(-1)^k}{\Gamma(\alpha+k+1)(r-1)!} \sum_{l=0}^{[\frac{n-k}{2}]} \binom{n+r-l-1}{l} (n+r-2l-1)_{r+k-1} \\ &\quad \times \int_0^\infty e^{-x} x^{n+\alpha-2l} dx \\ &= \frac{\Gamma(\alpha+n+1)}{(r-1)!} \sum_{k=0}^{n} \frac{(-1)^k}{\Gamma(\alpha+k+1)} \\ &\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(n+r-l-1)!}{l!(n-k-2l)!(\alpha+n)_{2l}} L_k^{(\alpha)}(x). \end{aligned} \tag{65}$$

Next, we let:

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,5} P_n^{(\alpha,\beta)}(x). \tag{66}$$

Then, from (e) of Proposition 1, and (61), (63) and integration by parts k times, we obtain:

$$\begin{aligned} C_{k,5} &= \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{k+\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)(r-1)!} \\ &\quad \times \int_{-1}^{1} F_{n+r}^{(r+k-1)}(x)(1-x)^{k+\alpha}(1+x)^{k+\beta}dx \\ &= \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{k+\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)(r-1)!} \\ &\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \binom{n+r-l-1}{l}(n+r-2l-1)_{r+k-1} \\ &\quad \times \int_{-1}^{1} x^{n-k-2l}(1-x)^{k+\alpha}(1+x)^{k+\beta}dx. \end{aligned} \tag{67}$$

Now, from (67), and using (d) in Proposition 2 and (6), we have:

$$\begin{aligned} C_{k,5} &= \frac{(-1)^n(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)(-2)^k}{(r-1)!\Gamma(k+\beta+1)} \\ &\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(n+r-l-1)!}{l!} \\ &\quad \times \sum_{s=0}^{n-k-2l} \frac{(-2)^s\Gamma(k+\beta+s+1)}{s!(n-k-2l-s)!\Gamma(2k+\alpha+\beta+s+2)} \\ &= \frac{(-1)^n\Gamma(k+\alpha+\beta+1)(-2)^k}{(r-1)!\Gamma(2k+\alpha+\beta+1)} \\ &\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(n+r-1-l)!}{l!(n-k-2l)!} \\ &\quad \times \sum_{s=0}^{n-k-2l} \frac{<2l+k-n>_s<k+\beta+1>_s 2^s}{<2k+\alpha+\beta+2>_s\, s!} \\ &= \frac{(-1)^n\Gamma(k+\alpha+\beta+1)(-2)^k}{(r-1)!\Gamma(2k+\alpha+\beta+1)} \\ &\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(n+r-1-l)!}{l!(n-k-2l)!} \\ &\quad \times {}_2F_1(2l+k-n, k+\beta+1; 2k+\alpha+\beta+2; 2). \end{aligned} \tag{68}$$

As we desired, we finally obtain:

$$\begin{aligned} \gamma_{n,r}(x) &= \frac{(-1)^n}{(r-1)!}\sum_{k=0}^{n} \frac{\Gamma(k+\alpha+\beta+1)(-2)^k}{\Gamma(2k+\alpha+\beta+1)} \\ &\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(n+r-1-l)!}{l!(n-k-2l)!}\, {}_2F_1(k+2l-n, k+\beta+1; 2k+\alpha+\beta+2; 2) P_n^{(\alpha,\beta)}(x). \end{aligned}$$

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