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Fourier–Zernike Series of Convolutions on Disks

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Abstract: This paper presents a systematic study of the analytic aspects of Fourier–Zernike series of convolutions of functions supported on disks. We then investigate different aspects of the presented theory in the cases of zero-padded functions.

Keywords: convolution; Zernike polynomials; Fourier–Zernike series

1. Introduction

The mathematical theory of convolution function algebras plays significant roles in classical harmonic analysis, representation theory, functional analysis, and operator theory; see [1–8] and the references therein. Over the last few decades, some new aspects of convolution function algebras have achieved significant popularity in modern harmonic analysis areas such as coorbit theory (including Gabor and wavelet analysis) [9–13] and recent applications in computational science and engineering [14–18].

In many applications in engineering, convolutions and correlations of functions on Euclidean spaces are required. This includes template matching in image processing for pattern recognition and protein docking [19–21] and characterizing how error probabilities propagate [22]. In some applications, the goal is not to recover the values of convolved functions, but rather their support, which is the Minkowski sum of the supports of the two functions being convolved [23]. In most of these applications, the functions of interest take non-negative values and as such can be normalized and treated as probability density functions (pdfs).

Usually, two approaches are taken to computing convolutions of pdfs on Euclidean space. First, if the functions are compactly supported, then their supports are enclosed in a solid cube with dimensions at least twice the size of the support of the functions, and periodic versions of the functions are constructed. In this way, convolution of these periodic functions on the d-torus can be used to replace convolution on d-dimensional Euclidean space. The benefit of this is that the spectrum is discretized, and fast Fourier transform (FFT) methods can be used to compute the convolutions. This approach is computationally attractive, but in this periodization procedure, the natural invariance of integration on Euclidean space under rotation transformations is lost when moving to the torus. This can be a significant issue in rotation matching problems.

A second approach is to take the original compactly-supported functions and replace them with functions on Euclidean space that have rapidly-decaying tails, but for which convolutions can be computed in closed form. For example, replacing each of the given functions with a sum of Gaussian distributions allows the convolution of the given functions to be computed as a sum of convolutions of Gaussians, which have simple closed-form expressions as Gaussians. The problem with this approach is that the resulting functions are not compactly supported. Moreover, if N Gaussians are used to describe each input function, then N^2 Gaussians result after the convolution.

An altogether different approach is explored here. Rather than periodizing the given functions or extending their support to the whole of Euclidean space, we consider functions that are supported on disks in the plane (and by natural extension, to balls in higher dimensional Euclidean spaces). The basic idea is that in polar coordinates, each function is expanded in an orthonormal basis consisting of Zernike polynomials in the radial direction and Fourier basis in the angular direction. These basis elements are orthonormal on the unit disk. Each input function to the convolution procedure is scaled to have support on the disk of a radius of one half and zero-padded on the unit disk. The result of the convolution (or correlation) then is a function that is supported on the unit disk. Since the convolution integral for compactly-supported functions can be restricted from all of the Euclidean space to the support of the functions, it is only this integral over the support that is performed when using Fourier–Zernike expansions. Hence, the behavior of these functions outside of disks becomes irrelevant to the final result. We work out how the Fourier–Zernike coefficients of the original functions appear in the convolution.

This article contains four sections. Section 2 is devoted to fixing notation and gives a brief summary of the convolution of functions on $\mathbb{R}^2$ and polar Fourier analysis. In Section 3, we present analytic aspects of the general theory of Fourier–Zernike series for functions defined on disks. Section 4 is dedicated to study the presented theory of Fourier–Zernike series for convolution of functions supported on disks. As the main result, we present a constructive closed form for Fourier–Zernike coefficients of convolution functions supported on disks. We then employ this closed form to present a constructive Fourier–Zernike approximation for convolution of zero-padded functions on $\mathbb{R}^2$.

2. Preliminaries and Notations

Throughout this section, we shall present preliminaries and the notation.

2.1. General Notations

For $d \in \mathbb{N}$ and $a > 0$, let $\mathbb{B}^d_a := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 \leq a\}$, where:

$$\|\mathbf{x}\|_2 := \left(\sum_{\ell=1}^{d} |x_\ell|^2\right)^{1/2},$$

for $\mathbf{x} := (x_1, \cdots, x_d)^T \in \mathbb{R}^d$. We then put $\mathbb{B}^d := \mathbb{B}^d_1$, that is the unit ball in $\mathbb{R}^d$.

It should be mentioned that each function $f \in L^1(\mathbb{R}^d)$ satisfies the following integral decomposition:

$$\int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{S}^{d-1}} \int_0^\infty f(r\mathbf{u}) r^{d-1}\, dr d\mathbf{u}. \tag{1}$$

Furthermore, if $f \in L^1(\mathbb{R}^d)$ is supported on $\mathbb{B}^d_a$, we then have:

$$\int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{S}^{d-1}} \int_0^a f(r\mathbf{u}) r^{d-1}\, dr d\mathbf{u}.$$

Let $d \in \mathbb{N}$, $a > 0$, and $b := a/2$. Let $f_1, f_2 \in L^2(\mathbb{R}^d)$ with $\mathrm{supp}(f_1), \mathrm{supp}(f_2) \subseteq \mathbb{B}^d_b$. Then, we have:

$$\mathrm{supp}(f_1 * f_2) \subseteq \mathrm{supp}(f_1) + \mathrm{supp}(f_2) \subseteq \mathbb{B}^d_b + \mathbb{B}^d_b \subseteq \mathbb{B}^d_a,$$

where:

$$(f_1 * f_2)(\mathbf{x}) := \int_{\mathbb{R}^d} f_1(\mathbf{y}) f_2(\mathbf{x} - \mathbf{y}) d\mathbf{y}, \tag{2}$$

for $\mathbf{x} \in \mathbb{R}^d$.

Let $d \in \mathbb{N}$ and C be a convex and compact set in $\mathbb{R}^d$. Let $\mathrm{f} : C \to \mathbb{C}$ be a continuous function. Then, there exists a canonical extension of f from C to $\mathbb{R}^d$ by zero-padding, still denoted by $f : \mathbb{R}^d \to \mathbb{C}$, such that $f(x) = \mathrm{f}(x)$ for all $x \in C$, and $f(x) = 0$ for all $x \notin C$.

Let $\mathrm{f}_k : C \to \mathbb{C}$ with $k \in \{1,2\}$ be continuous functions. We then define the canonical windowed convolution of f_1 with f_2, denoted by $\mathrm{f}_1 \circledast \mathrm{f}_2$, by:

$$(\mathrm{f}_1 \circledast \mathrm{f}_2)(\mathbf{x}) := (f_1 * f_2)(\mathbf{x}) = \int_{\mathbb{R}^d} f_1(\mathbf{y}) f_2(\mathbf{x}-\mathbf{y}) d\mathbf{y}, \tag{3}$$

where f_k is the canonical extension of f_k from C to $\mathbb{R}^d$. We may also denote $\mathrm{f}_1 \circledast \mathrm{f}_2$ by $f_1 \circledast f_2$, as well.

Since each f_k is supported on C, we deduce that $f_1 * f_2$ is supported on $C + C$. Hence, we get:

$$(\mathrm{f}_1 \circledast \mathrm{f}_2)(\mathbf{x}) = \int_C f_1(\mathbf{y}) f_2(\mathbf{x}-\mathbf{y}) d\mathbf{y} = \int_C \mathrm{f}_1(\mathbf{y}) f_2(\mathbf{x}-\mathbf{y}) d\mathbf{y}, \tag{4}$$

for all $\mathbf{x} \in C + C$.

Let $a > 0$ and $b := a/2$. Furthermore, let $C := \mathbb{B}_b^d$. Then, C is a convex and compact set in $\mathbb{R}^d$. Furthermore, we have $C + C \subseteq \mathbb{B}_a^d$. Then, for continuous functions $\mathrm{f}_k : \mathbb{B}_b^d \to \mathbb{C}$ with $k \in \{1,2\}$, the convolution $f_1 * f_2$ is supported on $\mathbb{B}_a^d$. Hence, we can write:

$$(\mathrm{f}_1 \circledast \mathrm{f}_2)(\mathbf{x}) = \int_{\mathbb{B}_b^d} \mathrm{f}_1(\mathbf{y}) f_2(\mathbf{x}-\mathbf{y}) d\mathbf{y}, \tag{5}$$

for all $\mathbf{x} \in \mathbb{B}_a^d$. Then, using the formula (1), we get:

$$(\mathrm{f}_1 \circledast \mathrm{f}_2)(\mathbf{x}) = \int_{\mathbb{S}^{d-1}} \int_0^{a/2} \mathrm{f}_1(r\mathbf{u}) f_2(\mathbf{x}-r\mathbf{u}) r^{d-1}\, dr d\mathbf{u}, \tag{6}$$

for all $\mathbf{x} \in \mathbb{B}^d$.

The Case $d = 2$

In this case, each function $f \in L^1(\mathbb{R}^2)$ satisfies the following integral decomposition:

$$\int_{\mathbb{R}^2} f(\mathbf{x}) d\mathbf{x} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} f(r,\theta) r\, dr d\theta. \tag{7}$$

Furthermore, if $f \in L^1(\mathbb{R}^2)$ is supported on $\mathbb{B}_a^2$, we then have:

$$\int_{\mathbb{R}^2} f(\mathbf{x}) d\mathbf{x} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{a} f(r,\theta) r\, dr d\theta.$$

2.2. Fourier–Zernike Analysis

Zernike polynomials are mostly used to express wavefront data in optical tests; see [24,25] and the references therein. The radial Zernike function Z_{nm}, where $m \in \mathbb{Z}$ and $n \geq 0$ is an integer with $n \geq |m|$ and $n - |m|$ even (or equivalently, n and $|m|$ are congruence modulo two, denoted by $|m| \overset{2}{\equiv} n$), is a polynomial in r given by [26,27]:

$$Z_{nm}(r) := \sum_{j=0}^{\frac{n-|m|}{2}} (-1)^j \frac{(n-j)!}{j!(\frac{n+|m|}{2}-j)!(\frac{n-|m|}{2}-j)} r^{n-2j}. \tag{8}$$

It has $(n - |m|)/2$ zeros between zero and one.

Furthermore, for each $m \in \mathbb{Z}$ and $|m| \leq n$ with $|m| \overset{2}{\equiv} n$, we have:

$$\int_0^1 Z_{nm}(r) J_m(\alpha r) r dr = (-1)^{\frac{n-m}{2}} \cdot \frac{J_{n+1}(\alpha)}{\alpha}, \tag{9}$$

for each $0 \neq \alpha \in \mathbb{R}$.

For a fixed $m \in \mathbb{Z}$, we have the following orthogonality relation:

$$\int_0^1 Z_{n_1 m}(r) Z_{n_2 m}(r) r dr = \frac{1}{2n_1 + 2} \cdot \delta_{n_1, n_2}, \tag{10}$$

for integer n_j, $j \in \{1, 2\}$ with $n_j \geq |m|$ and $n_j - |m|$ even.

Therefore, for each $a > 0$ and $m \in \mathbb{Z}$, we conclude:

$$\int_0^a Z_{n_1 m}(a^{-1} r) Z_{n_2 m}(a^{-1} r) r dr = \frac{a^2}{2n_1 + 2} \cdot \delta_{n_1, n_2}, \tag{11}$$

for integer n_j, $j \in \{1, 2\}$ with $n_j \geq |m|$ and $n_j - |m|$ even.

Then, for a given $a > 0$ and each $m \in \mathbb{Z}$, the set:

$$\mathcal{E}_m^a := \left\{ \mathcal{Z}_{nm}^a : n \geq |m| \geq 0 \text{ and } |m| \overset{2}{\equiv} n \right\}, \tag{12}$$

forms an orthonormal basis for the Hilbert function space $L^2([0, a], r dr)$, where:

$$\mathcal{Z}_{nm}^a(r) := \frac{\sqrt{2n+2}}{a} \cdot Z_{nm}(a^{-1} r).$$

In detail, for integer n_j, $j \in \{1, 2\}$ with $n_j \geq |m|$ and $n_j - |m|$ even, we have:

$$\int_0^a \mathcal{Z}_{n_1 m}^a(r) \mathcal{Z}_{n_2 m}^a(r) r dr = \delta_{n_1, n_2}.$$

Furthermore, for each $m \in \mathbb{Z}$ and $|m| \leq n$ with $|m| \overset{2}{\equiv} n$, we have:

$$\int_0^a Z_{nm}(a^{-1} r) J_m(\alpha r) r dr = a(-1)^{\frac{n-m}{2}} \cdot \frac{J_{n+1}(a\alpha)}{\alpha}, \tag{13}$$

for each $a > 0$ and $0 \neq \alpha \in \mathbb{R}$, where J_q is the q^{th} order Bessel function of the first kind, for each $q \in \mathbb{Z}$.

Hence, any function $v : [0, a] \to \mathbb{R}$ satisfies the following expansion:

$$v(r) = \sum_{\{n : |m| \leq n \text{ and } m \overset{2}{\equiv} n\}} \left(\int_0^a v(s) \mathcal{Z}_{nm}^a(s) s ds \right) \mathcal{Z}_{nm}^a(r), \tag{14}$$

for $r \in [0, a]$.

We then can define the Fourier–Zernike basis element V_{nm}^a in the polar form as follows:

$$V_{nm}^a(r, \theta) := \frac{\sqrt{2n+2}}{a} Z_{nm}(a^{-1} r) \mathcal{Y}_m(\theta) = a^{-1} \sqrt{\frac{n+1}{\pi}} Z_{nm}(a^{-1} r) \exp(\mathrm{i} m\theta), \tag{15}$$

for $m \in \mathbb{Z}$, and $n \geq 0$ is an integer with $n \geq |m|$ and $n - |m|$ even.

Then, any restricted 2D integrable function $f(r, \theta)$ defined on $r \leq a$ can be expanded with respect to V_{nm}^a as defined in (15) via:

$$f(r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\{n : |m| \leq n \text{ and } |m| \overset{2}{\equiv} n\}} C_{n,m}^a(f) V_{nm}^a(r, \theta), \tag{16}$$

where:

$$C_{n,m}^a(f) := \int_0^a \int_0^{2\pi} f(r, \theta) \overline{V_{nm}^a(r, \theta)} r dr d\theta. \tag{17}$$

The Case $a = 1$

In this case, any integrable function $v : [0, 1] \to \mathbb{R}$ satisfies the following expansion:

$$v(r) = \sum_{\{n:|m|\le n \text{ and } m \overset{2}{\equiv} n\}} \left(\int_0^1 v(s)\mathcal{Z}^1_{nm}(s)sds \right) \mathcal{Z}^1_{nm}(r), \tag{18}$$

for $r \in [0, 1]$, where:

$$Z^1_{nm}(r) = \sqrt{2n+2}Z_{nm}(r).$$

Furthermore, Fourier–Zernike basis elements V^1_{nm} in the polar form have the following form:

$$V^1_{nm}(r,\theta) := \sqrt{2n+2}Z_{nm}(r) \cdot \mathcal{Y}_m(\theta) = \sqrt{\frac{n+1}{\pi}} \cdot Z_{nm}(r) \cdot \exp(im\theta), \tag{19}$$

for $m \in \mathbb{Z}$, and $n \ge 0$ is an integer with $n \ge |m|$ and $n - |m|$ even.

Hence, any restricted 2D integrable function $f(r,\theta)$ defined on $r \le 1$ can be expanded with respect to V^1_{nm} as defined in (15) via:

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\le n \text{ and } |m| \overset{2}{\equiv} n\}} C^1_{n,m}(f)V^1_{nm}(r,\theta), \tag{20}$$

where:

$$C^1_{n,m}(f) := \int_0^1 \int_0^{2\pi} f(r,\theta)\overline{V^1_{nm}(r,\theta)}rdrd\theta. \tag{21}$$

3. Fourier–Zernike Series of Functions Supported on Disks

This section is dedicated to studying the analytical aspects of Fourier–Zernike series of functions supported on disks (2D balls). We shall present a unified method for computing the Fourier–Zernike coefficients of functions supported on disks.

First, we need some preliminary results.

Proposition 1. *Let $r, a > 0$, $0 \le s \le a$, and $0 < \alpha, \theta \le 2\pi$. We then have:*

$$e^{\mathrm{i}rs\cos(\alpha-\theta)} = \sqrt{2\pi} \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\le n \text{ and } |m| \overset{2}{\equiv} n\}} \sqrt{2n+2}\frac{\mathrm{i}^m(-1)^{\frac{n-m}{2}}J_{n+1}(ar)}{r}e^{-\mathrm{i}m\alpha}V^a_{nm}(s,\theta). \tag{22}$$

Proof. Let $\mathbf{x} = s\mathbf{u}_\theta$ and $\boldsymbol{\omega} = r\mathbf{u}_\alpha$. By the Jacobi–Anger expansion, we can write:

$$e^{\mathrm{i}rs\cos(\alpha-\theta)} = e^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{x}} = \sum_{m=-\infty}^{\infty} \mathrm{i}^m J_m(rs)e^{\mathrm{i}m\theta}e^{-\mathrm{i}m\alpha}. \tag{23}$$

Let $m \in \mathbb{Z}$. Expanding $J_m(rs)$ with respect to s, using (14), we can write:

$$J_m(rs) = \sum_{\{n:|m|\le n \text{ and } |m| \overset{2}{\equiv} n\}} \left(\int_0^a \mathcal{Z}^a_{nm}(p)J_m(rp)pdp \right) \mathcal{Z}^a_{nm}(s).$$

Using (13), we can write:

$$\begin{aligned} \int_0^a \mathcal{Z}^a_{nm}(p)J_m(rp)pdp &= \frac{\sqrt{2n+2}}{a} \int_0^a Z_{nm}(a^{-1}p)J_m(rp)pdp \\ &= \frac{\sqrt{2n+2}}{r}(-1)^{\frac{n-m}{2}}J_{n+1}(ar). \end{aligned}$$

We then deduce that:

$$J_m(rs) = r^{-1} \cdot \sum_{\{n:|m|\leq n \text{ and } |m| \overset{2}{\equiv} n\}} \sqrt{2n+2}(-1)^{\frac{n-m}{2}} J_{n+1}(ar)\mathcal{Z}^a_{nm}(s). \tag{24}$$

Applying Equation (24) in (23), we get:

$$\begin{aligned} e^{\mathrm{i}rs\cos(\alpha-\theta)} &= \sum_{m=-\infty}^{\infty} \mathrm{i}^m J_m(rs) e^{\mathrm{i}m\theta} e^{-\mathrm{i}m\alpha} \\ &= r^{-1} \cdot \sum_{m=-\infty}^{\infty} \mathrm{i}^m \left(\sum_{\{n:|m|\leq n \text{ and } |m| \overset{2}{\equiv} n\}} \sqrt{2n+2}(-1)^{\frac{n-m}{2}} J_{n+1}(ar)\mathcal{Z}^a_{nm}(s) \right) e^{\mathrm{i}m\theta} e^{-\mathrm{i}m\alpha} \\ &= \sqrt{2\pi} \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\leq n \text{ and } |m| \overset{2}{\equiv} n\}} \sqrt{2n+2} \frac{\mathrm{i}^m (-1)^{\frac{n-m}{2}} J_{n+1}(ar)}{r} e^{-\mathrm{i}m\alpha} V^a_{nm}(s,\theta). \end{aligned}$$

□

We then conclude the following consequences.

Corollary 1. *Let $a > 0$ and $m \in \mathbb{Z}$, $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. We then have:*

$$\int_0^a \int_0^{2\pi} e^{\mathrm{i}rs\cos(\alpha-\theta)} \overline{V^a_{nm}(s,\theta)} s ds d\theta = 2\sqrt{\pi(n+1)} \frac{\mathrm{i}^m (-1)^{\frac{n-m}{2}} J_{n+1}(ar)}{r} e^{-\mathrm{i}m\alpha}, \tag{25}$$

for all $r > 0$ and $0 < \alpha \leq 2\pi$.

For an integral vector $\mathbf{k} := (k_1, k_2)^T \in \mathbb{Z}^2$, let:

$$\rho(\mathbf{k}) = \rho(k_1, k_2) := \sqrt{k_1^2 + k_2^2},$$

and $0 \leq \Phi(\mathbf{k}) = \Phi(k_1, k_2) < 2\pi$ be given by:

$$k_1 = \rho(k_1, k_2)\cos\Phi(k_1, k_2), \qquad k_2 = \rho(k_1, k_2)\sin\Phi(k_1, k_2).$$

We may denote $\rho(\mathbf{k})$ with $|\mathbf{k}|$, as well.

Corollary 2. *Let $a > 0$ and $\mathbf{k} \in \mathbb{Z}^2$. Furthermore, let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. We then have:*

$$\int_0^a \int_0^{2\pi} e^{\pi \mathrm{i} a^{-1} s \mathbf{u}_\theta^T \mathbf{k}} \overline{V^a_{nm}(s\mathbf{u}_\theta)} s ds d\theta = 2\sqrt{n+1} \frac{\mathrm{i}^m (-1)^{\frac{n-m}{2}} J_{n+1}(\pi|\mathbf{k}|)}{a^{-1}\sqrt{\pi}|\mathbf{k}|} e^{-\mathrm{i}m\Phi(\mathbf{k})}. \tag{26}$$

Proof. Let $a > 0$ and $\mathbf{k} \in \mathbb{Z}^2$. Suppose $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. Applying Equation (25), for $r := a^{-1}\pi|\mathbf{k}|$ and $\alpha := \Phi(\mathbf{k})$, we get:

$$\begin{aligned}
\int_0^a \int_0^{2\pi} e^{\pi \mathrm{i} a^{-1} s \mathbf{u}_\theta^T \mathbf{k}} \overline{V_{nm}^a(s\mathbf{u}_\theta)} s ds d\theta &= \int_0^a \int_0^{2\pi} e^{\pi \mathrm{i} a^{-1} s|\mathbf{k}|\cos(\Phi(\mathbf{k})-\theta)} \overline{V_{nm}^a(s\mathbf{u}_\theta)} s ds d\theta \\
&= \int_0^a \int_0^{2\pi} e^{\mathrm{i} r s \cos(\alpha-\theta)} \overline{V_{nm}^a(s\mathbf{u}_\theta)} s ds d\theta \\
&= 2\sqrt{\pi(n+1)} \frac{\mathrm{i}^m (-1)^{\frac{n-m}{2}} J_{n+1}(ar)}{r} e^{-\mathrm{i} m \alpha} \\
&= 2\sqrt{\pi(n+1)} \frac{\mathrm{i}^m (-1)^{\frac{n-m}{2}} J_{n+1}(\pi|\mathbf{k}|)}{a^{-1}\pi|\mathbf{k}|} e^{-\mathrm{i} m \Phi(\mathbf{k})} \\
&= 2\sqrt{n+1} \frac{\mathrm{i}^m (-1)^{\frac{n-m}{2}} J_{n+1}(\pi|\mathbf{k}|)}{a^{-1}\sqrt{\pi}|\mathbf{k}|} e^{-\mathrm{i} m \Phi(\mathbf{k})}.
\end{aligned}$$

□

The next result presents a closed form for Fourier–Zernike coefficients of functions defined on disks.

Theorem 1. *Let $a > 0$ and $\Omega_a := [-a, a]^2$. Let $f \in L^2(\Omega_a)$ be a function supported in $\mathbb{B}_a^2$. Furthermore, let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. We then have:*

$$C_{n,m}^a(f) = \sum_{\mathbf{k} \in \mathbb{Z}^2} c_a(\mathbf{k}; n, m) \widehat{f}(\mathbf{k}), \tag{27}$$

where, for each $\mathbf{k} \in \mathbb{Z}^2$, we have:

$$\widehat{f}(\mathbf{k}) := \int_{-a}^{a} \int_{-a}^{a} f(x_1, x_2) e^{-\pi \mathrm{i} a^{-1}(k_1 x_1 + k_2 x_2)} dx_1 dx_2, \tag{28}$$

and:

$$c_a(\mathbf{k}; n, m) := \sqrt{n+1} \frac{\mathrm{i}^m (-1)^{\frac{n-m}{2}} J_{n+1}(\pi|\mathbf{k}|)}{2a\sqrt{\pi}|\mathbf{k}|} e^{-\mathrm{i} m \Phi(\mathbf{k})}. \tag{29}$$

Proof. Let $a > 0$ and $\Omega_a := [-a, a]^2$. Let $f \in L^2(\Omega_a)$ be a function supported in $\mathbb{B}_a^2$. Hence, we have:

$$f(\mathbf{x}) = \frac{1}{4a^2} \sum_{\mathbf{k} \in \mathbb{Z}^2} \widehat{f}(\mathbf{k}) e^{\pi \mathrm{i} a^{-1} \mathbf{x}^T \mathbf{k}}, \tag{30}$$

for all $\mathbf{x} = (x_1, x_2)^T \in \Omega_a$, where for $\mathbf{k} = (k_1, k_2)^T \in \mathbb{Z}^2$, we have:

$$\widehat{f}(\mathbf{k}) = \int_{-a}^{a} \int_{-a}^{a} f(x_1, x_2) e^{-\pi \mathrm{i} a^{-1}(k_1 x_1 + k_2 x_2)} dx_1 dx_2.$$

Hence, using (26), we get:

$$\begin{aligned}
C_{n,m}^a(f) &= \int_0^a \int_0^{2\pi} f(s\mathbf{u}_\theta) \overline{V_{nm}^a(s\mathbf{u}_\theta)} s ds d\theta \\
&= \int_0^a \int_0^{2\pi} \left(\frac{1}{4a^2} \sum_{\mathbf{k} \in \mathbb{Z}^2} \widehat{f}(\mathbf{k}) e^{\pi \mathrm{i} a^{-1} s \mathbf{u}_\theta^T \mathbf{k}} \right) \overline{V_{nm}^a(s\mathbf{u}_\theta)} s ds d\theta \\
&= \frac{1}{4a^2} \sum_{\mathbf{k} \in \mathbb{Z}^2} \widehat{f}(\mathbf{k}) \left(\int_0^a \int_0^{2\pi} e^{\pi \mathrm{i} a^{-1} s \mathbf{u}_\theta^T \mathbf{k}} \overline{V_{nm}^a(s\mathbf{u}_\theta)} s ds d\theta \right) = \sum_{\mathbf{k} \in \mathbb{Z}^2} c_a(\mathbf{k}; n, m) \widehat{f}(\mathbf{k}).
\end{aligned}$$

□

Corollary 3. *Let $a > 0$ and $\Omega_a := [-a,a]^2$. Let $f \in L^2(\Omega_a)$ be a function supported in $\mathbb{B}_a^2$. We then have:*

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\leq n \text{ and } |m|\overset{2}{\equiv}n\}} C_{n,m}^a(f) V_{nm}^a(r,\theta),$$

for $0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$.

Remark 1. *Equation (27) guarantees that the Fourier–Zernike coefficients of functions supported on disks can be computed from the standard Fourier coefficients $\widehat{f}(\mathbf{k})$, which can be implemented by FFT.*

The next result presents a closed form for Fourier–Zernike coefficients of functions supported on disks.

Theorem 2. *Let $a > 0$ and $f \in L^1(\mathbb{R}^2)$ be a function supported on $\mathbb{B}_a^2$. We then have:*

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\leq n \text{ and } |m|\overset{2}{\equiv}n\}} C_{n,m}^a(f) V_{nm}^a(r,\theta),$$

for $0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$, with:

$$C_{n,m}^a(f) = \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k};n,m)\widehat{f}(\mathbf{k}), \tag{31}$$

where, for each $\mathbf{k} := (k_1,k_2)^T \in \mathbb{Z}^2$, we have:

$$\widehat{f}(\mathbf{k}) := \int_{-a}^{a}\int_{-a}^{a} f(x_1,x_2)e^{-\pi \mathrm{i} a^{-1}(k_1x_1+k_2x_2)}dx_1dx_2. \tag{32}$$

The next result gives an explicit closed form for Fourier–Zernike coefficients of zero-padded functions.

Proposition 2. *Let $a > 0$ and $f \in L^1(\mathbb{R}^2)$ be a continuous function. Let $R(f)$ be the restriction of f to the disk $\mathbb{B}_a^2$ and $E(f)$ be the extension of $R(f)$ to the rectangle $\Omega_a := [-a,a]^2$ by zero-padding. Furthermore, let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. We then have:*

$$C_{n,m}^a(R(f)) = \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k};n,m)\widehat{E(f)}(\mathbf{k}), \tag{33}$$

where, for $\mathbf{k} := (k_1,k_2)^T \in \mathbb{Z}^2$:

$$\widehat{E(f)}(\mathbf{k}) = \int_{-a}^{a}\int_{-a}^{a} f(x_1,x_2)e^{-\pi \mathrm{i} a^{-1}(k_1x_1+k_2x_2)}dx_1dx_2. \tag{34}$$

Let $\mathcal{R} := \{\rho(k_1,k_2) : k_1,k_2 \in \mathbb{Z}\}$. For each $r \in \mathcal{R}$, let:

$$\Theta_r := \{\Phi(i,j) : r = \rho(i,j),\ \ i,j \in \mathbb{Z}\}.$$

Proposition 3. *With the above assumptions, we have:*

1. $\mathbb{N} \cup \{0\} \subseteq \mathcal{R} \subseteq \sqrt{\mathbb{N}} := \{\sqrt{n} : n \in \mathbb{N} \cup \{0\}\}$.
2. *$\mathcal{R}$ is a discrete subset of $[0,\infty)$.*
3. *For each $r \in \mathcal{R}$, the set Θ_r is a finite subset of $[0,2\pi)$.*

4. $\mathbb{Z}^2 = \bigcup_{r\in\mathcal{R}}\{(r\cos\theta, r\sin\theta)^T : \theta \in \Theta_r\}$.

Proof. (1)–(3) are straightforward.

(4) Let $\mathbf{x} \in \bigcup_{r\in\mathcal{R}}\{(r\cos\theta, r\sin\theta)^T : \theta \in \Theta_r\}$. Suppose $r \in \mathcal{R}$ and $\theta \in \Theta_r$ with $\mathbf{x} = (r\cos\theta, r\sin\theta)^T$. Hence, $\theta = \Phi(i,j)$ with $\rho(i,j) = r$, for some $i, j \in \mathbb{Z}$. We then have:

$$r\cos\theta = \rho(i,j)\cos\Phi(i,j) = \sqrt{i^2+j^2}\frac{i}{\sqrt{i^2+j^2}} = i \in \mathbb{Z},$$

$$r\sin\theta = \rho(i,j)\sin\Phi(i,j) = \sqrt{i^2+j^2}\frac{j}{\sqrt{i^2+j^2}} = j \in \mathbb{Z}.$$

Thus, we deduce that $\mathbf{x} = (r\cos\theta, r\sin\theta)^T \in \mathbb{Z}^2$. Therefore, we get $\bigcup_{r\in\mathcal{R}}\{(r\cos\theta, r\sin\theta)^T : \theta \in \Theta_r\} \subseteq \mathbb{Z}^2$. Conversely, let $\mathbf{x} = (k_1, k_2)^T \in \mathbb{Z}^2$ be given. We then have $k_1, k_2 \in \mathbb{Z}$, and hence, we get $k_1 = \rho(k_1,k_2)\cos\Phi(k_1,k_2)$ and $k_2 = \rho(k_1,k_2)\sin\Phi(k_1,k_2)$. Then, we conclude that $\mathbf{x} = (r\cos\theta, r\sin\theta)^T$, with $r := \rho(k_1,k_2)$ and $\theta := \Phi(k_1,k_2)$. This implies that $\mathbf{x} \in \bigcup_{r\in\mathcal{R}}\{(r\cos\theta, r\sin\theta)^T : \theta \in \Theta_r\}$, and hence, $\mathbb{Z}^2 \subseteq \bigcup_{r\in\mathcal{R}}\{(r\cos\theta, r\sin\theta)^T : \theta \in \Theta_r\}$. □

We then present the following polarized version of Theorem 1.

Theorem 3. *Let $a > 0$ and $\Omega_a := [-a,a]^2$. Let $f \in L^2(\Omega_a)$ be a function supported on $\mathbb{B}_a^2$. Furthermore, let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \stackrel{2}{\equiv} |m|$. We then have:*

$$C_{n,m}^a(f) = \sum_{\tau\in\mathcal{R}}\sum_{\alpha\in\Phi_\tau} A_{mn}^a(\tau,\alpha)\widehat{f}(\tau\mathbf{u}_\alpha), \tag{35}$$

where:

$$A_{mn}^a(\tau,\alpha) := c_a(\tau\mathbf{u}_\alpha; n, m). \tag{36}$$

Proof. Let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \stackrel{2}{\equiv} |m|$. First, suppose that $\tau \in \mathcal{R}$ and $\alpha \in \Phi_\tau$. Let $\mathbf{k} := \tau\mathbf{u}_\alpha = (\tau\cos\alpha, \tau\sin\alpha)^T \in \mathbb{Z}^2$. Thus, $|\mathbf{k}| = \tau$ and $\Phi(\mathbf{k}) = \alpha$. Therefore, using (27), we get:

$$\begin{aligned} C_{n,m}(f) &= \sum_{\mathbf{k}\in\mathbb{Z}^2} c(\mathbf{k}; n, m)\widehat{f}(\mathbf{k}) \\ &= \sum_{\tau\in\mathcal{R}}\sum_{\alpha\in\Phi_\tau} c(\tau\mathbf{u}_\alpha; n, m)\widehat{f}(\tau\mathbf{u}_\alpha) = \sum_{\tau\in\mathcal{R}}\sum_{\alpha\in\Phi_\tau} A_{mn}(\tau,\alpha)\widehat{f}(\tau\mathbf{u}_\alpha). \end{aligned}$$

□

Theorem 4. *Let $a > 0$ and $f \in L^1(\mathbb{R}^2)$ be a function supported on $\mathbb{B}_a^2$. Furthermore, let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \stackrel{2}{\equiv} |m|$. We then have:*

$$C_{n,m}^a(f) = \sum_{\tau\in\mathcal{R}}\sum_{\alpha\in\Phi_\tau} A_{mn}^a(\tau,\alpha)\widehat{f}(\tau\mathbf{u}_\alpha), \tag{37}$$

where:

$$A_{mn}^a(\tau,\alpha) := c_a(\tau\mathbf{u}_\alpha; n, m). \tag{38}$$

The next result gives a polarized version for explicit closed form of Fourier–Zernike coefficients for zero-padded functions.

Proposition 4. *Let $a > 0$ and $f \in L^1(\mathbb{R}^2)$ be a continuous function. Let $R(f)$ be the restriction of f to the unit disk $\mathbb{B}^2_a$ and $E(f)$ be the canonical extension of $R(f)$ to the rectangle $\Omega_a := [-a,a]^2$ by zero-padding. Furthermore, let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. We then have:*

$$C^a_{n,m}(R(f)) = \sum_{\tau\in\mathcal{R}} \sum_{\alpha\in\Phi_\tau} A^a_{mn}(\tau,\alpha)\widehat{E(f)}(\tau\mathbf{u}_\alpha), \tag{39}$$

where:

$$\widehat{E(f)}(\tau\mathbf{u}_\alpha) = \int_{-a}^{a}\int_{-a}^{a} f(x_1,x_2)e^{-\pi\mathrm{i}\tau a^{-1}(x_1\cos\alpha + x_2\sin\alpha)}dx_1dx_2. \tag{40}$$

Theorem 5. *Let $a > 0$ and $f \in L^1(\mathbb{R}^2)$ be a continuous function. Let $R(f)$ be the restriction of f to the unit disk $\mathbb{B}^2_a$ and $E(f)$ be the canonical extension of $R(f)$ to the rectangle $\Omega_a := [-a,a]^2$ by zero-padding. We then have:*

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\leq n \text{ and } |m|\overset{2}{\equiv} n\}} C^a_{n,m}(R(f))V^a_{nm}(r,\theta),$$

for $0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$, where:

$$C^a_{n,m}(R(f)) = \sum_{\tau\in\mathcal{R}} \sum_{\alpha\in\Phi_\tau} A^a_{mn}(\tau,\alpha)\widehat{E(f)}(\tau\mathbf{u}_\alpha). \tag{41}$$

4. Fourier–Zernike Series for Convolution of Functions Supported on Disks

We then continue by investigating analytical aspects of Fourier–Zernike series as a constructive approximation for the convolution of functions supported on disks.

The following theorem introduces a constructive method for computing the Fourier–Zernike coefficients of the convolution of functions supported on disks.

Theorem 6. *Let $a > 0$ and $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1,2\}$ be functions supported on $\mathbb{B}^2_{a/2}$. Furthermore, let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. The Fourier–Zernike coefficient $C^a_{n,m}(f_1 * f_2)$ of $f_1 * f_2$ is given by:*

$$C^a_{n,m}(f_1 * f_2) = \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k};n,m)\widehat{f_1}(\mathbf{k})\widehat{f_2}(\mathbf{k}), \tag{42}$$

where, for $j \in \{1,2\}$, $\mathbf{k} \in \mathbb{Z}^2$, $m \in \mathbb{Z}$, and $n \in \mathbb{N}$, we have:

$$\widehat{f_j}(\mathbf{k}) := \int_{-a}^{a}\int_{-a}^{a} f_j(x_1,x_2)e^{-\pi\mathrm{i}a^{-1}(k_1x_1+k_2x_2)}dx_1dx_2, \tag{43}$$

and:

$$c_a(\mathbf{k};n,m) := \sqrt{n+1}\frac{\mathrm{i}^m(-1)^{\frac{n-m}{2}}J_{n+1}(\pi|\mathbf{k}|)}{2a\sqrt{\pi}|\mathbf{k}|}e^{-\mathrm{i}m\Phi(\mathbf{k})}. \tag{44}$$

Proof. Let $a > 0$ and $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1,2\}$ be functions supported on $\mathbb{B}^2_{a/2}$. Then, $f_1 * f_2$ is supported on $\mathbb{B}^2_a$. Let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. Regarding each f_j as a function supported on $\mathbb{B}^2_a$, by the convolution property of the Fourier transform, we have:

$$(\widehat{f_1 * f_2})(\mathbf{k}) = \widehat{f_1}(\mathbf{k})\widehat{f_2}(\mathbf{k}), \tag{45}$$

for each $\mathbf{k} \in \mathbb{Z}^2$. Then, applying (45) in Equation (27), we get:

$$\begin{aligned} C^a_{n,m}(f_1 * f_2) &= \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; n, m)\widehat{f_1 * f_2}(\mathbf{k}) \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; n, m)\widehat{f}_1(\mathbf{k})\widehat{f}_2(\mathbf{k}). \end{aligned}$$

□

Corollary 4. *Let $a > 0$ and $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1, 2\}$ be functions supported on $\mathbb{B}^2_{a/2}$. We then have:*

$$(f_1 * f_2)(r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\le n \text{ and } |m|\overset{2}{\equiv}n\}} C^a_{n,m}(f_1 * f_2)V^a_{nm}(r, \theta), \tag{46}$$

where:

$$C^a_{n,m}(f_1 * f_2) = \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; n, m)\widehat{f}_1(\mathbf{k})\widehat{f}_2(\mathbf{k}). \tag{47}$$

We then present the following polarized version of Theorem 6.

Theorem 7. *Let $a > 0$ and $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1, 2\}$ be functions supported on $\mathbb{B}^2_{a/2}$. Furthermore, $m \in \mathbb{Z}$ and $n \ge |m|$ with $n \overset{2}{\equiv} |m|$. We then have:*

$$C^a_{n,m}(f_1 * f_2) = \sum_{\tau\in\mathcal{R}} \sum_{\alpha\in\Phi_\tau} A^a_{mn}(\tau, \alpha)\widehat{f}_1(\tau\mathbf{u}_\alpha)\widehat{f}_2(\tau\mathbf{u}_\alpha). \tag{48}$$

Corollary 5. *Let $a > 0$ and $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1, 2\}$ be functions supported on $\mathbb{B}^2_{a/2}$. We then have:*

$$f_1 * f_2(r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\le n \text{ and } |m|\overset{2}{\equiv}n\}} C^a_{n,m}(f_1 * f_2)V^a_{nm}(r, \theta), \tag{49}$$

where:

$$C^a_{n,m}(f_1 * f_2) = \sum_{\tau\in\mathcal{R}} \sum_{\alpha\in\Phi_\tau} A^a_{mn}(\tau, \alpha)\widehat{f}_1(\tau\mathbf{u}_\alpha)\widehat{f}_2(\tau\mathbf{u}_\alpha). \tag{50}$$

Next, we present a closed form for Fourier–Zernike coefficients of zero-padded functions.

Theorem 8. *Let $a > 0$ and $b := a/2$. Suppose $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1, 2\}$ are continuous functions. Let $R(f_j)$ be the restriction of f_j to $\mathbb{B}^2_b$ and $E(f_j)$ be the canonical extension of $R(f_j)$ to the rectangle $\Omega_a := [-a, a]^2$ by zero-padding. Furthermore, let $m \in \mathbb{Z}$ and $n \ge |m|$ with $n \overset{2}{\equiv} |m|$. We then have:*

$$C^a_{n,m}(f_1 \circledast f_2) = \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; n, m)\widehat{f}_1[\mathbf{k}]\widehat{f}_2[\mathbf{k}], \tag{51}$$

where, for $\mathbf{k} := (k_1, k_2)^T \in \mathbb{Z}^2$:

$$\widehat{f}_j[\mathbf{k}] := \int_0^b \int_0^{2\pi} f_j(r, \theta)e^{-\pi i a^{-1} r(k_1\cos\theta + k_2\sin\theta)} r dr d\theta. \tag{52}$$

Proof. Let $a > 0$ and $b := a/2$. Suppose $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1, 2\}$ are continuous functions. Let $R(f_j)$ be the restriction of f_j to the disk $\mathbb{B}^2_b$ and $E(f_j)$ be the extension of $R(f_j)$ to the rectangle $\Omega_a := [-a, a]^2$ by zero-padding. We then have:

$$f_1 \circledast f_2 = E(f_1) * E(f_2).$$

Let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. Then, using Equation (42), we get:

$$\begin{aligned} C^a_{n,m}(f_1 \circledast f_2) &= C^a_{n,m}(E(f_1) * E(f_2)) \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; n, m)\widehat{E(f_1)}(\mathbf{k})\widehat{E(f_2)}(\mathbf{k}), \end{aligned}$$

with:

$$\widehat{E(f_j)}(\mathbf{k}) = \int_{-a}^{a}\int_{-a}^{a} E(f_j)(x_1, x_2)e^{-\pi \mathrm{i} a^{-1}(k_1x_1+k_2x_2)}dx_1dx_2,$$

for $j \in \{1, 2\}$. Since each $E(f_j)$ is an extension of $R(f_j)$ to the rectangle Ω_a by zero-padding and $R(f_j)$ is the restriction of f_j to the disk $\mathbb{B}^2_b \subseteq [-b, b]^2$, we can write:

$$\begin{aligned} \widehat{E(f_j)}(\mathbf{k}) &= \int_{-a}^{a}\int_{-a}^{a} E(f_j)(x_1, x_2)e^{-\pi \mathrm{i} a^{-1}(k_1x_1+k_2x_2)}dx_1dx_2 \\ &= \int_{\Omega_a} E(f_j)(x_1, x_2)e^{-\pi \mathrm{i} a^{-1}(k_1x_1+k_2x_2)}dx_1dx_2 \\ &= \int_{\mathbb{B}^2_b} f_j(x_1, x_2)e^{-\pi \mathrm{i} a^{-1}(k_1x_1+k_2x_2)}dx_1dx_2 \\ &= \int_0^b\int_0^{2\pi} f_j(r, \theta)e^{-\pi \mathrm{i} a^{-1}r(k_1\cos\theta+k_2\sin\theta)}rdrd\theta = \widehat{f}_j[\mathbf{k}]. \end{aligned}$$

□

Corollary 6. *Let $a > 0$ and $b := a/2$. Suppose $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1, 2\}$ are continuous functions. Let $R(f_j)$ be the restriction of f_j to $\mathbb{B}^2_b$ and $E(f_j)$ be the canonical extension of $R(f_j)$ to the rectangle $\Omega_a := [-a, a]^2$ by zero-padding. We then have:*

$$(f_1 \circledast f_2)(r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\leq n \text{ and } |m|\overset{2}{\equiv} n\}} C^a_{n,m}(f_1 \circledast f_2)V^a_{nm}(r, \theta), \tag{53}$$

where:

$$C^a_{n,m}(f_1 \circledast f_2) = \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; n, m)\widehat{f}_1[\mathbf{k}]\widehat{f}_2[\mathbf{k}]. \tag{54}$$

We then present the following polarized version of closed forms for Fourier–Zernike approximations of zero-padded functions.

Theorem 9. *Let $a > 0$ and $b := a/2$. Suppose $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1, 2\}$ are continuous functions. Let $R(f_j)$ be the restriction of f_j to $\mathbb{B}^2_b$ and $E(f_j)$ be the canonical extension of $R(f_j)$ to the rectangle $\Omega_a := [-a, a]^2$ by zero-padding. Furthermore, let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. We then have:*

$$C^a_{n,m}(f_1 \circledast f_2) = \sum_{\tau\in\mathcal{R}}\sum_{\alpha\in\Phi_\tau} A^a_{mn}(\tau, \alpha)\widehat{f}_1[\tau\mathbf{u}_\alpha]\widehat{f}_2[\tau\mathbf{u}_\alpha]. \tag{55}$$

Corollary 7. *Let $a > 0$ and $b := a/2$. Suppose $f_j \in L^1(\mathbb{R}^2)$ with $j \in \{1, 2\}$ are continuous functions. Let $R(f_j)$ be the restriction of f_j to $\mathbb{B}^2_b$ and $E(f_j)$ be the canonical extension of $R(f_j)$ to the rectangle $\Omega_a := [-a, a]^2$ by zero-padding. We then have:*

$$(f_1 \circledast f_2)(r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\leq n \text{ and } |m|\overset{2}{\equiv} n\}} C^a_{n,m}(f_1 \circledast f_2)V^a_{nm}(r, \theta), \tag{56}$$

Convolution Approximation of Fourier–Zernike Basis Elements

Let $a > 0$ and $b := a/2$. Suppose $f_j : \mathbb{R}^2 \to \mathbb{R}$ with $j \in \{1,2\}$ are continuous functions supported on the disk $\mathbb{B}_b^2$ with the associated Fourier–Zernike coefficients $\left\{C_{n,m}^b(f_j) : m \in \mathbb{Z},\ n \in \mathbb{I}_m\right\}$, with $\mathbb{I}_m := \{n : |m| \leq n \text{ and } |m| \stackrel{2}{\equiv} n\}$. Hence, we can write:

$$f_j = \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\leq n \text{ and } |m|\stackrel{2}{\equiv}n\}} C_{n,m}^b(f_j) V_{nm}^b, \tag{57}$$

where:

$$C_{n,m}^b(f_j) = \sum_{\mathbf{k}\in\mathbb{Z}^2} c_b(\mathbf{k}; n, m) \widehat{f_j}(\mathbf{k}),$$

for $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \stackrel{2}{\equiv} |m|$.

Using the linearity of convolutions, as linear operators, we get:

$$\begin{aligned} f_1 * f_2 &= \left(\sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\leq n \text{ and } |m|\stackrel{2}{\equiv}n\}} C_{n,m}^b(f_1) V_{nm}^b \right) * \left(\sum_{m'=-\infty}^{\infty} \sum_{\{n':|m'|\leq n' \text{ and } |m'|\stackrel{2}{\equiv}n'\}} C_{n',m'}^b(f_2) V_{n'm'}^b \right) \\ &= \sum_{m=-\infty}^{\infty} \sum_{\{n:|m|\leq n \text{ and } |m|\stackrel{2}{\equiv}n\}} \sum_{m'=-\infty}^{\infty} \sum_{\{n':|m'|\leq n' \text{ and } |m'|\stackrel{2}{\equiv}n'\}} C_{n,m}^b(f_1) C_{n',m'}^b(f_2) V_{nm}^b \circledast V_{n'm'}^b, \end{aligned}$$

where $V_{nm}^b \circledast V_{n'm'}^b$ is the standard convolution of Fourier–Zernike basis elements, considering them as functions defined on $\mathbb{R}^2$ by zero-padding and supported in $\mathbb{B}_b^2$.

Therefore, the convolution of Fourier–Zernike basis elements can be viewed as pre-computed kernels.

Proposition 5. *Let $a > 0$ and $b := a/2$. Suppose $\mathbf{k} \in \mathbb{Z}^2$, $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \stackrel{2}{\equiv} |m|$. We then have:*

$$\widehat{V_{nm}^b}[\mathbf{k}] = 2\sqrt{n+1}\,\mathrm{i}^{-m} e^{\mathrm{i}m\Phi(\mathbf{k})} a(-1)^{n-m} \frac{J_{n+1}(\pi|\mathbf{k}|/2)}{\sqrt{\pi}|\mathbf{k}|}. \tag{58}$$

Proof. Let $m \in \mathbb{Z}$ and $n \geq |m|$ with $n \overset{2}{\equiv} |m|$. Regarding V^b_{nm} as a function defined on $\mathbb{R}^2$ by zero-padding and supported on $\mathbb{B}^2_b$, still denoted by V^b_{nm}, for each $\mathbf{k} \in \mathbb{Z}^2$, we can write:

$$\begin{aligned}
\widehat{V^b_{nm}}[\mathbf{k}] &= \widehat{V^b_{nm}}[|\mathbf{k}|\mathbf{u}_{\Phi(\mathbf{k})}] \\
&= \int_{\mathbb{B}^2_b} V_{nm}(\mathbf{x}) e^{-\pi \mathrm{i}|\mathbf{k}|\mathbf{u}^T_{\Phi(\mathbf{k})}\mathbf{x}} d\mathbf{x} \\
&= \int_0^{2\pi}\int_0^b V^b_{nm}(r,\theta) e^{-\pi \mathrm{i} a^{-1} r|\mathbf{k}|\mathbf{u}^T_{\Phi(\mathbf{k})}\mathbf{u}_\theta} r dr d\theta \\
&= \int_0^{2\pi}\int_0^b V^b_{nm}(r,\theta)\left(\sum_{l=-\infty}^{\infty} \mathrm{i}^{-l} J_l(a^{-1}\pi r|\mathbf{k}|) e^{-\mathrm{i}l\theta} e^{\mathrm{i}l\Phi(\mathbf{k})}\right) r dr d\theta \\
&= \frac{\sqrt{2n+2}}{b\sqrt{2\pi}} \int_0^{2\pi}\int_0^b Z_{nm}(b^{-1}r) e^{\mathrm{i}m\theta}\left(\sum_{l=-\infty}^{\infty} \mathrm{i}^{-l} J_l(a^{-1}\pi r|\mathbf{k}|) e^{-\mathrm{i}l\theta} e^{\mathrm{i}l\Phi(\mathbf{k})}\right) r dr d\theta \\
&= \frac{\sqrt{2n+2}}{b\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \mathrm{i}^{-l} e^{\mathrm{i}l\Phi(\mathbf{k})}\left(\int_0^{2\pi}\int_0^b Z_{nm}(b^{-1}r) J_l(a^{-1}\pi r|\mathbf{k}|) e^{\mathrm{i}m\theta} e^{-\mathrm{i}l\theta} r dr d\theta\right) \\
&= \frac{\sqrt{2n+2}}{b\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} \mathrm{i}^{-l} e^{\mathrm{i}l\Phi(\mathbf{k})}\left(\int_0^{2\pi} e^{\mathrm{i}m\theta} e^{-\mathrm{i}l\theta} d\theta\right)\left(\int_0^b Z_{nm}(b^{-1}r) J_l(a^{-1}\pi r|\mathbf{k}|) r dr\right) \\
&= \frac{\sqrt{2\pi(2n+2)}}{b} \sum_{l=-\infty}^{\infty} \mathrm{i}^{-l} e^{\mathrm{i}l\Phi(\mathbf{k})} \delta_{ml}\left(\int_0^b Z_{nm}(b^{-1}r) J_l(a^{-1}\pi r|\mathbf{k}|) r dr\right) \\
&= \frac{\sqrt{2\pi(2n+2)}}{b} \mathrm{i}^{-m} e^{\mathrm{i}m\Phi(\mathbf{k})}\left(\int_0^b Z_{nm}(b^{-1}r) J_m(a^{-1}\pi r|\mathbf{k}|) r dr\right).
\end{aligned}$$

Hence, we get:

$$\widehat{V^b_{nm}}[\mathbf{k}] = \frac{\sqrt{2\pi(2n+2)}}{b} \mathrm{i}^{-m} e^{\mathrm{i}m\Phi(\mathbf{k})}\left(\int_0^b Z_{nm}(b^{-1}r) J_m(a^{-1}\pi r|\mathbf{k}|) r dr\right). \tag{59}$$

Applying (13) in (59), we can write:

$$\int_0^b Z_{nm}(b^{-1}r) J_m(a^{-1}\pi r|\mathbf{k}|) r dr = b(-1)^{\frac{n-m}{2}} \frac{J_{n+1}(ba^{-1}\pi|\mathbf{k}|)}{a^{-1}\pi|\mathbf{k}|} = a^2(-1)^{\frac{n-m}{2}} \frac{J_{n+1}(\pi|\mathbf{k}|/2)}{2\pi|\mathbf{k}|}.$$

which implies that:

$$\begin{aligned}
\widehat{V^b_{nm}}[\mathbf{k}] &= \frac{\sqrt{2\pi(2n+2)}}{b} \mathrm{i}^{-m} e^{\mathrm{i}m\Phi(\mathbf{k})}\left(\int_0^b Z_{nm}(b^{-1}r) J_m(a^{-1}\pi r|\mathbf{k}|) r dr\right) \\
&= \frac{\sqrt{2\pi(2n+2)}}{b} \mathrm{i}^{-m} e^{\mathrm{i}m\Phi(\mathbf{k})} a^2 (-1)^{\frac{n-m}{2}} \frac{J_{n+1}(\pi|\mathbf{k}|/2)}{2\pi|\mathbf{k}|} \\
&= 2\sqrt{n+1}\,\mathrm{i}^{-m} e^{\mathrm{i}m\Phi(\mathbf{k})} a (-1)^{n-m} \frac{J_{n+1}(\pi|\mathbf{k}|/2)}{\sqrt{\pi}|\mathbf{k}|}.
\end{aligned}$$

□

Proposition 6. *Let $a > 0$ and $b := a/2$. Furthermore, let $m, m' \in \mathbb{Z}$ and $n \in \mathbb{I}_m$, $n' \in \mathbb{I}_{m'}$. Then, for each $\ell \in \mathbb{Z}$ and $k \in \mathbb{I}_\ell$, we have:*

$$C^a_{k,\ell}(V^b_{nm} \circledast V^b_{n'm'}) = \frac{a^2\sqrt{(n+1)(n'+1)}}{\mathrm{i}^{(m+m')}(-1)^{m-n-m'+n'}\pi} \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; k, \ell) \frac{e^{\mathrm{i}(m+m')\Phi(\mathbf{k})} J_{n'+1}(\pi|\mathbf{k}|/2) J_{n'+1}(\pi|\mathbf{k}|/2)}{|\mathbf{k}|}. \tag{60}$$

Proof. Let $a > 0$ and $b := a/2$. Furthermore, let $m, m' \in \mathbb{Z}$ and $n \in \mathbb{I}_m$, $n' \in \mathbb{I}_{m'}$. Regarding V^b_{nm} and $V^b_{n'm'}$ as functions supported on $\mathbb{B}^2_b$, $V^b_{nm} \circledast V^b_{n'm'}$ is a function supported on the disk $\mathbb{B}^2_a$. Suppose $\ell \in \mathbb{Z}$ and $k \in \mathbb{I}_\ell$. Using (51), we have:

$$\begin{aligned} C^a_{k,\ell}(V^b_{nm} \circledast V^b_{n'm'}) &= \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; k, \ell) \widehat{V^b_{nm}}[\mathbf{k}] \widehat{V^b_{n'm'}}[\mathbf{k}] \\ &= \frac{a^2\sqrt{(n+1)(n'+1)}}{\mathrm{i}^{(m+m')}(-1)^{m-n-m'+n'}\pi} \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; k, \ell) \frac{e^{\mathrm{i}(m+m')\Phi(\mathbf{k})} J_{n'+1}(\pi|\mathbf{k}|/2) J_{n'+1}(\pi|\mathbf{k}|/2)}{|\mathbf{k}|}. \end{aligned}$$

□

Theorem 10. *Let $a > 0$ and $b := a/2$. Suppose $m, m' \in \mathbb{Z}$, $n \in \mathbb{I}_m$ and $n' \in \mathbb{I}_{m'}$. We then have:*

$$V^b_{nm} \circledast V^b_{n'm'}(r, \theta) = \sum_{\ell=\infty}^{\infty} \sum_{k\in\mathbb{I}_\ell} C^a_{k,\ell}(V^b_{nm} \circledast V^b_{n'm'}) V^a_{k,\ell}(r, \theta), \tag{61}$$

where for each $\ell \in \mathbb{Z}$ and $k \in \mathbb{I}_\ell$, we have:

$$C^a_{k,\ell}(V^b_{nm} \circledast V^b_{n'm'}) = \frac{a^2\sqrt{(n+1)(n'+1)}}{\mathrm{i}^{(m+m')}(-1)^{m-n-m'+n'}\pi} \sum_{\mathbf{k}\in\mathbb{Z}^2} c_a(\mathbf{k}; k, \ell) \frac{e^{\mathrm{i}(m+m')\Phi(\mathbf{k})} J_{n'+1}(\pi|\mathbf{k}|/2) J_{n'+1}(\pi|\mathbf{k}|/2)}{|\mathbf{k}|}. \tag{62}$$

5. Conclusions

The mathematical foundations for computing convolutions of functions supported on disks in the plane are derived. The motivation for this work is the way that the Fourier–Zernike basis transforms under rotation, which is not shared by the multi-dimensional Fourier series of periodized functions. Extensions to functions supported on balls in d-dimensional Euclidean space with the Fourier series for the angular direction being replaced by hyper-spherical harmonics follow in a natural way.

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