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Extensions of Móricz Classes and Convergence of Trigonometric Sine Series in L^1-Norm

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Abstract: In this paper, the extensions of classes $\widetilde{S}$, $\widetilde{C}$ and $\widetilde{B}V$ are made by defining the classes $\widetilde{S}_r$, $\widetilde{C}_r$ and $\widetilde{B}V_r$, $r = 0, 1, 2, \ldots$ It is also shown that class $\widetilde{S}_r$ is a subclass of $\widetilde{C}_r \cap \widetilde{B}V_r$. Moreover, the results on L^1-convergence of r times differentiated trigonometric sine series have been obtained by considering the r^{th} $(r = 0, 1, 2, \ldots)$ derivative of modified sine sum under the new extended class $\widetilde{C}_r \cap \widetilde{B}V_r$.

Keywords: Dirichlet kernel; L^1-convergence; modified sine sum

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1. Introduction

Consider the trigonometric sine series

$$\sum_{k=1}^{\infty} a_k \sin kx \tag{1}$$

where $a_0, a_1, a_2, \ldots$ are the real coefficients. The nth partial sum, S_n, of Series (1) is represented as

$$S_n(x) = \sum_{k=1}^{n} a_k \sin kx = -\sum_{k=1}^{n} b_k (\cos kx)' \tag{2}$$

where the prime denotes derivatives and $b_k = \frac{a_k}{k}$. Also, $f(x) = \lim_{n\to\infty} S_n(x)$.

Various conditions are given in the literature (see [1–9]), which guarantee that Series (1) is a Fourier series.

In 1984, Teljakovskii [9] introduced a class $\widetilde{S}$, as follows:

Class $\widetilde{S}$ [9]. A null sequence $\{a_k\}$ is said to belong to class $\widetilde{S}$ if there exists a non-increasing sequence $\{B_k\}$ of numbers *s.t.*

$$|\Delta b_k| \leq B_k \quad \forall k = 1, 2, 3, \ldots$$
$$\sum_{k=1}^{\infty} k\, B_k < \infty.$$

where $b_k = \frac{a_k}{k}$, $\Delta b_k = b_k - b_{k+1}$ and proved the following result:

Theorem 1 [9]. *If $\{a_k\} \in \widetilde{S}$, then Series (1) is the Fourier series of some function $f \in L^1(0, \pi)$.*

In 1989, Móricz [5] introduced new classes $\widetilde{B}V$ and $\widetilde{C}$ of the coefficient sequences for the sine series.

Class $\widetilde{B}V$ **[5].** A null sequence $\{a_k\}$ belongs to $\widetilde{B}V$ if

$$\sum_{k=1}^{\infty} k\, |\Delta b_k| < \infty \tag{3}$$

Class $\widetilde{C}$ [5]. A null sequence $\{a_k\}$ belongs to class $\widetilde{C}$ if for every $\varepsilon > 0$ there exists $\delta > 0$, independent of n, and such that for all n,

$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta b_k D_k'(x) \right| dx \le \varepsilon. \tag{4}$$

Here, $D_k'(x)$ is the first derivative of Dirichlet kernel $\left(D_k(x) = \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}} \right)$.
Equation (4) implies that, for $1 \le n \le N$,

$$\int_0^\delta \left| \sum_{k=n}^{N} \Delta b_k D_k'(x) \right| dx \le 2\varepsilon.$$

The following result was proved by Móricz [7].

Theorem 2 [5]. *If $\{a_k\} \in \widetilde{B}V$, then*

$$\|u_n - f\| \to 0 \quad (n \to \infty) \text{ if and only if } \{a_k\} \in \widetilde{C}.$$

where $u_n(x) = S_n(x) + b_{n+1}D_n'(x)$.

The classes $\widetilde{S}$, $\widetilde{B}V$ and $\widetilde{C}$ seem to be more appropriate for the sine series than the classes S ([7,8]) BV [10], and C [3] in the ordinary sense. Also, Móricz [5] has proved that $\widetilde{S} \subset \widetilde{B}V \cap \widetilde{C}$.

Motivated by the aforesaid authors, new extended classes $\widetilde{S}_r$, $\widetilde{B}V_r$, and $\widetilde{C}_r$ $(r = 0, 1, 2, \ldots)$ are defined in this paper as follows:

Class $\widetilde{S}_r$. A sequence $\{a_k\}$ is said to belong to class $\widetilde{S}_r$ $(r = 0, 1, 2, \ldots)$ if $a_k \to 0$ as $k \to \infty$, and there exists a non-increasing sequence $\{B_k\}$ of numbers *s.t.*

$$|\Delta b_k| \le B_k \quad \forall k = 1, 2, 3, \ldots$$
$$\sum_{k=1}^{\infty} k^{r+1} B_k < \infty, \; r = 0, 1, 2, 3, \ldots$$

where $b_k = \frac{a_k}{k}$, $r = 0, 1, 2, 3, \ldots$

$B_k \downarrow 0$ and $\sum_{k=1}^{\infty} k^{r+1} B_k < \infty$, implies that $k^{r+2} B_k = o(1)$ as $k \to \infty$ $(r = 0, 1, 2, \ldots)$.

Remark 1. *For $r = 0$, $\widetilde{S}_r = \widetilde{S}$.*

Remark 2. *Obviously, $\widetilde{S}_{r+1} \subset \widetilde{S}_r$, but the converse need not be true.*

Example 1. *Consider a sequence $\Delta b_n = \frac{1}{n^{r+3}}$, $r = 0, 1, 2, \ldots$ and $n \in N$.*

$$a_n = nb_n = n \sum_{k=n}^{\infty} \Delta b_k \le \sum_{k=n}^{\infty} \frac{k}{k^{r+3}} = \sum_{k=n}^{\infty} \frac{1}{k^{r+2}} \to 0 \text{ as } n \to \infty.$$

Choose $B_n = \frac{1}{n^{r+3}}$, $r = 0, 1, 2, \ldots$ $\forall n$. Clearly, $B_n \downarrow 0$ as $n \to \infty$ and $|\Delta b_n| \le B_n \, \forall n$.
Consider the series

$$\sum_{n=1}^{\infty} n^{r+1} B_n = \sum_{n=1}^{\infty} n^{r+1} \frac{1}{n^{r+3}} \approx \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{which is convergent.}$$

This implies $\{a_n\} \in \widetilde{S}_r$.

But the series $\sum\limits_{n=1}^{\infty} n^{r+2} B_n \approx \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent.

This implies that $\{a_n\}$ does not belong to class $\widetilde{S}_{r+1}$.

Class $\widetilde{B}V_r$. A null sequence $\{a_k\}$ belongs to $\widetilde{B}V_r$, $(r = 0, 1, 2, \ldots)$ if

$$\sum_{k=1}^{\infty} k^{r+1} |\Delta b_k| < \infty$$

Remark 3. *For $r = 0$, $\widetilde{B}V_r = \widetilde{B}V$.*

Remark 4. *Clearly, $\widetilde{B}V_{r+1} \subset \widetilde{B}V_r$, $(r = 0, 1, 2, \ldots)$, but the converse may not be true.*

Class $\widetilde{C}_r$. A null sequence $\{a_k\}$ belongs to class $\widetilde{C}_r$ $(r = 0, 1, 2, \ldots)$, if for every $\varepsilon > 0$, there exists $\delta > 0$, independent of n, and such that for all n,

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx \leq \varepsilon$$

Here, $D_k^{r+1}(x)$ is the $(r+1)^{th}$ derivative of Dirichlet kernel.

Equation (4) implies, for $1 \leq n \leq N$,

$$\int_0^{\delta} \left| \sum_{k=n}^{N} \Delta b_k D_k^{r+1}(x) \right| dx \leq 2\varepsilon$$

Remark 5. *For $r = 0$, $\widetilde{C}_r = \widetilde{C}$.*

Remark 6. *It is obvious that $\widetilde{C}_{r+1} \subset \widetilde{C}_r$ but the converse need not be true.*

Example 2. *Define $\Delta b_n = \frac{1}{n^{r+3}}$, $r = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$*

$$a_n = n b_n = n \sum_{k=n}^{\infty} \Delta b_k \leq \sum_{k=n}^{\infty} \frac{k}{k^{r+3}} = \sum_{k=n}^{\infty} \frac{1}{k^{r+2}} \to 0 \text{ as } n \to \infty.$$

Consider, the integral

$$\begin{aligned} \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{r+2}(x) \right| dx &= \sum_{k=n}^{\infty} \frac{1}{n^{r+3}} \int_0^{\pi} |D_k^{r+2}(x)|\, dx = O\left(\sum_{k=n}^{\infty} \frac{1}{n^{r+3}} \left(n^{r+2} \log n\right) \right) \\ &= O\left(\sum_{k=n}^{\infty} \frac{\log n}{n} \right) \end{aligned}$$

which is divergent.

However,

$$\begin{aligned} \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx &= \sum_{k=n}^{\infty} \frac{1}{n^{r+3}} \int_0^{\pi} |D_k^{r+1}(x)|\, dx = O\left(\sum_{k=n}^{\infty} \frac{1}{n^{r+3}} \left(n^{r+1} \log n\right) \right) \\ &= O\left(\sum_{k=n}^{\infty} \frac{\log n}{n^2} \right) \text{ which is convergent.} \end{aligned}$$

Therefore $\{a_n\} \in \widetilde{C}_r$.

Lemmas related to the main results are given in Section 2. The Section 3 comprises the main results of this paper. Firstly, in this section, we have shown that the new extended class $\widetilde{S}_r$ is a subclass of $\widetilde{C}_r \cap \widetilde{B}V_r$ $(r = 0, 1, 2, \ldots)$. Moreover, the theorems are presented concerning the L^1 convergence of trigonometric sine series using modified sine sum [11], defined as

$$\beta_n(x) = \sum_{k=1}^{n} \left(\frac{a_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2 \left(\frac{a_j}{j} \right) \right) k \sin kx \tag{5}$$

under the extended classes of numerical sequences.

2. Lemmas

Lemma 1. *[6] Let $n \geq 1$ and r be a nonnegative integer $x \in [\varepsilon, \pi]$. Then, $|D_n^r(x)| \leq \frac{Cn^r}{x}$, where C denotes a positive absolute constant.*

Lemma 2. *[6] $\|D_n^r(x)\|_{L^1} = O(n^r \log n)$, $r = 0, 1, 2, \ldots$ where $D_n^r(x)$ represents the r^{th} derivative of the Dirichlet kernel.*

3. Main Results

Theorem 3. *The following relation holds $\widetilde{S}_r \subset \widetilde{C}_r \cap \widetilde{B}V_r$ for each $r \in \{0, 1, 2, \ldots\}$.*

Proof. It is plain that $\widetilde{S}_r \subset \widetilde{B}V_r$.

In order to prove that $\widetilde{S}_r \subset \widetilde{C}_r$ we take a sequence $\{a_k\}$ in $\widetilde{S}_r$ and consider

$$\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx; \text{ where } b_k = \frac{a_k}{k}$$

If we apply summation by parts, we obtain

$$\begin{aligned} &\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx \\ \leq \lim_{N \to \infty} &\left[\sum_{k=n}^{N-1} \Delta B_k \int_0^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta b_j}{B_j} D_j^{r+1}(x) \right| dx + B_N \int_0^{\pi} \left| \sum_{K=0}^{N} \frac{\Delta b_k}{B_k} D_k^{r+1}(x) \right| dx \right. \\ &\left. + B_n \int_0^{\pi} \left| \sum_{K=0}^{n-1} \frac{\Delta b_k}{B_k} D_k^{r+1}(x) \right| dx \right] \end{aligned}$$

Clearly $\left| \frac{\Delta b_k}{B_k} \right| \leq 1$. Now, if we first apply Bernstein's inequality [12] and then Sidon Fomin's inequality ([1,7]), we get

$$\int_0^{\pi} \left| \sum_{k=0}^{n} \frac{\Delta b_k}{B_k} D_k^{(r+1)}(x) \right| dx \leq M(n+1)^{r+2}, \quad r = 0, 1, 2, \ldots$$

$$\begin{aligned} \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx &\leq \lim_{N \to \infty} \left\{ \sum_{k=n}^{N-1} \Delta B_k (k+1)^{r+2} + B_N (N+1)^{r+2} \right\} + n^{r+2} B_n \\ &= \sum_{k=n}^{\infty} \left[(k+1)^{r+2} - k^{r+2} \right] B_k + n^{r+2} B_n \\ &= O\left(\sum_{k=n}^{\infty} k^{r+1} B_k \right) + n^{r+2} B_n \end{aligned}$$

So, by given hypothesis, we have

$$\int_0^\pi \left| \sum_{k=n}^\infty \Delta b_k D_k^{r+1}(x) \right| dx \le \frac{\varepsilon}{2} \text{ if } n \text{ is large enough say } n \ge n_0. \tag{6}$$

For any $1 \le n \le N$, we can estimate as follows:

$$\begin{aligned} \int_0^\delta \left| \sum_{k=n}^N \Delta b_k D_k^{r+1}(x) \right| dx &\le \int_0^\delta \left| \sum_{k=n}^{n_0} \Delta b_k D_k^{r+1}(x) \right| dx + \int_0^\delta \left| \sum_{k=n_0}^N \Delta b_k D_k^{r+1}(x) \right| dx \\ &\le \tfrac{1}{2}\delta \sum_{k=1}^{n_0} k(k+1)^{r+1} |\Delta b_k| + \tfrac{\varepsilon}{2} < \varepsilon \end{aligned}$$

provided δ is small enough. This proves that $\{a_k\} \in \widetilde{C}_r$. □

Theorem 4. *Let $\{a_k\}$ be a sequence of numbers belonging to the class $\widetilde{C} \cap \widetilde{B}V$ and if $\lim_{n\to\infty} a_n \log n = 0$, then*

$$\|\beta_n - f\| = o(1), \ n \to \infty.$$

Proof. The modified trigonometric sine sum is given by

$$\begin{aligned} \beta_n(x) &= \sum_{k=1}^n \left(\tfrac{a_{k+1}}{k+1} + \sum_{j=k}^n \Delta^2 \left(\tfrac{a_j}{j} \right) \right) k \ \sin kx \\ &= \sum_{k=1}^n a_k \sin kx + \left(\tfrac{a_{n+2}}{n+2} - \tfrac{a_{n+1}}{n+1} \right) \sum_{k=1}^n k \sin kx \\ &= - \sum_{k=1}^n b_k (\cos kx)' - (b_{n+2} - b_{n+1}) D_n'(x) \end{aligned}$$

By using the summation by parts, we get

$$\beta_n = - \sum_{k=1}^n \Delta b_k D_k'(x) - b_n D_n'(x) - (b_{n+2} - b_{n+1}) D_n'(x)$$

Under the given hypothesis and Lemma 1, series $\sum_{k=1}^n \Delta b_k D_k'(x)$ converges absolutely and $b_n D_n'(x) \to 0$ as $n \to \infty$.

Hence $\lim_{n\to\infty} \beta_n(x) = f(x)$ exists in $(0, \pi)$.

Next, consider

$$\begin{aligned} \|f(x) - \beta_n(x)\| &= \| \sum_{k=n+1}^\infty a_k \sin kx - \left(\tfrac{a_{n+2}}{n+2} - \tfrac{a_{n+1}}{n+1} \right) \sum_{k=1}^n k \sin kx \| \\ &= \int_0^\pi \left| - \sum_{k=n+1}^\infty b_k (\cos kx)' - (b_{n+1} - b_{n+2}) D_n'(x) \right| dx \end{aligned} \tag{7}$$

By using Abel's transformation, we have

$$\begin{aligned} &= \int_0^\pi \left| - \sum_{k=n+1}^\infty \Delta b_k D_k'(x) + b_{n+2} D_n'(x) \right| dx \\ &= \int_0^\pi \left| \sum_{k=n+1}^\infty \Delta b_k D_k'(x) \right| dx + \tfrac{n}{n+2} a_{n+2} \log n \end{aligned} \tag{8}$$

The second term of the above equation is of $o(1)$ as $a_n \log n = 0\ as\ n \to \infty$. For the remaining part, let $\varepsilon > 0$, then there exists $\delta > 0$, such that

$$\int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k'(x) \right| dx < \frac{\varepsilon}{2} \text{ for all } n \geq 0. \tag{9}$$

Then

$$\begin{aligned} \int_0^\pi \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k'(x) \right| dx &= \int_0^\delta \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k'(x) \right| dx + \int_\delta^\pi \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k'(x) \right| dx \\ &\leq \tfrac{\epsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta b_k| \int_\delta^\pi |D_k'(x)| dx \\ &\leq \tfrac{\epsilon}{2} + C \sum_{k=n+1}^{\infty} k\, |\Delta b_k| \int_\delta^\pi dx/x^2 \\ &\leq \tfrac{\epsilon}{2} + C\delta^{-1} \sum_{k=n+1}^{\infty} k\, |\Delta b_k| \leq \varepsilon \end{aligned} \tag{10}$$

This proves that $\|f(x) - \beta_n(x)\| = o(1)\ as\ n \to \infty$. □

Theorem 5. *Let $\{a_k\}$ be a sequence of numbers belonging to the class $\widetilde{C} \cap \widetilde{B}V$, and if $\lim_{n\to\infty} a_n \log\, n = 0$, then*

$$\|S_n - f\| = o(1), \quad n \to \infty.$$

Proof. $\|S_n - f\| \leq \|S_n - \beta_n\| + \|\beta_n - f\|$

$$\begin{aligned} &\leq |b_{n+1}| \int_0^\pi |D_n'(x)| dx + |b_{n+2}| \int_0^\pi |D_n'(x)| dx + o(1) \\ &\leq |a_{n+1}| \log n + |a_{n+2}| \log n (by\ Lemma\ 2) \\ &= o(1),\ n \to \infty \end{aligned}$$

□

Theorem 6. *Let $\{a_k\}$ be a sequence of numbers belonging to the class $\widetilde{C}_r \cap \widetilde{B}V_r$ and if $n^r\, a_n \log n = 0,\ as\, n\, \infty,\ for\ each\ r = 0, 1, 2, \ldots$ Then*

$$\|\beta_n^r(x) - f^r(x)\| = o(1), \quad n \to \infty$$

Here, $f^r(x)$ is the rth derivative of f(x), where $r = 0, 1, 2, \ldots$

Proof. Consider the modified trigonometric sine sum as

$$\begin{aligned} \beta_n(x) &= \sum_{k=1}^{n} \left(\tfrac{a_{k+1}}{k+1} + \sum_{j=k}^{n} \Delta^2 \left(\tfrac{a_j}{j} \right) \right) k\ \sin kx \\ &= \sum_{k=1}^{n} a_k \sin kx + \left(\tfrac{a_{n+2}}{n+2} - \tfrac{a_{n+1}}{n+1} \right) \sum_{k=1}^{n} k \sin kx \end{aligned} \tag{11}$$

Taking r-times differentiation of $\beta_n(x)$, we get

$$\begin{aligned} \beta_n^r(x) &= S_n^r(x) + \left(\tfrac{a_{n+2}}{n+2} - \tfrac{a_{n+1}}{n+1}\right) \sum_{k=1}^{n} k^{r+1} \sin\left(kx + \tfrac{r\pi}{2}\right) \\ &= \sum_{k=1}^{n} k^r a_k \sin\left(kx + \tfrac{r\pi}{2}\right) + \left(\tfrac{a_{n+1}}{n+1} - \tfrac{a_{n+2}}{n+2}\right) \sum_{k=1}^{n} k^{r+1} \cos\left(kx + \tfrac{(r+1)\pi}{2}\right) \\ &= -\sum_{k=1}^{n} k^{r+1} b_k \cos\left(kx + \tfrac{(r+1)\pi}{2}\right) + (b_{n+1} - b_{n+2}) D_n^{r+1}(x) \end{aligned} \tag{12}$$

If we apply Abel's transformation on the first term of above equation, we get

$$\begin{aligned} \beta_n^r(x) &= -\sum_{k=1}^{n-1} \Delta b_k D_k^{r+1}(x) - b_n D_n^{r+1}(x) + (b_{n+1} - b_{n+2}) D_n^{r+1}(x) \\ &= -\sum_{k=1}^{n} \Delta b_k D_k^{r+1}(x) - b_{n+2} D_n^{r+1}(x) \end{aligned} \tag{13}$$

The series $\sum_{k=1}^{\infty} \Delta b_k D_k^{r+1}(x)$ converges absolutely and $b_n D_n^{r+1}(x) \to 0$ as $n \to \infty$ using Lemma 1 and given hypothesis.

Therefore $\lim_{n\to\infty} \beta_n^r(x) = f^r(x)$ exists in $(0, \pi)$.

Next, consider

$$\begin{aligned} \|f(x) - \beta_n(x)\| &= \left\| \sum_{k=n+1}^{\infty} a_k \sin kx - \left(\tfrac{a_{n+2}}{n+2} - \tfrac{a_{n+1}}{n+1}\right) \sum_{k=1}^{n} k \sin kx \right\| \\ \|f^r(x) - \beta_n^r(x)\| &= \left\| \sum_{k=n+1}^{\infty} k^r a_k \sin\left(kx + \tfrac{r\pi}{2}\right) \right. \\ &\quad \left. - \left(\tfrac{a_{n+2}}{n+2} - \tfrac{a_{n+1}}{n+1}\right) \sum_{k=1}^{n} k^{r+1} \sin\left(kx + \tfrac{r\pi}{2}\right) \right\| \\ &= \left\| \sum_{k=n+1}^{\infty} k^r a_k \sin\left(kx + \tfrac{r\pi}{2}\right) + \left(\tfrac{a_{n+2}}{n+2} - \tfrac{a_{n+1}}{n+1}\right) \sum_{k=1}^{n} k^{r+1} \cos\left(kx + \tfrac{(r+1)\pi}{2}\right) \right\| \\ &= \int_0^{\pi} \left| -\sum_{k=n+1}^{\infty} k^{r+1} b_k \cos\left(kx + \tfrac{(r+1)\pi}{2}\right) + (b_{n+2} - b_{n+1}) D_n^{r+1}(x) \right| dx \end{aligned}$$

If we apply Abel's transformation, we obtain

$$\begin{aligned} &= \int_0^{\pi} \left| -\sum_{k=n+1}^{\infty} \Delta b_k D_k^{r+1}(x) + b_{n+1} D_n^{r+1}(x) - b_{n+1} D_n^{r+1}(x) + b_{n+2} D_n^{r+1}(x) \right| dx \\ &\le \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx + |b_{n+2}| \int_0^{\pi} \left| D_n^{r+1}(x) \right| dx \\ &\le \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx + \tfrac{a_{n+2}}{n+2} n^{r+1} \log n \end{aligned}$$

The second term of the above equation are of $o(1)$ as $n^r a_n \log\, n = 0$ as $n \to \infty$. For the remaining part, let $\varepsilon > 0$, then there exists $\delta > 0$, such that $\int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx < \varepsilon/2$ for all $n \geq 0$. Then

$$\begin{aligned} \int_0^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx &= \int_0^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx + \int_{\delta}^{\pi} \left| \sum_{k=n+1}^{\infty} \Delta b_k D_k^{r+1}(x) \right| dx \\ &\le \tfrac{\varepsilon}{2} + \sum_{k=n+1}^{\infty} |\Delta b_k| \int_{\delta}^{\pi} \left| D_k^{r+1}(x) \right| dx \\ &\le \tfrac{\varepsilon}{2} + C \sum_{k=n+1}^{\infty} k^{r+1} |\Delta b_k| \int_{\delta}^{\pi} dx / x^{r+2} \\ &\le \tfrac{\varepsilon}{2} + C\delta^{-(r+1)} \sum_{k=n+1}^{\infty} k^{r+1} |\Delta b_k| \le \varepsilon \qquad \text{(by given hypothesis)} \end{aligned}$$

Therefore, $\|f^r(x) - \beta_n^r(x)\|_{L^1} = o(1)$ as $n \to \infty$. $\square$

Remark 7. *For $r = 0$, Theorem 6 reduces to Theorem 4.*

Theorem 7. *Let $\{a_k\}$ be a sequence of numbers belonging to the class $\widetilde{C}_r \cap \widetilde{B}V_r$ and if $n^r a_n \log n = o(1)$ as $n \to \infty$. Then*

$$\|S_n^r(x) - f^r(x)\| = o(1), \ n \to \infty.$$

where $r = 0, 1, 2 \ldots$.

Proof. $\|S_n^r - f^r\| \leq \|S_n^r - \beta_n^r\| + \|\beta_n^r - f^r\|$

$$\begin{aligned} &\leq |b_{n+2}| \int_0^\pi |D_n^{r+1}(x)| dx + |b_{n+1}| \int_0^\pi |D_n^{r+1}(x)| dx + o(1) \\ &\leq \tfrac{|a_{n+2}|}{n+2} n^{r+1} \log n + \tfrac{|a_{n+1}|}{n+1} n^{r+1} \log n \qquad (\text{by Lemma } 2) \\ &= o(1) \text{ as } n \to \infty \end{aligned} \tag{14}$$

□

Remark 8. *For $r = 0$, Theorem 7 reduces to Theorem 5.*

Remark 9. *Combining Theorem 6 and Theorem 7 with Theorem 3, the following result holds:*

Corollary 1. *If $\{a_k\} \in \widetilde{S}_r$ $(r = 0, 1, 2, 3, \ldots)$ and if $n^r a_n \log n = o(1)$ as $n \to \infty$. Then*

(i) $\|\beta_n^r(x) - f^r(x)\| = o(1), \ n \to \infty.$

(ii) $\|S_n^r(x) - f^r(x)\| = o(1), \ n \to \infty.$

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