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Results on Meromorphic Functions Partially Sharing Some Values in an Angular Domain

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Abstract: By using the Tsuji characteristic of meromorphic function in an angular domain, we investigate two meromorphic functions partially sharing some values in an angle region, and obtain one main result and a series of corollaries that are improvements and generalization of the previous results given by Zheng, Cao-Yi, Li-Yi and Xuan.

Keywords: meromorphic function; Tsuji characteristic; angular domain

MSC: 30D30; 30D35

1. Introduction and Main Results

In this article, the main aim is to discuss two meromorphic functions partially sharing some values in an angle region. Thus, we will use some basic symbols and notations of Nevanlinna theory which can be found in [1–3]. We use $\mathbb{C}$ to denote the whole complex plane, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, and the subset $\mathbb{X} \subset \mathbb{C}$.

In [4], Nevanlinna first studied the uniqueness of meromorphic functions in $\mathbb{C}$ and proved the well-known 5 IM theorem: *If two non-constant meromorphic functions f and g share five distinct values $a_j (j = 1, \dots, 5)$ IM on $\mathbb{C}$, then $f(z) \equiv g(z)$.*

After his wonderful works, there are lots of papers about the problem on meromorphic functions sharing values and sets on $\mathbb{C}$ (see [3]). For example, the problems on uniqueness of meromorphic functions sharing one, two, three or some sets on $\mathbb{C}$ were studied by Lahiri et al (including [3,5–7]). In 2010, Zheng [8] pointed out: the problem on how to extend some important uniqueness results in the complex plane to an angular domain is very interesting. Around 2003, Zheng [9,10] firstly investigated the value distribution of meromorphic functions in an angular domain. In the past few decades, the problem about the uniqueness of meromorphic functions in an angular domain attracted many investigations, and they studied the uniqueness theorems of meromorphic functions sharing values or sets in an angular domain, and obtained a series of interesting and important results (see [9–18]).

In [16], Mao-Liu considered the uniqueness of meromorphic functions in an angular region by using a different method, and obtained

Theorem 1 (see [16])**.** *Let f, g be two meromorphic functions in the unit disc $\mathbb{D}$, $a_1, a_2, \dots, a_5 \in \widehat{\mathbb{C}}$ be 5 distinct values, and $\Delta(\theta_0, \delta) (0 < \delta < \pi)$ be an angular domain such that for some $a \in \widehat{\mathbb{C}}$,*

$$\limsup_{r \to 1^-} \frac{\log n(r, \Delta(\theta_0, \delta/2), f(z) = a)}{\log \frac{1}{1-r}} = \tau > 1.$$

If f and g share $a_j (j = 1, 2, \dots, 5)$ IM in $\Delta(\theta_0, \delta)$, then $f(z) \equiv g(z)$.

Meanwhile, Cao-Yi [11] discussed the uniqueness of meromorphic functions sharing five values in an angular region and obtained:

Theorem 2 (see [11], Theorem 1.3)**.** *Let $\Omega = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leq 2\pi$, and let f and g be two transcendental meromorphic functions satisfying*

$$\lim_{r\to\infty} \frac{S_{\alpha,\beta}(r,f)}{\log(rT(r,f))} = \infty, \quad (r \notin E),$$

where $S_{\alpha,\beta}(r,f)$ is the angular characteristic function of meromorphic function f. Suppose that f and g share five distinct values $a_j \in \widehat{\mathbb{C}}(j = 1,2,3,4,5)$ IM in Ω. Then, $f(z) \equiv g(z)$.

Remark 1. *This theorem can be seen as 5 IM theorem in an angular domain.*

In [8], Zheng investigated the uniqueness problem about meromorphic function sharing five distinct values in an angular domain, by using the Tsjui's characteristic function.

Theorem 3 (see [8], Theroem 2.9.1)**.** *Let f and g be two nonconstant meromorphic functions in an angular domain $\Omega(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}(0 < \beta - \alpha < 2\pi)$, and*

$$\limsup_{r\to\infty} \frac{\mathfrak{T}_{\alpha,\beta}(r,f)}{\log r} = \infty. \tag{1}$$

If f and g share five distinct values $a_j \in \widehat{\mathbb{C}}(j = 1,2,3,4,5)$ IM in $\Omega(\alpha,\beta)$, then $f(z) \equiv g(z)$.

Remark 2. *If $f(z)$ satisfies Label (1), then we say that $f(z)$ is transcendental with respect to the Tsuji characteristic.*

Let us introduce the Tsuji characteristic as follows (also see [8]). Assume that $f(z)$ is a meromorphic function in an angular domain $\Omega(\alpha,\beta)$. Define

$$\mathfrak{M}_{\alpha,\beta}(r,f) = \frac{1}{2\pi}\int_{\arcsin(r^{-\omega})}^{\pi-\arcsin(r^{-\omega})} \log^{+}\left|f(re^{i(\alpha+\omega^{-1}\theta)}\sin^{\omega^{-1}}\theta)\right| \frac{1}{r^{\omega}\sin^{2}\theta}d\theta,$$

$$\mathfrak{N}_{\alpha,\beta}(r,f) = \sum_{1<|b_n|<r(\sin(\omega(\beta_n-\alpha)))^{\omega-1}} \left(\frac{\sin\omega(\beta_n-\alpha)}{|b_n|^{\omega}} - \frac{1}{r^{\omega}}\right),$$

where $\omega = \frac{\pi}{\beta-\alpha}$,

$$\Xi(\alpha,\beta;r) = \{z = te^{i\theta} : \alpha < \theta < \beta, 1 < t \leq r(\sin(\omega(\theta-\alpha)))^{\frac{1}{\omega}}\},$$

and b_n are the poles of $f(z)$ in $\Xi(\alpha,\beta;r)$ appearing often according to their multiplicities and then Tsuji characteristic of f is

$$\mathfrak{T}_{\alpha,\beta}(r,f) = \mathfrak{M}_{\alpha,\beta}(r,f) + \mathfrak{N}_{\alpha,\beta}(r,f).$$

In [17], Xuan further discussed the problem on sharing some values and uniqueness of meromorphic functions in an angular region, and obtained the following result.

Theorem 4 (see [17], Corollary 2.7)**.** *Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions, and let $f(z)$ be of finite order λ and such that, for some $a \in \mathbb{C}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. For m pair of real numbers $\{\alpha_j, \beta_j\}$ satisfying*

$$\sum_{j=1}^{m}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma}\sqrt{\frac{\delta}{2}},$$

and

$$-\pi \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \cdots \le \alpha_m < \beta_m \le \pi,$$

where $\sigma = \max\{\omega, \mu\}$ *and* $\omega = \max\{\frac{\pi}{\beta_1-\alpha_1}, \ldots, \frac{\pi}{\beta_m-\alpha_m}\}$, *assume that* $a_j (j = 1, 2, \ldots, q)$ *are* q *distinct complex numbers or* ∞ *satisfying*

$$k_1 \ge k_2 \ge \cdots \ge k_p, \tag{2}$$

and $\overline{E}_{k_j)}(a_j, \Omega, f) = \overline{E}_{k_j)}(a_j, \Omega, g)$ $(j = 1, 2, \ldots, q)$, *where* $\Omega = \{z : \alpha < \arg z < \beta\}$. *If* q *and* $k_j (j = 1, 2, \ldots, p)$ *satisfy one of the following cases:*

(i) $q = 7$,
(ii) $q = 6$ *and* $k_3 \ge 2$,
(iii) $q = 5$, $k_3 \ge 3$ *and* $k_5 \ge 2$,
(iv) $q = 5$ *and* $k_4 \ge 4$,
(v) $q = 5$, $k_3 \ge 5$ *and* $k_4 \ge 3$,
(vi) $q = 5$, $k_3 \ge 6$ *and* $k_4 \ge 2$.

Then, $f(z) \equiv g(z)$.

From the above results, we find that these forms of sharing values such as IM, CM and $\overline{E}_{k_j)}(a_j, \Omega, f) = \overline{E}_{k_j)}(a_j, \Omega, g)$ show that the distinct zeros of $f - a_j$ and $g - a_j$ in a angular domain Ω are the same, or the distinct zeros of $f - a_j$ and $g - a_j$ in a angular domain Ω with an order less than a positive integer are the same. Thus, a natural question is: *what would have happened if the distinct zeros of* $f - a_j$ *and* $g - a_j$ *with order* $\le k_j$ *in an angular domain* Ω *are not the same*?

Proceeding from the above question, our main aim of this article is to further study the problem on partially sharing some values for meromorphic functions in an angular domain. In order to state our results, let us introduce the following definition.

Definition 1. *For* $A \subset \Omega$ *and* $a \in \widehat{\mathbb{C}}$, *we use* $\overline{\mathfrak{N}}^A_{\alpha,\beta}(r, \frac{1}{f-a})$ *to denote the reduced counting function of those zeros of* $f - a$ *in the angular domain* Ω *which belong to the set* A.

Next, our main theorem is listed below.

Theorem 5. *Suppose that* f *and* g *are two transcendental meromorphic functions with respect to the Tsuji characteristic in an angular domain* $\Omega(\alpha, \beta)$, *and* $a_1, \ldots, a_q (q \ge 5)$ *are* q *distinct complex numbers or* ∞. *Let* $k_j (j = 1, \ldots, q)$ *and* t *be positive integers or infinity satisfying Label (2),* $1 \le t \le q$ *and* $\delta_j (\ge 0) \in \mathbb{R} (j = 1, 2, \ldots, q)$ *satisfy*

$$(1 + \frac{1}{k_t}) \sum_{j=t}^{q} \frac{1}{1 + k_j} + 3 + \sum_{j=1}^{q} \delta_j < (q - t - 1)(1 + \frac{1}{k_t}) + t. \tag{3}$$

Set $A_j = \overline{E}_{k_j)}(a_j, \Omega, f) \setminus \overline{E}_{k_j)}(a_j, \Omega, g)$ *for* $j = 1, 2, \ldots, q$. *If*

$$\overline{\mathfrak{N}}^{A_j}_{\alpha,\beta}(r, \frac{1}{f - a_j}) \le \delta_j \mathfrak{T}_{\alpha,\beta}(r, f) \tag{4}$$

and

$$\begin{aligned} &\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r, \frac{1}{f-a_j})}{\sum_{j=1}^{q} \overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r, \frac{1}{g-a_j})} \\ > &\frac{k_t}{(1 + k_t) \sum_{j=t}^{q} \frac{k_j}{1+k_j} - 2(1 + k_t) + (t - 2 - \sum_{j=1}^{q} \delta_j) k_t}, \end{aligned} \tag{5}$$

then $f(z) \equiv g(z)$.

2. Some Lemmas

To prove our main result, some lemmas are required which are listed below.

Lemma 1 (see [8], p. 59)**.** *(The Tsuji second fundamental theorem). Let f be a nonconstant meromorphic function in an angular domain $\Omega(\alpha,\beta)$. Let $a_1, a_2, \ldots, a_q$ be q distinct complex numbers in the extended complex plane $\widehat{\mathbb{C}}$. Then,*

$$(q-2)\mathfrak{T}_{\alpha,\beta}(r,f) < \sum_{j=1}^{q} \overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a_j}) + Q_{\alpha,\beta}(r,f),$$

where

$$Q_{\alpha,\beta}(r,f) = \mathfrak{M}_{\alpha,\beta}\left(r,\frac{f'}{f}\right) + \sum_{j=1}^{q} \mathfrak{M}_{\alpha,\beta}\left(r,\frac{f'}{f-a_j}\right) + O(1).$$

Lemma 2 (see [8], Lemma 2.5.4)**.** *Let $f(z)$ be a meromorphic function in $\Omega(\alpha,\beta)$. Then, for $0<r<R$,*

$$\mathfrak{M}_{\alpha,\beta}\left(r,\frac{f^{(p)}}{f}\right) \le K\left[\log^{+}\mathfrak{T}_{\alpha,\beta}(R,f) + \log\frac{R}{R-r} + 1\right].$$

Furthermore, $Q_{\alpha,\beta}(r,f) = O(\log r + \log^{+}\mathfrak{T}_{\alpha,\beta}(r,f))$ as $r \to +\infty$ possibly except a set of r with finite linear measure.

Remark 3. *Throughout this paper, we say that $Q_{\alpha,\beta}(r,f)$ is the error term associated with the Tsuji characteristic for the angle $\Omega(\alpha,\beta)$. We assume that it may not be the same at each occurrence.*

By utilizing the method of discussing multiple values which is given by Yang [2], one can get the following lemma easily.

Lemma 3 (see [8])**.** *Suppose that f is a meromorphic function in an angular domain Ω and $\omega = \frac{\pi}{\beta-\alpha}$, and a is an arbitrary complex number, and k is a positive integer. Then,*

$$(i)\quad \overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a}) \le \frac{k}{k+1}\overline{\mathfrak{N}}^{k)}_{\alpha,\beta}(r,\frac{1}{f-a}) + \frac{1}{k+1}\mathfrak{N}_{\alpha,\beta}(r,\frac{1}{f-a}),$$

$$(ii)\quad \overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a}) \le \frac{k}{k+1}\overline{\mathfrak{N}}^{k)}_{\alpha,\beta}(r,\frac{1}{f-a}) + \frac{1}{k+1}\mathfrak{T}_{\alpha,\beta}(r,f) + O(1).$$

Remark 4 (see [8])**.** *Let $\overline{\mathfrak{N}}^{k)}_{\alpha,\beta}(r,\frac{1}{f-a})$ denote the distinct zeros of $f(z)-a$ in Ω, whose multiplicities are $\le k$, and $\overline{\mathfrak{N}}^{(k}_{\alpha,\beta}(r,\frac{1}{f-a})$ denote the distinct zeros of $f(z)-a$ in Ω, whose multiplicities are $> k$.*

Lemma 4 (see [8], Lemma 2.3.3)**.** *Let $f(z)$ be a meromorphic function in $\Omega(\alpha,\beta)$, and for any real number $\varepsilon > 0$, $\Omega_\varepsilon = \Omega(\alpha+\varepsilon,\beta-\varepsilon)$. Then, for $\varepsilon > 0$, we have*

$$\mathfrak{N}_{\alpha,\beta}(r,f) \le \omega\frac{N(r,\Omega,f)}{r^{\omega}} + \omega^2\int_1^r \frac{N(t,\Omega,f)}{t^{\omega+1}}dt,$$

and

$$\mathfrak{N}_{\alpha,\beta}(r,f) \ge \omega c^{\omega}\frac{N(cr,\Omega_\varepsilon,f)}{r^{\omega}} + \omega^2 c^{\omega}\int_1^{cr} \frac{N(t,\Omega_\varepsilon,f)}{t^{\omega+1}}dt,$$

where $0<c<1$ is a constant depending on ε, $\omega = \frac{\pi}{\beta-\alpha}$ and $N(t,\Omega,f) = \int_1^r \frac{n(t,\Omega,f)}{t}dt$, $n(t,\Omega,f)$ is the number of poles of $f(z)$ in $\Omega \cap \{z : 1 < |z| \le t\}$.

3. The Proof of Theorem 5

Proof. The reduction to absurdity will be employing below. Assume that $f \not\equiv g$. Then, in view of Lemmas 1–3, for any integer $t(1 \le t \le q)$, it yields

$$\begin{aligned}
(q-2)\mathfrak{T}_{\alpha,\beta}(r,f) &\le \sum_{j=1}^{q} \overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a_j}) + Q_{\alpha,\beta}(r,f) \\
&= \sum_{j=1}^{q} \left\{ \overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j}) + \overline{\mathfrak{N}}^{(k_j+1}_{\alpha,\beta}(r,\frac{1}{f-a_j}) \right\} + Q_{\alpha,\beta}(r,f) \\
&\le \sum_{j=1}^{q} \left\{ \overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j}) + \frac{1}{1+k_j}\mathfrak{N}^{(k_j+1}_{\alpha,\beta}(r,\frac{1}{f-a_j}) \right\} + Q_{\alpha,\beta}(r,f) \\
&\le \sum_{j=1}^{q} \left\{ \frac{k_j}{1+k_j}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j}) + \frac{1}{1+k_j}\mathfrak{N}_{\alpha,\beta}(r,\frac{1}{f-a_j}) \right\} + Q_{\alpha,\beta}(r,f) \\
&\le \sum_{j=1}^{t-1} \left(\frac{k_j}{1+k_j} - \frac{k_t}{1+k_t} \right) \overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j}) + \left(\sum_{j=1}^{q} \frac{1}{1+k_j} \right) \mathfrak{T}_{\alpha,\beta}(r,f) \\
&\quad + \sum_{j=1}^{q} \frac{k_t}{1+k_t}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j}) + Q_{\alpha,\beta}(r,f) \\
&\le \sum_{j=1}^{q} \frac{k_t}{1+k_t}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j}) \\
&\quad + \left(t-1-\frac{(t-1)k_t}{1+k_t} + \sum_{j=t}^{q} \frac{1}{1+k_j} \right) \mathfrak{T}_{\alpha,\beta}(r,f) + Q_{\alpha,\beta}(r,f),
\end{aligned}$$

that is,

$$\left(\sum_{j=t}^{q} \frac{k_j}{1+k_j} - 2 + \frac{(t-1)k_t}{1+k_t} \right) \mathfrak{T}_{\alpha,\beta}(r,f) \le \sum_{j=1}^{q} \frac{k_t}{1+k_t}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j}) + Q_{\alpha,\beta}(r,f). \tag{6}$$

Similarly,

$$\left(\sum_{j=t}^{q} \frac{k_j}{1+k_j} - 2 + \frac{(t-1)k_t}{1+k_t} \right) \mathfrak{T}_{\alpha,\beta}(r,g) \le \sum_{j=1}^{q} \frac{k_t}{1+k_t}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{g-a_j}) + Q_{\alpha,\beta}(r,g). \tag{7}$$

In view of $A_j = \overline{E}_{k_j)}(a_j,f) \setminus \overline{E}_{k_j)}(a_j,g)$, set $B_j = \overline{E}_{k_j)}(a_j,f) \setminus A_j$ for $j = 1,2,\ldots,q$. Then, from (4), it yields

$$\begin{aligned}
\sum_{j=1}^{q} \overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j}) &= \sum_{j=1}^{q} \overline{\mathfrak{N}}^{A_j}_{\alpha,\beta}(r,\frac{1}{f-a_j}) + \sum_{j=1}^{q} \overline{\mathfrak{N}}^{B_j}_{\alpha,\beta}(r,\frac{1}{f-a_j}) \\
&\le \sum_{j=1}^{q} \delta_j \mathfrak{T}_{\alpha,\beta}(r,f) + \mathfrak{N}_{\alpha,\beta}(r,\frac{1}{f-g}) \\
&\le \left(1 + \sum_{j=1}^{q} \delta_j \right) \mathfrak{T}_{\alpha,\beta}(r,f) + \mathfrak{T}_{\alpha,\beta}(r,g) + O(1).
\end{aligned}$$

Since f, g are transcendental with respect to the Tsuji characteristic, thus, in view of (6) and (7), for $r \to +\infty$, we obtain

$$\begin{aligned}&\left(\sum_{j=t}^{q}\frac{k_j}{1+k_j}-2+\frac{(t-1)k_t}{1+k_t}+o(1)\right)\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j})\\ \leq&\left(1+\sum_{j=1}^{q}\delta_j\right)\sum_{j=1}^{q}\frac{k_t}{1+k_t}\left(\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j})+(1+o(1))\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{g-a_j})\right),\end{aligned} \tag{8}$$

and by combining with

$$1\geq\frac{k_1}{k_1+1}\geq\frac{k_2}{k_2+1}\geq\cdots\geq\frac{k_q}{k_q+1}\geq\frac{1}{2},$$

then we can deduce from (8) that

$$\begin{aligned}&\left\{\sum_{j=t}^{q}\frac{k_j}{1+k_j}-2+\frac{(t-1)k_t}{1+k_t}-\frac{k_t}{1+k_t}\left(1+\sum_{j=1}^{q}\delta_j\right)+o(1)\right\}\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j})\\ \leq&(1+o(1))\frac{k_t}{1+k_t}\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{g-a_j}),\end{aligned}$$

this means

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{f-a_j})}{\sum_{j=1}^{q}\overline{\mathfrak{N}}^{k_j)}_{\alpha,\beta}(r,\frac{1}{g-a_j})}\leq\frac{\frac{k_t}{1+k_t}}{\sum_{j=t}^{q}\frac{k_j}{1+k_j}-2+(t-2-\sum_{j=1}^{q}\delta_j)\frac{k_t}{1+k_t}}.$$

This is a contradiction with (5). Thus, the conclusion of Theorem 5 holds, that is, $f(z)\equiv g(z)$. Therefore, this is completely the proof of Theorem 5. □

4. Consequences

In view of Theorem 5, it is easy to obtain the following consequences.

Corollary 1. *When $t=1$, $k_j=\infty$ for $j=1,2,\ldots,q$ and*

$$\lambda=\liminf_{r\to\infty}\frac{\sum_{j=1}^{q}\overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{f-a_j})}{\sum_{j=1}^{q}\overline{\mathfrak{N}}_{\alpha,\beta}(r,\frac{1}{g-a_j})}>\frac{1}{q-3}.$$

If $\overline{\mathfrak{N}}^{A_j}_{\alpha,\beta}(r,\frac{1}{f-a_j})\leq\delta_j\mathfrak{T}_{\alpha,\beta}(r,f)$, and $\delta_j(\geq 0)$ satisfy $0\leq\sum_{j=1}^{q}\delta_j<k-3-\frac{1}{\lambda}$, then $f(z)\equiv g(z)$.

Remark 5. *When $\overline{E}(a_j,\Omega,f)\subseteq\overline{E}(a_j,\Omega,g)$ and $q=5$, thus $A_1=A_2=\cdots=A_5=\emptyset$. Thus, if we choose $\delta_j=0$ for $j=1,2,\ldots,5$ and any constant λ such that $0\leq 2-\frac{1}{\lambda}$ in Corollary 1, then we immediately obtain $f\equiv g$. Especially, if $q=5$ and $\overline{E}(a_j,\Omega,f)=\overline{E}(a_j,\Omega,g)$, then $\lambda=1$ and $\delta_j=0$ for $j=1,2,\ldots,5$. We can obtain $f\equiv g$. Thus, Corollary 1 is an improvement of Theorem 1.3 and Theorem 1.4.*

Corollary 2. *Suppose that f, g are two transcendental meromorphic functions with respect to the Tsuji characteristic in an angular domain $\Omega(\alpha,\beta)$, and $a_1,\ldots,a_q(q\geq 5)$ are q distinct complex numbers or ∞. Let $k_j(j=1,\ldots,q)$ be positive integers or infinity with $k_1\geq k_2\geq\cdots\geq k_q$, if $\overline{E}_{k_j)}(a_j,\Omega,f)\subseteq\overline{E}_{k_j)}(a_j,\Omega,g)$ and*

$$\sum_{j=2}^{q}\frac{k_j}{1+k_j}-\frac{k_1}{\lambda(1+k_1)}-2>0,$$

where λ is stated as in Corollary 1, then $f(z) \equiv g(z)$.

Remark 6. *Let $\sharp S$ denote the cardinality of a set S. Then,*

(i) *if $q = 6$ and $k_1 = k_2 = \cdots = k_6 \geq 3$, and $\overline{E}_{k_j)}(a_j, \Omega, f) \subseteq \overline{E}_{k_j)}(a_j, \Omega, g)$ and $\sharp\overline{E}_{k_j)}(a_j, \Omega, f) = \frac{1}{2}\sharp\overline{E}_{k_j)}(a_j, \Omega, g)$, for $j = 1, \ldots, q$, that is, $\lambda = \frac{1}{2}$, then we have $f(z) \equiv g(z)$ in view of Corollary 2;*

(ii) *if $q = 5$ and $k_1 = k_2 = \cdots = k_5 \geq 5$, and $\overline{E}_{k_j)}(a_j, \Omega, f) \subseteq \overline{E}_{k_j)}(a_j, \Omega, g)$ and $\sharp\overline{E}_{k_j)}(a_j, \Omega, f) = \frac{2}{3}\sharp\overline{E}_{k_j)}(a_j, \Omega, g)$, for $j = 1, \ldots, q$, that is, $\lambda = \frac{2}{3}$, then we have $f(z) \equiv g(z)$ in view of Corollary 2;*

(iii) *if $q = 5$ and $k_1 = k_2 = \cdots = k_5 \geq 3$, and $\overline{E}_{k_j)}(a_j, \Omega, f) \subseteq \overline{E}_{k_j)}(a_j, \Omega, g)$ and $\sharp\overline{E}_{k_j)}(a_j, \Omega, f) = \frac{8}{9}\sharp\overline{E}_{k_j)}(a_j, \Omega, g)$, for $j = 1, \ldots, q$, that is, $\lambda = \frac{8}{9}$, then we have $f(z) \equiv g(z)$ in view of Corollary 2.*

This shows that Corollary 2 is an improvement of Theorem 1.4 in a way.

Corollary 3. *Under the hypothesis of Corollary 2, if $\overline{E}_{k_j)}(a_j, \Omega, f) = \overline{E}_{k_j)}(a_j, \Omega, g)$ and*

$$\sum_{j=2}^{q} \frac{k_j}{1+k_j} - \frac{k_1}{1+k_1} - 2 > 0,$$

then $f(z) \equiv g(z)$.

Corollary 4. *Suppose that f, g are two transcendental meromorphic functions with respect to the Tsuji characteristic in an angular domain $\Omega(\alpha, \beta)$, and $a_1, \ldots, a_q (q \geq 5)$ be q distinct complex numbers or ∞. Let $k_j (j = 1, \ldots, q)$ and t be positive integers or infinity with $k_1 \geq k_2 \geq \cdots \geq k_q$ and $1 \leq t \leq q$, if $\overline{E}_{k_j)}(a_j, \Omega, f) \subseteq \overline{E}_{k_j)}(a_j, \Omega, g)$ and*

$$\sum_{j=t}^{q} \frac{k_j}{1+k_j} - 2 + \frac{(t - 2 - \frac{1}{\lambda})k_t}{1+k_t} > 0, \tag{9}$$

where λ is stated as in Corollary 1, then $f(z) \equiv g(z)$.

Remark 7. *Supposing that $\overline{E}_{k_j)}(a_j, \Omega, f) = \overline{E}_{k_j)}(a_j, \Omega, g)$ and $t = 3$ in Corollary 4, thus (9) can be represented as*

$$\sum_{j=3}^{q} \frac{k_j}{1+k_j} > 2.$$

Thus, this shows that we have improved Theorem 1.4.

Theorem 6. *Let the other assumptions in Theorem 5 and Corollaries 1–4 remain unchanged under removing the condition that $f(z)$ is transcendental in Tsuji sense, and let $f(z)$ satisfy*

$$\limsup_{r \to \infty} \frac{N(r, \Omega_\varepsilon, f = a)}{r^\omega \log r} = \infty, \tag{10}$$

for some $a \in \widehat{\mathbb{C}}$ and $\varepsilon > 0$, where $\omega = \frac{\pi}{\beta - \alpha}$, $N(t, \Omega, f) = \int_1^r \frac{n(t, \Omega, f)}{t} dt$, and $n(t, \Omega, f)$ is the number of poles of $f(z)$ in $\Omega \cap \{z : 1 < |z| \leq t\}$. Then, $f(z) \equiv g(z)$.

Proof of Theorem 6. If $f(z)$ satisfies the condition (10), then it is easy to get that f is transcendental in Tsuji sense (see [8]). Thus, by utilizing the conclusions of Theorem 5 and Corollaries 1–4, it is also easy to obtain the conclusions of Theorem 6. □

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