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Article

New Sufficient Condition for the Positive Definiteness of Fourth Order Tensors

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Abstract: In this paper, we give a new Z-eigenvalue localization set for Z-eigenvalues of structured fourth order tensors. As applications, a sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative fourth order tensors is obtained and a new Z-eigenvalue based sufficient condition for the positive definiteness of fourth order tensors is also presented. Finally, numerical examples are given to verify the efficiency of our results.

Keywords: fourth order tensor; bound; nonnegative tensor; Z-eigenvalue; positive definiteness

MSC: 15A18; 15A42; 15A69

1. Introduction

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an m-th order n dimensional real square tensor, x be a real n-vector. Then, let $N = \{1, 2, \ldots, n\}$, we define the following real n-vector:

$$\mathcal{A}x^{m-1} = \left(\sum_{i_2, \cdots, i_m = 1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m} \right)_{i \in N}, \ x^{[m-1]} = (x_i^{m-1})_{i \in N}.$$

If there exists a real vector x and a real number λ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called H-eigenvalue of $\mathcal{A}$ and x is called H-eigenvector of $\mathcal{A}$ associated with λ. If there exists a real vector x and a real number λ such that

$$\mathcal{A}x^{m-1} = \lambda x, \ \ x^T x = 1.$$

then λ is called Z-eigenvalue of $\mathcal{A}$ and x is called Z-eigenvector of $\mathcal{A}$ associated with λ [1,2].

An mth-degree homogeneous polynomial form of n variables

$$f(x) = \mathcal{A}x^m = \sum_{i_2, \cdots, i_m = 1}^{n} a_{i_1 \ldots i_m} x_{i_1} \ldots x_{i_m} \tag{1}$$

is positive definite, i.e., $f(x) > 0$, if and only if the real symmetric tensor $\mathcal{A}$ is positive definite [2]. When m is even, an eigenvalue method is given to verify the positive definiteness of $\mathcal{A}$.

Theorem 1 ([2])**.** *Let $\mathcal{A}$ be an even-order real symmetric tensor. Then*

(1) $\mathcal{A}$ is positive definite if and only if all of its H-eigenvalues are positive;

(2) $\mathcal{A}$ is positive definite if and only if all of its Z-eigenvalues are positive.

From Theorem 1, we can verify the positive definiteness of $\mathcal{A}$ by the H-eigenvalues or the Z-eigenvalues of $\mathcal{A}$. But when m and n are large, it is difficult to compute all the H-eigenvalues (or Z-eigenvalues) or the smallest H-eigenvalue (or Z-eigenvalue) of an order m dimension n real tensor $\mathcal{A}$. Based on the Geršhgorin-type theorem for H-eigenvalues, which is introduced in [2], Li et al. provided some sufficient conditions for the positive definiteness of an even-order real symmetric tensor [3], and some improved results are obtained in [4–8].

First, let us recall the definitions of strictly diagonally dominant (SDD) tensors and quasi-doubly SDD (QDSDD) tensors [7].

Definition 1. *A tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ is called a strictly diagonally dominant (SDD) tensor if for $i \in N$,*

$$|a_{i \dots i}| > R_i(\mathcal{A}).$$

Definition 2. *A tensor $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{R}^{[m,n]}$ is called a quasi-doubly strictly diagonally dominant (QDSDD) tensor if for $i, j \in N, i \neq j$,*

$$(|a_{i \dots i}| - R_i^j(\mathcal{A}))|a_{j \dots j}| > |a_{ij \dots j}| R_j(\mathcal{A}).$$

The following useful theorem is given in [7].

Theorem 2 ([7]). *Let $\mathcal{A}$ be an even-order real symmetric tensor with all positive diagonal entries.*
(1) If $\mathcal{A}$ is strictly diagonally dominant, then $\mathcal{A}$ is positive definite;
(2) If $\mathcal{A}$ is quasi-doubly strictly diagonally dominant, then $\mathcal{A}$ is positive definite.

Positive definiteness of fourth order tensors has important applications in signal processing, automatic control, and magnetic resonance imaging [9–12]. Recently, in order to preserve positive definiteness for a fourth order tensor, a ternary quartics approach is proposed in [13]. Extending the Riemannian framework from 2nd order tensors to the space of 4th order tensors, a riemannian approach is given to guarantee positive definiteness for a fourth order tensor [14]. In [11], the authors explain the definition of the smallest Z-eigenvalue and present a computational method for calculating it. Very recently, much literature has focused on the properties of Z-eigenvalues of tensors [15–24], but there are no Z-eigenvalues based sufficient conditions for the positive definiteness of an even-order real symmetric tensor.

In this paper, based on the Z-eigenvalue localization sets of structured fourth order tensors, a new sufficient condition for the positive definiteness of fourth order tensors is given.

2. New Z-Eigenvalue Localization Set for Structured Fourth Order Tensors

In this section, a Geršhgorin-type theorem for Z-eigenvalues of structured fourth order tensors is obtained. For any $k \in N$, let

$$\Delta^k = \{(i_2 i_3 i_4) : \text{there are at least two } i_h = k \text{ for } h = 2,3,4\},$$

$$\Delta^{\bar{k}} = \{(i_2, i_3, i_4) : \text{there are at most one } i_h = k \text{ for } h = 2,3,4\},$$

then,

$$R_i(\mathcal{A}) = \sum_{i_2, i_3, i_4 \in N} |a_{i i_2 i_3 i_4}| = r_i^{\Delta^{\bar{k}}}(\mathcal{A}) + r_i^{\Delta^k}(\mathcal{A}),$$

where

$$r_i^{\Delta^k}(\mathcal{A}) = \sum_{(i_2, i_3, i_4) \in \Delta^k} |a_{i i_2 i_3 i_4}|, \; r_i^{\Delta^{\bar{k}}}(\mathcal{A}) = \sum_{(i_2, i_3, i_4) \in \Delta^{\bar{k}}} |a_{i i_2 i_3 i_4}|,$$

where we assume that

$$\Delta^{kt} = \{(i_2 i_3 i_4) : (i_2 i_3 i_4) \in \Delta^k \text{ and } i_h = t \text{ for some } h = 2,3,4\},$$

$$\Delta^{k\bar{t}} = \{(i_2 i_3 i_4) : (i_2 i_3 i_4) \in \Delta^k \text{ and } i_h \neq t \text{ for any } h = 2,3,4\},$$

$$\Delta^{\bar{k}t} = \{(i_2, i_3, i_4) : (i_2 i_3 i_4) \in \Delta^{\bar{k}} \text{ and } i_h = t \text{ for some } h = 2,3,4\},$$

and

$$\Delta^{\overline{kt}} = \{(i_2, i_3, i_4) : (i_2 i_3 i_4) \in \Delta^{\bar{k}} \text{ and } i_h \neq t \text{ for any } h = 2,3,4\}.$$

We give our main results in this section as follows.

Theorem 3. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[4,n]}$ with*

$$\beta_i^{\Delta^{1i}}(\mathcal{A}) = \ldots = \beta_i^{\Delta^{ni}}(\mathcal{A}) = C_i(constant),\ i \in N.$$

Then

$$\sigma(\mathcal{A}) \subseteq \Upsilon(\mathcal{A}) = \bigcup_{i,j \in N, i \neq j} \Upsilon_{ij}(\mathcal{A}),$$

where

$$\Upsilon_{ij}(\mathcal{A}) = \{z \in \mathbb{R} : \left(|z - C_i| - r_i^{\Delta^{\bar{k}i}}(\mathcal{A})\right) |z - C_j| \leq \left(\beta_i^{\Delta^{k\bar{i}}}(\mathcal{A}) + r_i^{\Delta^{\overline{ki}}}(\mathcal{A})\right) \left(\beta_j^{\Delta^{k\bar{j}}}(\mathcal{A}) + r_j^{\Delta^{\overline{kj}}}(\mathcal{A}) + r_j^{\Delta^{\bar{k}j}}(\mathcal{A})\right)\},$$

and

$$\beta_i^{\Delta^{ki}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4) \in \Delta^{ki}} |a_{ii_2i_3i_4}| = C_i,\ r_i^{\Delta^{\bar{k}i}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4) \in \Delta^{\bar{k}i}} |a_{ii_2i_3i_4}|,$$

$$\beta_i^{\Delta^{k\bar{i}}}(\mathcal{A}) = \max_{k \in N} \left\{ \sum_{(i_2,i_3,i_4) \in \Delta^{k\bar{i}}} |a_{ii_2i_3i_4}| \right\},\ r_i^{\Delta^{\overline{ki}}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4) \in \Delta^{\overline{ki}}} |a_{ii_2i_3i_4}|,$$

$$\beta_j^{\Delta^{kj}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4) \in \Delta^{kj}} |a_{ji_2i_3i_4}| = C_j,\ r_j^{\Delta^{\bar{k}j}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4) \in \Delta^{\bar{k}j}} |a_{ji_2i_3i_4}|,$$

$$\beta_j^{\Delta^{k\bar{j}}}(\mathcal{A}) = \max_{k \in N} \left\{ \sum_{(i_2,i_3,i_4) \in \Delta^{k\bar{j}}} |a_{ji_2i_3i_4}| \right\},\ r_j^{\Delta^{\overline{kj}}}(\mathcal{A}) = \sum_{(i_2,i_3,i_4) \in \Delta^{\overline{kj}}} |a_{ji_2i_3i_4}|.$$

Proof. Let λ be a Z-eigenvalue of $\mathcal{A}$ with corresponding Z-eigenvector $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n \backslash \{0\}$, i.e.,

$$\mathcal{A}x^3 = \lambda x, \text{ and } x^T x = 1. \tag{2}$$

Let $|x_t| \geq |x_s| \geq \max_{i \in N, i \neq t,s} |x_i|$, then for any $k \in N$, we have

$$\begin{aligned} & \left(\lambda - (\sum_{(i_2,i_3,i_4) \in \Delta^{1t}} a_{ti_2i_3i_4} x_1^2 + \ldots + \sum_{(i_2,i_3,i_4) \in \Delta^{nt}} a_{ti_2i_3i_4} x_n^2) \right) x_t \\ = & \sum_{(i_2,i_3,i_4) \in \Delta^{k\bar{t}}} a_{ti_2i_3i_4} x_{i_2} x_{i_3} x_{i_4} + \sum_{(i_2,i_3,i_4) \in \Delta^{\bar{k}t}} a_{ti_2i_3i_4} x_{i_2} x_{i_3} x_{i_4} \\ + & \sum_{(i_2,i_3,i_4) \in \Delta^{\bar{k}\bar{t}}} a_{ti_2i_3i_4} x_{i_2} x_{i_3} x_{i_4}. \end{aligned}$$

Taking modulus in the above equation, and using the triangle inequality and $x^T x = 1$, we get

$$\begin{aligned} & |\lambda - C_t||x_t| \\ \leq & \sum_{(i_2,i_3,i_4) \in \Delta^{k\bar{t}}} |a_{ti_2i_3i_4}||x_{i_2|}|x_{i_3}||x_{i_4}| + \sum_{(i_2,i_3,i_4) \in \Delta^{\bar{k}}} |a_{ti_2i_3i_4}||x_{i_2|}|x_{i_3}||x_{i_4}| \\ \leq & \sum_{(i_2,i_3,i_4) \in \Delta^{k\bar{t}}} |a_{ti_2i_3i_4}||x_k|^2|x_t| + \sum_{(i_2,i_3,i_4) \in \Delta^{\bar{k}t}} |a_{ti_2i_3i_4}||x_s| \\ + & \sum_{(i_2,i_3,i_4) \in \Delta^{\bar{k}\bar{t}}} |a_{ti_2i_3i_4}||x_s|. \end{aligned}$$

Therefore,

$$\left(|\lambda - C_t| - r_t^{\Delta^{\bar{k}t}}(\mathcal{A}) \right) |x_t| \leq \left(\beta_t^{\Delta^{k\bar{t}}}(\mathcal{A}) + r_t^{\Delta^{\bar{k}\bar{t}}}(\mathcal{A}) \right) |x_s|. \tag{3}$$

If $|x_s| = 0$, then $|\lambda - C_t| - r_t^{\Delta^{\bar{k}t}}(\mathcal{A}) \leq 0$, and it is obvious that $\lambda \in \Upsilon(\mathcal{A})$.
If $|x_s| > 0$, from equality (2), we similarly get

$$(|\lambda - C_s|) |x_s| \leq \left(\beta_s^{\Delta^{k\bar{s}}}(\mathcal{A}) + r_s^{\Delta^{\bar{k}\bar{s}}}(\mathcal{A}) + r_s^{\Delta^{\bar{k}s}}(\mathcal{A}) \right) |x_t|. \tag{4}$$

Multiplying inequalities (3) with (4), we have

$$\begin{aligned} & \left(|\lambda - C_t| - r_t^{\Delta^{\bar{k}t}}(\mathcal{A}) \right) |\lambda - C_s| \\ \leq & \left(\beta_t^{\Delta^{k\bar{t}}}(\mathcal{A}) + r_t^{\Delta^{\bar{k}\bar{t}}}(\mathcal{A}) \right) \left(\beta_s^{\Delta^{k\bar{s}}}(\mathcal{A}) + r_s^{\Delta^{\bar{k}\bar{s}}}(\mathcal{A}) + r_s^{\Delta^{\bar{k}s}}(\mathcal{A}) \right). \end{aligned}$$

Thus, we complete the proof. □

3. Upper Bound for the Z-Spectral Radius of Weakly Symmetric Nonnegative Tensors

In this section, we obtain a sharp upper bound for weakly symmetric nonnegative tensors. Firstly, let us recall the definition of the Z-spectral radius of tensor $\mathcal{A}$.

Definition 3 ([2]). *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. The Z-spectral radius $\rho(\mathcal{A})$ of $\mathcal{A}$ is defined as*

$$\rho(\mathcal{A}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\},$$

where $\sigma(\mathcal{A})$, called the Z-spectrum of $\mathcal{A}$, is the set of all Z-eigenvalues of $\mathcal{A}$.

A tensor $\mathcal{A}$ is called *weakly symmetric* if the associated homogeneous polynomial $\mathcal{A}x^m$ satisfies

$$\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}.$$

We need the following Perron-Frobenius Theorem for the Z-eigenvalue of nonnegative tensors [23].

Lemma 1. *Suppose that the m-order n-dimensional tensor $\mathcal{A}$ is weakly symmetric, nonnegative and irreducible. Then $\rho(\mathcal{A})$ is a positive Z-eigenvalue with a positive Z-eigenvector.*

Based on the above Lemma, we give the main result of this section.

Theorem 4. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[4,n]}$ be weakly symmetric, nonnegative and irreducible and*

$$\max_{i \in N}\{\beta_i^{\Delta^{1i}}(\mathcal{A}), \ldots, \beta_i^{\Delta^{ni}}(\mathcal{A})\} = C_i(constant),\ i \in N.$$

Then

$$\rho(\mathcal{A}) \leq \max\{\max_{i,j \in N, i \neq j} \frac{1}{2}\left(C_i + r_i^{\Delta^{\bar{k}i}}(\mathcal{A}) + C_j + \Lambda_{ij}^{\frac{1}{2}}(\mathcal{A})\right), \max_{i \in N} C_i\},$$

where

$$\Lambda_{ij}(\mathcal{A}) = \left(C_i + r_i^{\Delta^{\bar{k}i}}(\mathcal{A}) - C_j\right)^2 + 4\left(\beta_i^{\Delta^{k\bar{i}}}(\mathcal{A}) + r_i^{\Delta^{\overline{ki}}}(\mathcal{A})\right)\left(\beta_j^{\Delta^{k\bar{j}}}(\mathcal{A}) + r_j^{\Delta^{\overline{kj}}}(\mathcal{A}) + r_j^{\Delta^{\bar{k}j}}(\mathcal{A})\right).$$

Proof. Let $x > 0$ be a Z-eigenvector of $\mathcal{A}$ corresponding to $\rho(\mathcal{A})$, if $\rho(\mathcal{A}) \geq \max_{i \in N} C_i$, then from the proof of Theorem 3, there exist $t, s \in N$, $s \neq t$ such that

$$\begin{aligned} & \left(\lambda - C_t - r_t^{\Delta^{\bar{k}t}}(\mathcal{A})\right)(\lambda - C_s) \\ \leq\ & \left(\beta_t^{\Delta^{k\bar{t}}}(\mathcal{A}) + r_t^{\Delta^{\overline{kt}}}(\mathcal{A})\right)\left(\beta_s^{\Delta^{k\bar{s}}}(\mathcal{A}) + r_s^{\Delta^{\overline{ks}}}(\mathcal{A}) + r_s^{\Delta^{\bar{k}s}}(\mathcal{A})\right). \end{aligned}$$

Then, solving for $\rho(\mathcal{A})$ we get

$$\rho(\mathcal{A}) \leq \frac{1}{2}\left(C_t + r_t^{\Delta^{\bar{k}t}}(\mathcal{A}) + C_s + \Lambda_{ts}^{\frac{1}{2}}(\mathcal{A})\right).$$

Thus, we complete the proof. □

4. Z-Eigenvalue Based Sufficient Condition for the Positive Definiteness of Fourth Order Tensors

In this section, we provide a new checkable sufficient condition for the positive definiteness of fourth order tensors, which is based on the inclusion set for Z-eigenvalues of structured fourth order tensors.

Theorem 5. *Let $\mathcal{A} \in \mathbb{R}^{[4,n]}$ with $\beta_i^{\Delta^{1i}}(\mathcal{A}) = \ldots = \beta_i^{\Delta^{ni}}(\mathcal{A}) = C_i > 0$ be a symmetric tensor. If for all $i, j \in N$, $j \neq i$,*

$$(C_i - r_i^{\Delta^{\bar{k}i}}(\mathcal{A}))C_j > (\beta_i^{\Delta^{k\bar{i}}}(\mathcal{A}) + r_i^{\Delta^{\overline{ki}}}(\mathcal{A}))(\beta_j^{\Delta^{k\bar{j}}}(\mathcal{A}) + r_j^{\Delta^{\overline{kj}}}(\mathcal{A}) + r_j^{\Delta^{\bar{k}j}}(\mathcal{A})),$$

then $\mathcal{A}$ is positive definite.

Proof. Assume that $\lambda \leq 0$ is a Z-eigenvalue of $\mathcal{A}$. From Theorem 3, we have $\lambda \in \Upsilon(\mathcal{A})$, hence, there are $i_0, j_0 \in N$ such that

$$\begin{aligned} & \left(|\lambda - C_{i_0}| - r_{i_0}^{\Delta^{\bar{k}i_0}}(\mathcal{A})\right)|\lambda - C_{j_0}| \\ \leq\ & \left(\beta_{i_0}^{\Delta^{k\overline{i_0}}}(\mathcal{A}) + r_{i_0}^{\Delta^{\overline{ki_0}}}(\mathcal{A})\right)\left(\beta_{j_0}^{\Delta^{k\overline{j_0}}}(\mathcal{A}) + r_{j_0}^{\Delta^{\overline{kj_0}}}(\mathcal{A}) + r_{j_0}^{\Delta^{\bar{k}j_0}}(\mathcal{A})\right). \end{aligned}$$

From $C_{i_0} > 0$ for all $i_0 \in N$, we get

$$\begin{aligned} & \left(|\lambda - C_{i_0}| - r_{i_0}^{\Delta^{\bar{k}i_0}}(\mathcal{A})\right)|\lambda - C_{j_0}| \\ \geq & \left(C_{i_0} - r_{i_0}^{\Delta^{\bar{k}i_0}}(\mathcal{A})\right)C_{j_0} \\ > & \left(\beta_{i_0}^{\Delta^{k\overline{i_0}}}(\mathcal{A}) + r_{i_0}^{\Delta^{\overline{ki_0}}}(\mathcal{A})\right)\left(\beta_{j_0}^{\Delta^{k\overline{j_0}}}(\mathcal{A}) + r_{j_0}^{\Delta^{\overline{kj_0}}}(\mathcal{A}) + r_{j_0}^{\Delta^{\bar{k}j_0}}(\mathcal{A})\right). \end{aligned}$$

This is a contradiction. Hence, $\lambda > 0$. Then, the symmetric tensor $\mathcal{A}$ is positive definite. □

Theorem 6. *Let $\mathcal{A} \in \mathbb{R}^{[4,n]}$ with $\beta_i^{\Delta^{1i}}(\mathcal{A}) > 0, \ \ldots, \ \beta_i^{\Delta^{ni}}(\mathcal{A}) > 0$ be a symmetric tensor,*

$$\min_{i \in N}\{\beta_i^{\Delta^{1i}}(\mathcal{A}), \ldots, \beta_i^{\Delta^{ni}}(\mathcal{A})\} = C_i.$$

Assume $\mathcal{B}$ is a symmetric tensor whose $(i_1 i_2 i_3 i_4)$-th entry is respectively defined as follows:

$$(\mathcal{B})_{i_1 i_2 i_3 i_4} = \begin{cases} b_{iijj} = b_{ijij} = b_{ijji} = \frac{1}{3}C_i, \ i \neq j, \\ b_{iiii} = C_i, \ i \in N, \\ a_{i_1 i_2 i_3 i_4}, \ \text{otherwise}, \end{cases} \tag{5}$$

If $\mathcal{B}$ is positive definite, then $\mathcal{A}$ is positive definite.

Proof. Let $x \in \mathbb{R}^n$ be a nonzero vector. Since $\mathcal{B}$ is positive definite, from the definition of the positive definiteness of symmetric tensors, we have

$$\mathcal{B}x^4 > 0.$$

Then, we have

$$\begin{aligned} 0 < \mathcal{A}x^4 &= \mathcal{B}x^4 + \sum_{i,j \in N}(a_{iijj} - b_{iijj})x_i^2 x_j^2 \\ &+ \sum_{i,j \in N}(a_{ijij} - b_{ijij})x_i^2 x_j^2 + \sum_{i,j \in N}(a_{ijji} - b_{ijji})x_i^2 x_j^2. \end{aligned}$$

Thus $\mathcal{A}$ is positive definite. □

By Theorems 5 and 6, we have the following sufficient condition for the positive definiteness of symmetric fourth order tensors.

Theorem 7. *Let $\mathcal{A} \in \mathbb{R}^{[4,n]}$ with $\beta_i^{\Delta^{1i}}(\mathcal{A}) > 0, \ \ldots, \ \beta_i^{\Delta^{ni}}(\mathcal{A}) > 0$ be a symmetric tensor,*

$$\min_{i \in N}\{\beta_i^{\Delta^{1i}}(\mathcal{A}), \ldots, \beta_i^{\Delta^{ni}}(\mathcal{A})\} = C_i.$$

If for all $i, j \in N$, $j \neq i$,

$$(C_i - r_i^{\Delta^{\bar{k}i}}(\mathcal{A}))C_j > (\beta_i^{\Delta^{k\bar{i}}}(\mathcal{A}) + r_i^{\Delta^{\overline{ki}}}(\mathcal{A}))(\beta_j^{\Delta^{k\bar{j}}}(\mathcal{A}) + r_j^{\Delta^{\overline{kj}}}(\mathcal{A}) + r_j^{\Delta^{\bar{k}j}}(\mathcal{A})),$$

then $\mathcal{A}$ is positive definite.

Based on the above theorem, we introduce the definition of Z-eigenvalue based quasi-doubly strictly diagonally dominated(Z-QDSDD) symmetric fourth order tensors.

Definition 4. *Let $\mathcal{A} \in \mathbb{R}^{[4,n]}$ with $\beta_i^{\Delta^{1i}}(\mathcal{A}) > 0, \ \ldots, \ \beta_i^{\Delta^{ni}}(\mathcal{A}) > 0$ be a symmetric tensor,*

$$\min_{i \in N}\{\beta_i^{\Delta^{1i}}(\mathcal{A}), \ldots, \beta_i^{\Delta^{ni}}(\mathcal{A})\} = C_i.$$

Then, the fourth order tensor $\mathcal{A}$ is called Z-eigenvalue based quasi-doubly strictly diagonally dominated(Z-QDSDD), if for all $i, j \in N$, $j \neq i$,

$$(C_i - r_i^{\Delta^{\bar{k}i}}(\mathcal{A}))C_j > (\beta_i^{\Delta^{k\bar{i}}}(\mathcal{A}) + r_i^{\Delta^{\overline{ki}}}(\mathcal{A}))(\beta_j^{\Delta^{k\bar{j}}}(\mathcal{A}) + r_j^{\Delta^{\overline{kj}}}(\mathcal{A}) + r_j^{\Delta^{\bar{k}j}}(\mathcal{A})),$$

5. Numerical Examples

In this section, some examples are given to show the efficiency of our results. First, an example is given to show the efficiency of the result in Theorem 3.

Example 1. *Consider the tensor $\mathcal{A} = (a_{i_1 i_2 i_3 i_4})$ of order 4 dimension 2 with entries defined as follows:*

$$a_{1111} = a_{1122} = 1, \ a_{1211} = a_{1222} = -1,$$
$$a_{2211} = a_{2222} = 2, \ a_{2111} = a_{2122} = -2,$$

and other $a_{i_1 i_2 i_3 i_4} = 0$. By computation, we get that, $\sigma(\mathcal{A}) = \{\ 0,\ 3\}$.

By Theorem 3.3 of [15], we have

$$\mathcal{L}(\mathcal{A}) = \{z \in \mathbb{R} : |z| \leq 5\}.$$

By Theorem 5 of [24], we have

$$\mathcal{K}(\mathcal{A}) = \{z \in \mathbb{R} : |z| \leq 6.5615\}.$$

By Theorem 3,

$$\beta_1^{\Delta^{k1}}(\mathcal{A}) = C_1 = 1,$$
$$\beta_1^{\Delta^{k\bar{1}}}(\mathcal{A}) = \max\{|a_{1211}|, |a_{1222}|\} = 1,$$
$$\beta_2^{\Delta^{k2}}(\mathcal{A}) = C_2 = 2,$$
$$\beta_2^{\Delta^{k\bar{2}}}(\mathcal{A}) = \max\{|a_{2111}|, |a_{2122}|\} = 2,$$
$$r_1^{\Delta^{\bar{k}1}}(\mathcal{A}) = r_2^{\Delta^{\bar{k}1}}(\mathcal{A}) = 0,$$
$$r_1^{\Delta^{\overline{k1}}}(\mathcal{A}) = r_2^{\Delta^{\overline{k1}}}(\mathcal{A}) = 0,$$

then we have

$$\Upsilon(\mathcal{A}) = \{z \in \mathbb{R} : |z-1||z-2| \leq 2\}.$$

The Z-eigenvalue inclusion sets $\Upsilon(\mathcal{A})$ and the exact Z-eigenvalues are drawn in Figure 1. We can see that, $\Upsilon(\mathcal{A})$ can capture all Z-eigenvalues of $\mathcal{A}$, and the Z-eigenvalue inclusion set $\Upsilon(\mathcal{A})$ is located on the right side of the coordinate axis, which is better than the Z-eigenvalue inclusion sets $\mathcal{K}(\mathcal{A})$ and $\mathcal{L}(\mathcal{A})$.

We now show the efficiency of the new upper bound in Theorem 4 by the following example.

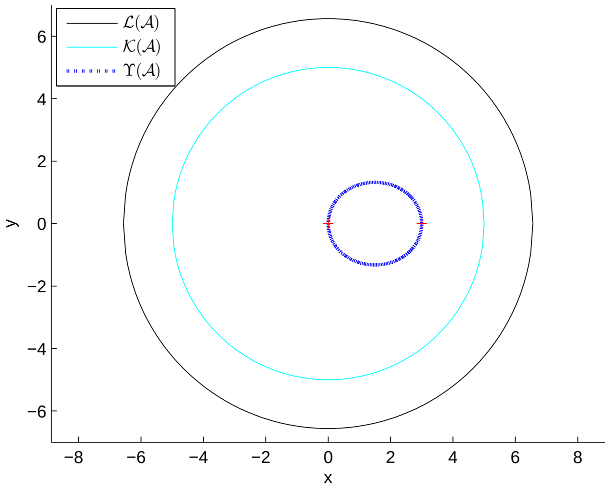

Figure 1. Comparisons of Z-eigenvalue inclusion sets.

Example 2. *Let* $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,2]}$ *be a symmetric tensor defined as follows:*

$$\begin{aligned} &a_{1111} = 0, \\ &a_{1211} = a_{1121} = a_{1112} = a_{2111} = 1, \\ &a_{2211} = a_{2121} = a_{2112} = a_{1122} = a_{1212} = a_{1221} = 1, \\ &a_{2222} = 4, \end{aligned}$$

while the other $a_{i_1 i_2 i_3 i_4} = 0$*. By Theorem 4.6 of [15], we have*

$$\rho(\mathcal{A}) \leq 7.$$

By Theorem 7 of [24], we have

$$\rho(\mathcal{A}) \leq 7.$$

By Theorem 4,

$$C_1 = 3,\ C_2 = 4$$

$$r_1^{\Delta^{\bar{k}1}}(\mathcal{A}) = \beta_1^{\Delta^{k\bar{1}}}(\mathcal{A}) = r_1^{\Delta^{\bar{k}\bar{1}}}(\mathcal{A}) = 0,$$

and

$$r_2^{\Delta^{\bar{k}2}}(\mathcal{A}) = \beta_1^{\Delta^{k\bar{1}}}(\mathcal{A}) = r_1^{\Delta^{\bar{k}\bar{1}}}(\mathcal{A}) = r_1^{\Delta^{\bar{k}1}}(\mathcal{A}) = 0,$$

then we have

$$\rho(\mathcal{A}) \leq 4.$$

In fact, $\rho(\mathcal{A}) = 4$*. Hence, the bound in Theorem 4 is sharper and could reach the true value of* $\rho(\mathcal{A})$ *in some cases.*

Finally, we now show the efficiency of result in Theorem 7 by the following example.

Example 3. *Let* $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}^{[4,2]}$ *be a symmetric tensor defined as follows:*

$$\begin{gathered} a_{1111} = 1,\ a_{2222} = 2, \\ a_{1112} = a_{1121} = a_{1211} = a_{2111} = 0.6, \\ a_{2221} = a_{2212} = a_{2122} = a_{1222} = 1, \\ a_{1122} = a_{1221} = a_{1212} = 10, \\ a_{2211} = a_{2121} = a_{2112} = 20. \end{gathered}$$

By computation, we get that,

$$a_{1111} = 1 < R_1(\mathcal{A}) = 32.8,\ a_{2222} = 2 < R_2(\mathcal{A}) = 63.6.$$

Hence, $\mathcal{A}$ is not a SDD tensor. Then, we cannot use Theorem 2 (1) to determine the positiveness of $\mathcal{A}$. We can get

$$(|a_{1111}| - R_1^2(\mathcal{A}))|a_{2222}| = -63.6 < |a_{1222}|R_2(\mathcal{A}) = 63.6,$$
$$(|a_{2222}| - R_2^1(\mathcal{A}))|a_{1111}| = -61 < |a_{2111}|R_1(\mathcal{A}) = 19.68.$$

Hence, $\mathcal{A}$ is not a QSDD tensor. Then, we cannot use Theorem 2 (2) to determine the positiveness of $\mathcal{A}$. However, it is easy to find

$$C_1 = 1,\ C_2 = 2,$$

and

$$\begin{aligned} & (C_1 - r_1^{\Delta^{\bar{k}1}}(\mathcal{A}))C_2 = 2 \\ > \quad & (\beta_1^{\Delta^{k\bar{1}}}(\mathcal{A}) + r_1^{\Delta^{\overline{k1}}}(\mathcal{A}))(\beta_2^{\Delta^{k\bar{2}}}(\mathcal{A}) + r_2^{\Delta^{\overline{k2}}}(\mathcal{A}) + r_2^{\Delta^{\bar{k}2}}(\mathcal{A})) = 1.8. \end{aligned}$$

In other words, $\mathcal{A}$ satisfies all the conditions of Theorem 7, i.e., $\mathcal{A}$ is a Z-QDSDD tensor. Hence, from Theorem 7, $\mathcal{A}$ is a positive definite tensor. In fact,

$$\sigma(\mathcal{A}) = \{\ 0.9908,\ 1.9669,\ 19.1249,\ 22.2080\}.$$

From the definition of positive definite tensors, $\mathcal{A}$ is positive definite.

6. Conclusions

In this paper, focused the fourth order tensors, a new Z-eigenvalue localization set for Z-eigenvalues of structured fourth order tensors is given. As an application, a sharper upper bound for the Z-spectral radius of weakly symmetric nonnegative fourth order tensors is obtained and a Z-eigenvalue based sufficient condition for the positive definiteness of structured fourth order tensors is also given. A positive definite diffusion tensor is a convex optimization problem with a convex quadratic objective function constrained by the nonnegativity requirement on the smallest Z-eigenvalue of the diffusivity function [11], but it is difficult to compute all the Z-eigenvalues or the smallest Z-eigenvalue of a fourth order tensor when n is large. Finally, we introduce the definition of Z-eigenvalue based doubly strictly diagonally dominated(Z-QDSDD) symmetric fourth order tensors and show that, if a tensor $\mathcal{A}$ is Z-QDSDD, then $\mathcal{A}$ is positive definite.

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