mathematics

MDPI

Article

On Discrete Fractional Solutions of Non-Fuchsian Differential Equations

Resat Yilmazer [1,*,†], Mustafa Inc [1,†] and Mustafa Bayram [2,†]

1 Department of Mathematics, Firat University, 23119 Elazig, Turkey; minc@firat.edu.tr
2 Department of Computer Engineering, Istanbul Gelisim University, 34315 Istanbul, Turkey; mbayram@gelisim.edu.tr
* Correspondence: ryilmazer@firat.edu.tr; Tel.: +90-424-237-0000
† These authors contributed equally to this work.

Received: 15 October 2018; Accepted: 5 December 2018; Published: 7 December 2018

check for updates

Abstract: In this article, we obtain new fractional solutions of the general class of non-Fuchsian differential equations by using discrete fractional nabla operator $\nabla^{\eta}(0 < \eta < 1)$. This operator is applied to homogeneous and nonhomogeneous linear ordinary differential equations. Thus, we obtain new solutions in fractional forms by a newly developed method.

Keywords: discrete fractional calculus; fractional nabla operator; non-Fuchsian equations

1. Introduction

The history of fractional mathematics dates back to Leibniz (1695). This field of work is rapidly increasing and, nowadays, it has many applications in science and engineering [1–4]. Heat transfer, diffusion and Schrödinger equation are some fields where fractional analysis is used.

A similar theory was started for discrete fractional analysis and the definition and properties of fractional sums and differences theory were developed. Many articles related to this topic have appeared lately [5–18].

In 1956 [5], differences of fractional order was first introduced by Kuttner. Difference of fractional order has attracted more interest in recent years.

Diaz and Osler [6], defined the notion of fractional difference as follows

$$\Delta^{\varsigma}\Phi(t) = \sum_{k=0}^{\infty}(-1)^k\binom{\varsigma}{k}\Phi(t+\varsigma-k)$$

where ς is any real number.

Granger and Joyeux [19] and Hosking [20], defined notion of the fractional difference as follows

$$\begin{aligned}\nabla^{\varsigma}\Phi(t) &= (1-q)^{\varsigma}\Phi(t)\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\varsigma+1)}{\Gamma(k+1)\Gamma(\varsigma-k+1)}q^k\Phi(t) \\ &= \sum_{k=0}^{\infty}(-1)\binom{\varsigma}{k}\Phi(t-k),\end{aligned}$$

where ς is any real number and $q\Phi(t) = \Phi(t-1)$ is the shift operator. Gray and Zhang [21], Acar and Atici [10] studied on a new definition and characteristics of the fractional difference.

2. Preliminary and Properties

In this section, we first present sufficient fundamental definitions and formulas so that the article is self-contained.

The rising factorial power $t^{\overline{m}}$ (*t to the m rising*, $m \in \mathbb{N}$) is defined by

$$t^{\overline{m}} = t\,(t+1)\,(t+2)\ldots(t+m-1)\,,\ t^{\overline{0}} = 1.$$

Let σ be any real number. Then "t to the σ rising" is defined to be

$$t^{\overline{\sigma}} = \frac{\Gamma(t+\sigma)}{\Gamma(t)},\ t \in \mathbb{R} - \{\ldots,-2,-1,0\}\,,\ 0^{\overline{\sigma}} = 0. \tag{1}$$

Also, the ∇ operator of Equation (1) is given by

$$\begin{aligned}\nabla(t^{\overline{\sigma}}) &= \nabla\frac{\Gamma(t+\sigma)}{\Gamma(t)} \\ &= \frac{\Gamma(t+\sigma)}{\Gamma(t)} - \frac{\Gamma(t-1+\sigma)}{\Gamma(t-1)} \\ &= \frac{(t-1+\sigma)\Gamma(t-1+\sigma)}{\Gamma(t)} - \frac{\Gamma(t-1+\sigma)}{\Gamma(t-1)} \\ &= \frac{\Gamma(t-1+\sigma)}{\Gamma(t-1)}\left(\frac{t-1+\sigma}{t-1} - 1\right) \\ &= \sigma\frac{\Gamma(t-1+\sigma)}{\Gamma(t)} \\ &= \sigma t^{\overline{\sigma-1}}\end{aligned} \tag{2}$$

where $\nabla u(t) = u(t) - u(t-1)$.

Let $\eta \in \mathbb{R}^+$ such that $m-1 \le \eta < m$, $m \in \mathbb{N}$. The ηth-order fractional nabla sum of g is given by

$$\nabla_b^{-\eta} g(t) = \frac{1}{\Gamma(\eta)}\sum_{s=b}^{t}(t-\delta(s))^{\overline{\eta-1}} g(s)\,, \tag{3}$$

where $t \in \mathbb{N}_b = \{b\} + \mathbb{N}_0 = \{b, b+1, b+2, \ldots\}\,,\ b \in \mathbb{R}$, $\delta(s) = s-1$ is backward jump operator. Also, we define the trivial sum by $\nabla_b^{-0} g(t) = g(t)$ for $t \in \mathbb{N}_b$.

The ηth-order Riemann-Liouville type nabla fractional difference of g is defined by

$$\begin{aligned}\nabla_b^{\eta} g(t) &= \nabla^m\left[\nabla^{-(m-\eta)} g(t)\right] \\ &= \nabla^m\left[\frac{1}{\Gamma(m-\eta)}\sum_{s=b}^{t}(t-\delta(s))^{\overline{m-\eta-1}} g(s)\right],\end{aligned} \tag{4}$$

where $g : \mathbb{N}_b^+ \longrightarrow \mathbb{R}$ [10].

Theorem 1 ([16])**.** *Let f and $g : \mathbb{N}_0^+ \longrightarrow \mathbb{R}$, $\gamma, \phi > 0$. Then*

$$\nabla^{-\gamma}\nabla^{-\phi} f(t) = \nabla^{-(\gamma+\phi)} f(t) = \nabla^{-\phi}\nabla^{-\gamma} f(t)\,, \tag{5}$$

$$\nabla^{\gamma}[hf(t) + kg(t)] = h\nabla^{\gamma} f(t) + k\nabla^{\gamma} g(t)\,,\ h, k \in \mathbb{R} \tag{6}$$

$$\nabla\nabla^{-\gamma} f(t) = \nabla^{-(\gamma-1)} f(t)\,, \tag{7}$$

$$\nabla^{-\gamma}\nabla f(t) = \nabla^{(1-\gamma)} f(t) - \binom{t+\gamma-2}{t-1} f(0)\,. \tag{8}$$

Lemma 1 (Power Rule [10]). *Let $v > 0$ and η be two real numbers so that $\frac{\Gamma(\eta+1)}{\Gamma(\eta+v+1)}$ is defined. Then,*

$$\nabla_b^{-v} (t-b+1)^{\bar{\eta}} = \frac{\Gamma(\eta+1)}{\Gamma(\eta+v+1)} (t-b+1)^{\overline{\eta+v}}, \quad t \in \mathbb{N}_b.$$

Lemma 2 (Leibniz Rule [10]). *For any $\eta > 0$, ηth-order fractional difference of the product fg is given in this form*

$$\nabla_0^{\eta} (fg)(t) = \sum_{m=0}^{t} \binom{\eta}{m} \left[\nabla_0^{\eta-m} f(t-m)\right] \left[\nabla^m g(t)\right], \tag{9}$$

where

$$\binom{\eta}{m} = \frac{\Gamma(\eta+1)}{\Gamma(m+1)\Gamma(\eta-m+1)}$$

and f, g are defined on $\mathbb{N}_0$, and t is a positive integer.

Lemma 3 (Index Law). *Let $g(t)$ is single-valued and analytic. Then*

$$(g_\gamma)_\eta = g_{\gamma+\eta} = (g_\eta)_\gamma \quad (g_\gamma \neq 0; g_\eta \neq 0; \gamma, \eta \in \mathbb{R}; t \in \mathbb{C}). \tag{10}$$

3. Main Results

We start by considering the following differential equation

$$\left(1+\frac{\ell}{x}\right)\frac{d^2y}{dx^2} + \left[a+\frac{b}{x}\left(1+\frac{\ell}{x}\right)\right]\frac{dy}{dx} + \left[c+\frac{d}{x}+\frac{\varepsilon}{x^2}\left(1+\frac{\ell}{x}\right)\right] y(x) = \psi \tag{11}$$

where ψ is a given function, $x \in \mathbb{C}\backslash\{0, -\ell\}$, and a, b, c, d, ε and ℓ are parameters.

Let

$$y(x) = x^{\tau} e^{\kappa x} w(x) \tag{12}$$

so that

$$\frac{dy}{dx} = x^{\tau-1} e^{\kappa x}\left[x\frac{dw}{dx} + (\tau+\kappa x) w(x)\right] \tag{13}$$

and

$$\frac{d^2y}{dx^2} = x^{\tau-2} e^{\kappa x}\left[x^2\frac{d^2w}{dx^2} + 2(\tau+\kappa x) x\frac{dw}{dx} + \left\{\kappa^2x^2 + 2\tau\kappa x + \tau(\tau-1)\right\} w(x)\right]. \tag{14}$$

By substituting (12)–(14) into the (11), we have

$$\begin{aligned} & x^2(x+\ell)\frac{d^2w}{dx^2} + \left[(2\tau+b)\ell + (2\tau+2\kappa\ell+b)x + (2\kappa+a)x^2\right] x\frac{dw}{dx} \\ & + \left[\{\tau(\tau+b-1)+\varepsilon\}\ell + \{\tau(\tau+2\kappa\ell+b-1)+\kappa b\ell+\varepsilon\} x \right. \\ & \left. + \left\{\kappa^2\ell + (2\tau+b)\kappa + \tau a + d\right\} x^2 + \left(\kappa^2+\kappa a+c\right) x^3\right] w(x) \\ = \; & x^{3-\tau} e^{-\kappa x}\psi(x), \quad x \in \mathbb{C}\backslash\{0, -\ell\}. \end{aligned} \tag{15}$$

Finally, we find it to be suitable to restrict the different parameters involved in (11) and (15) by means of the following equalities;

$$\begin{aligned} 2\tau + b &= 0, \\ \tau(\tau+b-1)+\varepsilon &= 0, \\ \kappa^2 + \kappa a + c &= 0, \end{aligned} \tag{16}$$

so that

$$\tau = -\frac{1}{2}b = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2}, \tag{17}$$

and

$$\kappa = \frac{-a \pm \sqrt{a^2-4c}}{2}. \tag{18}$$

Under the parametric constraints given by (16), the Equation (15) will immediately decrease to a simpler form

$$(x+\ell)\frac{d^2w}{dx^2} + [2\kappa\ell + (2\kappa + a)x]\frac{dw}{dx} + \left(\kappa^2\ell + \tau a + d\right)w(x) = x^{1-\tau}e^{-\kappa x}\psi(x) \tag{19}$$

where τ and κ are given by (17) and (18), respectively.

Theorem 2. *Let* $w, \psi \in \left\{w, \psi : 0 \neq \left|w_\eta(x)\right|, \left|\psi_\eta(x)\right| < \infty\right\}$, *and* $\eta \in \mathbb{R}$. *Then the nonhomogeneous linear differential equation*

$$w_2(\alpha x + \beta) + w_1(\gamma x + \nu\alpha + \delta) + \nu\gamma w(x) = \psi(x),\ x \neq -\frac{\beta}{\alpha},\ \alpha \neq 0,\ \nu \in \mathbb{R} \tag{20}$$

has particular solutions in the below forms:

$$w^I(x) = \left\{\left[\psi_{-q^{-1}\nu}(\alpha x+\beta)^{(\delta\alpha-\gamma\beta-\alpha^2)/\alpha^2}e^{\frac{\gamma}{\alpha}x}\right]_{-1}(\alpha x+\beta)^{(\gamma\beta-\delta\alpha)/\alpha^2}e^{-\frac{\gamma}{\alpha}x}\right\}_{-1+q^{-1}\nu} \tag{21}$$

$$\begin{aligned} w^{II}(x) &= (\alpha x+\beta)^{\frac{-(\gamma x+\delta)}{\alpha}-\nu+1} \\ &\times \left(\left\{\left[\psi(\alpha x+\beta)^{\frac{\gamma x+\delta}{\alpha}+\nu-1}\right]_{q^{-1}\nu}(\alpha x+\beta)^{\frac{-\delta\alpha+\gamma\beta}{\alpha^2}+1}e^{-\frac{\gamma}{\alpha}x}\right\}_{-1}\right. \\ &\times \left.(\alpha x+\beta)^{\frac{\delta\alpha-\gamma\beta}{\alpha^2}-2}e^{\frac{\gamma}{\alpha}x}\right)_{-1-q^{-1}\nu} \end{aligned} \tag{22}$$

where $w_n = \frac{d^nw}{dx^n}$ $(n = 0,1,2)$, $w_0 = w = w(x)$, $\alpha, \beta, \gamma, \nu, \delta$ *are given constants.*

Proof. For $\psi(x) \neq 0$,

(*i*) When we operate ∇^η to the both sides of (20), we have

$$\nabla^\eta[w_2(\alpha x+\beta)] + \nabla^\eta[w_1(\gamma x+\nu\alpha+\delta)] + \nabla^\eta(w\nu\gamma) = \nabla^\eta\psi \tag{23}$$

by using (9) and (10) we obtain

$$\nabla^\eta[w_2(\alpha x+\beta)] = w_{2+\eta}(\alpha x+\beta) + q\eta\alpha w_{1+\eta} \tag{24}$$

$$\nabla^\eta[w_1(\gamma x+\nu\alpha+\delta)] = w_{1+\eta}(\gamma x+\nu\alpha+\delta) + q\eta\gamma w_\eta \tag{25}$$

where q is a shift operator which is defined by $w(t-1) = qw(t)$. By substituting (24), (25) into the (23), we have

$$w_{2+\eta}(\alpha x+\beta) + [(\eta q+\nu)\alpha + \gamma x + \delta]w_{1+\eta} + (q\eta\gamma + \nu\gamma)w_\eta = \psi_\eta. \tag{26}$$

We choose η such that

$$q\eta\gamma + \nu\gamma = 0, \quad \eta = -q^{-1}\nu.$$

Then we obtain

$$w_{2-q^{-1}\nu}(\alpha x+\beta) + w_{1-q^{-1}\nu}(\gamma x+\delta) = \psi_{-q^{-1}\nu} \tag{27}$$

from (26).

Therefore, setting

$$w_{1-q^{-1}\nu} = u \quad \left(w = u_{-1+q^{-1}\nu}\right) \tag{28}$$

we have

$$u_1 + u\left(\frac{\gamma x+\delta}{\alpha x+\beta}\right) = \psi_{-q^{-1}\nu}\,(\alpha x+\beta)^{-1} \tag{29}$$

from (27). A particular solution of a first order ordinary differential Equation (29):

$$u = \left[\psi_{-q^{-1}\nu}\,(\alpha x+\beta)^{\left(\delta\alpha-\gamma\beta-\alpha^2\right)/\alpha^2} e^{\frac{\gamma}{\alpha}x}\right]_{-1} (\alpha x+\beta)^{(\gamma\beta-\delta\alpha)/\alpha^2} e^{-\frac{\gamma}{\alpha}x}. \tag{30}$$

Thus we obtain the solution (21) from (28) and (30).

(***ii***) Set

$$w = (\alpha x+\beta)^{\sigma} W(x) \tag{31}$$

The first and second derivations of (31) are acquired as follows:

$$w_1 = \sigma(\alpha x+\beta)^{\sigma-1}\alpha W + (\alpha x+\beta)^{\sigma} W_1 \tag{32}$$

$$w_2 = \sigma(\sigma-1)(\alpha x+\beta)^{\sigma-2}\alpha^2 W + 2\sigma(\alpha x+\beta)^{\sigma-1}\alpha W_1 + (\alpha x+\beta)^{\sigma} W_2. \tag{33}$$

Substitute (31)–(33) into (20), we have

$$\begin{aligned} & W_2(\alpha x+\beta) + W_1(2\sigma\alpha+\gamma x+\nu\alpha+\delta) \\ & +W\left(\frac{\alpha^2\sigma(\sigma-1)+\alpha\sigma(\gamma x+\nu\alpha+\delta)}{\alpha x+\beta}+\nu\gamma\right) \\ = \ & \psi(\alpha x+\beta)^{-\sigma}. \end{aligned} \tag{34}$$

Here, we choose σ such that

$$\alpha\sigma(\alpha\sigma-\alpha+\gamma x+\nu\alpha+\delta) = 0$$

that is $\sigma = 0$, $\sigma = \frac{-(\gamma x+\delta)}{\alpha} - \nu + 1$.

In the case $\sigma = 0$, we have the same results as *i*.

Let $\sigma = \frac{-(\gamma x+\delta)}{\alpha} - \nu + 1$. From (31) and (34)

$$w = (\alpha x+\beta)^{\frac{-(\gamma x+\delta)}{\alpha}-\nu+1} W \tag{35}$$

and

$$W_2(\alpha x+\beta) + W_1[\alpha(2-\nu)-\delta-\gamma x] + \nu\gamma W = \psi(\alpha x+\beta)^{\frac{\gamma x+\delta}{\alpha}+\nu-1} \tag{36}$$

respectively.

Applying the operator ∇^{η} to both members of (36), we have

$$\begin{aligned} & W_{2+\eta}(\alpha x+\beta) + W_{1+\eta}[\alpha(2-\nu+\eta q)-\delta-\gamma x] + W_{\eta}(-\gamma\eta q+\nu\gamma) \\ = \ & \left[\psi(\alpha x+\beta)^{\frac{\gamma x+\delta}{\alpha}+\nu-1}\right]_{\eta}. \end{aligned} \tag{37}$$

Choose η such that

$$-\gamma\eta q+\nu\gamma = 0, \quad \eta = q^{-1}\nu$$

we have then

$$W_{2+q^{-1}\nu}\left(\alpha x+\beta\right)+W_{1+q^{-1}\nu}\left[2\alpha-\left(\gamma x+\delta\right)\right]=\left[\psi\left(\alpha x+\beta\right)^{\frac{\gamma x+\delta}{\alpha}+\nu-1}\right]_{q^{-1}\nu} \tag{38}$$

from (37).

Therefore, setting

$$W_{1+q^{-1}\nu}=\vartheta,\quad W=\vartheta_{-1-q^{-1}\nu} \tag{39}$$

we have

$$\vartheta_1+\vartheta\left[\frac{2\alpha}{\alpha x+\beta}-\frac{\gamma x+\delta}{\alpha x+\beta}\right]=\left[\psi\left(\alpha x+\beta\right)^{\frac{\gamma x+\delta}{\alpha}+\nu-1}\right]_{q^{-1}\nu}\left(\alpha x+\beta\right)^{-1} \tag{40}$$

from (38). A particular solution of ordinary differential Equation (40) is given by

$$\vartheta=\left\{\left[\psi\left(\alpha x+\beta\right)^{\frac{\gamma x+\delta}{\alpha}+\nu-1}\right]_{q^{-1}\nu}\left(\alpha x+\beta\right)^{\frac{-\delta\alpha+\gamma\beta}{\alpha^2}+1}e^{-\frac{\gamma}{\alpha}x}\right\}_{-1}\left(\alpha x+\beta\right)^{\frac{\delta\alpha-\gamma\beta}{\alpha^2}-2}e^{\frac{\gamma}{\alpha}x}. \tag{41}$$

Thus we obtain the solution (22) from (35), (39) and (41). □

Furthermore, we can prove for the homogen part such that the homogeneous linear ordinary differential equation

$$w_2\left(\alpha x+\beta\right)+w_1\left(\gamma x+\nu\alpha+\delta\right)+\nu\gamma w\left(x\right)=0,\ x\neq-\frac{\beta}{\alpha},\ \alpha\neq0,\ \nu\in\mathbb{R} \tag{42}$$

has solutions of the forms

$$w^{\mathrm{I}}\left(x\right)=h\left[\left(\alpha x+\beta\right)^{\left(\gamma\beta-\delta\alpha\right)/\alpha^2}e^{-\frac{\gamma}{\alpha}x}\right]_{-1+q^{-1}\nu}, \tag{43}$$

$$w^{\mathrm{II}}\left(x\right)=h\left(\alpha x+\beta\right)^{\frac{-\left(\gamma x+\delta\right)}{\alpha}-\nu+1}\left[\left(\alpha x+\beta\right)^{\frac{\delta\alpha-\gamma\beta}{\alpha^2}-2}e^{\frac{\gamma}{\alpha}x}\right]_{-1-q^{-1}\nu} \tag{44}$$

where h is an arbitrary constant.

Now, in Theorem 1, we further set

$$\alpha=1,\ \beta=\ell,\ \gamma=2\kappa+a,\ \delta=2\kappa\ell-\nu,\ \nu=\frac{\kappa^2\ell+\tau a+d}{2\kappa+a} \tag{45}$$

and let

$$\psi\left(x\right)\to x^{1-\tau}e^{-\kappa x}\psi\left(x\right).$$

We thus find that the nonhomogeneous differential Equation (19) has a particular solution given by

$$\begin{aligned} w^{\mathrm{I}}\left(x\right) &= \left(\left\{\left[x^{1-\tau}e^{-\kappa x}\psi\left(x\right)\right]_{-q^{-1}\nu}\left(x+\ell\right)^{-\nu-a\ell-1}e^{\left(2\kappa+a\right)x}\right\}_{-1}\right. \\ &\quad \left.\left(x+\ell\right)^{\nu+a\ell}e^{-\left(2\kappa+a\right)x}\right)_{-1+q^{-1}\nu}, \end{aligned} \tag{46}$$

$$\begin{aligned} w^{\mathrm{II}}\left(x\right) &= \left(x+\ell\right)^{-\left(2\kappa+a\right)x-2\kappa\ell+1} \\ &\quad\times\left(\left\{\left[x^{1-\tau}e^{-\kappa x}\psi\left(x\right)\left(x+\ell\right)^{\left(2\kappa+a\right)x+2\kappa\ell-1}\right]_{q^{-1}\nu}\left(x+\ell\right)^{\nu+a\ell+1}e^{-\left(2\kappa+a\right)x}\right\}_{-1}\right. \\ &\quad\left.\times\left(x+\ell\right)^{-\nu-a\ell-2}e^{\left(2\kappa+a\right)x}\right)_{-1-q^{-1}\nu} \end{aligned} \tag{47}$$

and that the corresponding homogeneous linear differential equation

$$(x+\ell)\frac{d^2w}{dx^2}+[2\kappa\ell+(2\kappa+a)x]\frac{dw}{dx}+\left(\kappa^2\ell+\tau a+d\right)w(x)=0 \tag{48}$$

has solutions of the forms

$$w^{\mathrm{I}}(x)=h\left[(x+\ell)^{\nu+a\ell}e^{-(2\kappa+a)x}\right]_{-1+q^{-1}\nu}, \tag{49}$$

$$w^{\mathrm{II}}(x)=h(x+\ell)^{-(2\kappa+a)x-2\kappa\ell+1}\left[(x+\ell)^{-\nu-a\ell-2}e^{(2\kappa+a)x}\right]_{-1-q^{-1}\nu} \tag{50}$$

where h is an arbitrary constant.

Therefore, the linear differential Equation (11), has a particular solution in the following forms

$$\begin{aligned} y^{\mathrm{I}}(x) &= x^{\tau}e^{\kappa x}w(x) \\ &= x^{\tau}e^{\kappa x}\left(\left\{\left[x^{1-\tau}e^{-\kappa x}\psi(x)\right]_{-q^{-1}\nu}(x+\ell)^{-\nu-a\ell-1}e^{(2\kappa+a)x}\right\}_{-1}\right. \\ &\quad \left.(x+\ell)^{\nu+a\ell}e^{-(2\kappa+a)x}\right)_{-1+q^{-1}\nu} \quad x\in\mathbb{C}\backslash\{0,-\ell\},\ v\in\mathbb{R} \end{aligned} \tag{51}$$

and

$$\begin{aligned} y^{\mathrm{II}}(x) &= x^{\tau}e^{\kappa x}(x+\ell)^{-(2\kappa+a)x-2\kappa\ell+1} \\ &\quad \times\left(\left\{\left[x^{1-\tau}e^{-\kappa x}\psi(x)(x+\ell)^{(2\kappa+a)x+2\kappa\ell-1}\right]_{q^{-1}\nu}(x+\ell)^{\nu+a\ell+1}e^{-(2\kappa+a)x}\right\}_{-1}\right. \\ &\quad \left.\times(x+\ell)^{-\nu-a\ell-2}e^{(2\kappa+a)x}\right)_{-1-q^{-1}\nu} \end{aligned} \tag{52}$$

and that the corresponding homogeneous linear differential equation

$$\left(1+\frac{\ell}{x}\right)\frac{d^2y}{dx^2}+\left[a+\frac{b}{x}\left(1+\frac{\ell}{x}\right)\right]\frac{dy}{dx}+\left[c+\frac{d}{x}+\frac{\varepsilon}{x^2}\left(1+\frac{\ell}{x}\right)\right]y(x)=0 \tag{53}$$

has solutions given by

$$y^{\mathrm{I}}(x)=hx^{\tau}e^{\kappa x}\left[(x+\ell)^{\nu+a\ell}e^{-(2\kappa+a)x}\right]_{-1+q^{-1}\nu}, \tag{54}$$

$$y^{\mathrm{II}}(x)=hx^{\tau}e^{\kappa x}(x+\ell)^{-(2\kappa+a)x-2\kappa\ell+1}\left[(x+\ell)^{-\nu-a\ell-2}e^{(2\kappa+a)x}\right]_{-1-q^{-1}\nu} \tag{55}$$

where $h\in\mathbb{R}$, the parameters τ, κ and ν are given by (17), (18) and (45).

Remark 1. *First of all, when $\ell=0$, the differential Equation* (11) *reduces to the following version of the Tricomi equation:*

$$\frac{d^2y}{dx^2}+\left(a+\frac{b}{x}\right)\frac{dy}{dx}+\left(c+\frac{d}{x}+\frac{\varepsilon}{x^2}\right)y=\psi(x).$$

By setting

$$\ell=0,\quad a=b=0,\quad c=k^2,\quad d=n,\quad \varepsilon=\frac{1}{4}-m^2,$$

in the Equation (11), *we readily obtain the following Hydrogen atom equation:*

$$\frac{d^2y}{dx^2}+\left(k^2+\frac{n}{x}+\frac{\frac{1}{4}-m^2}{x^2}\right)y=\psi(x).$$

Example 1. *In the case $\alpha = 1, \beta = \gamma = v = 0, \delta = 2$ and $\psi(x) = x$, we have*

$$w(x) + \frac{2}{x} w_1 = 1 \ \ (x \neq 0) \tag{56}$$

from (20). Solution of Equation (56) is obtained as

$$\begin{aligned} w(x) &= \left\{ [x^2]_{-1} x^{-2} \right\} \\ &= \left\{ \frac{x^3}{3} x^{-2} \right\} \\ &= \frac{1}{6} x^2 \end{aligned} \tag{57}$$

by using (21). The function obtained in (57) provide the Equation (56).

4. Conclusions

In this article, we use the discrete fractional operator for the homogeneous and non-homogeneous non-Fuchsian differential equations. This solution of the equation has not been obtained before by using ∇ operator. We can obtain particular solutions of the same type linear singular ordinary and partial differential equations by using the discrete fractional nabla operator in future works.

Author Contributions: These authors contributed equally to this work.

Funding: This research received no external funding.

Acknowledgments: The authors gratefully thank the anonymous the editor and referees for valuable suggestions which improved the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley and Sons, Inc.: New York, NY, USA, 1993.
2. Oldham, K.; Spanier, J. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*; Dover Publications, Inc.: Mineola, NY, USA, 2002.
3. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
4. Baleanu, D.; Guven, Z.B.; Machado, J.A.T.; *New Trends in Nanotechnology and Fractional Calculus Applications*; Springer: Berlin/Heidelberg, Germany, 2010.
5. Kuttner, B. On differences of fractional order. *Proc. Lond. Math. Soc.* **1957**, *3*, 453–466. [CrossRef]
6. Diaz, J.B.; Osler, T.J. Differences of Fractional Order. *Am. Math. Soc.* **1974**, *28*, 185–202. [CrossRef]
7. Atici, F.M.; Eloe, P.W. A transform method in discrete fractional calculus. *Int. J. Differ. Equ.* **2007**, *2*, 165–176.
8. Atici , F.M.; Eloe, P.W. Discrete fractional calculus with the nabla operator. *Electr. J. Qual. Theory Differ. Equ.* **2009**, *3*, 1–12.
9. Lin, S.D.; Tu, S.T.; Srivastava, H.M. A Unified Presentation of Certain Families of Non-Fuchsian Differential Equations via Fractional Calculus Operators. *Comput. Math. Appl.* **2003**, *45*, 1861–1870. [CrossRef]
10. Acar, N.; Atici , F.M. Exponential functions of discrete fractional calculus. *Appl. Anal. Discrete Math.* **2013**, *7*, 343–353. [CrossRef]
11. Anastassiou, G.A. Right nabla discrete fractional calculus. *Int. J. Differ. Equ.* **2011**, *6*, 91–104.
12. Holm, M. Sum and Difference Compositions in Discrete Fractional Calculus. *COBO Math. J.* **2011**,*13*, 153–184. [CrossRef]
13. Yilmazer, R.; Ozturk, O. On Nabla Discrete Fractional Calculus Operator for a Modified Bessel Equation. *Therm. Sci.* **2018**, *22*, S203–S209. [CrossRef]

14. Yilmazer, R.; Inc, M.; Tchier, F.; Baleanu, D. Particular Solutions of the Confluent Hypergeometric Differential Equation by Using the Nabla Fractional Calculus Operator. *Entropy* **2016**, *18*, 49. [CrossRef]
15. Atici , F.M.; Sengül, S. Modeling with fractional difference equations. *J. Math. Anal. Appl.* **2010**, *369*, 1–9. [CrossRef]
16. Mohan, J.J. Solutions of perturbed nonlinear nabla fractional difference equations. *Novi Sad. J. Math.* **2013**, *43*, 125–138.
17. Mohan, J.J. Analysis of nonlinear fractional nabla difference equations. *Int. J. Anal. Appl.* **2015**, *7*, 79–95.
18. Yilmazer, R. *N*-fractional calculus operator N^{μ} method to a modified hydrogen atom equation. *Math. Commun.* **2010**, *15*, 489–501.
19. Granger, C.W.J.; Joyeux, R. An introduction to long-memory time series models and fractional differencing. *J. Time Ser. Anal.* **1980**, *1*, 15–29. [CrossRef]
20. Hosking, J.R.M. Fractional differencing. *Biometrika* **1981**, *68*, 165–176. [CrossRef]
21. Gray, H.L.; Zhang, N. On a New Definition of the Fractional Difference. *Math. Comput.* **1988**, *50*, 513–529. [CrossRef]

© 2018 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).