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Classification Theorems of Ruled Surfaces in Minkowski Three-Space

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Abstract: By generalizing the notion of the pointwise 1-type Gauss map, the generalized 1-type Gauss map has been recently introduced. Without any assumption, we classified all possible ruled surfaces with the generalized 1-type Gauss map in a 3-dimensional Minkowski space. In particular, null scrolls do not have the proper generalized 1-type Gauss map. In fact, it is harmonic.

Keywords: ruled surface; null scroll; Minkowski space; pointwise 1-type Gauss map; generalized 1-type Gauss map; conical surface of *G*-type

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1. Introduction

Thanks to Nash's imbedding theorem, Riemannian manifolds can be regarded as submanifolds of Euclidean space. The notion of finite-type immersion has been used in studying submanifolds of Euclidean space, which was initiated by B.-Y. Chen by generalizing the eigenvalue problem of the immersion [1]. An isometric immersion x of a Riemannian manifold M into a Euclidean space $\mathbb{E}^m$ is said to be of finite-type if it has the spectral decomposition as:

$$x = x_0 + x_1 + \cdots + x_k,$$

where x_0 is a constant vector and $\Delta x_i = \lambda_i x_i$ for some positive integer k and $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. Here, Δ denotes the Laplacian operator defined on M. If $\lambda_1, \ldots, \lambda_k$ are mutually different, M is said to be of k-type. Naturally, we may assume that a finite-type immersion x of a Riemannian manifold into a Euclidean space is of k-type for some positive integer k.

The notion of finite-type immersion of the submanifold into Euclidean space was extended to the study of finite-type immersion or smooth maps defined on submanifolds of a pseudo-Euclidean space $\mathbb{E}^m_s$ with the indefinite metric of index $s \geq 1$. In this sense, it is very natural for geometers to have interest in the finite-type Gauss map of submanifolds of a pseudo-Euclidean space [2–4].

We now focus on surfaces of the Minkowski space $\mathbb{E}^3_1$. Let M be a surface in the 3-dimensional Minkowski space $\mathbb{E}^3_1$ with a non-degenerate induced metric. From now on, a surface M in $\mathbb{E}^3_1$ means non-degenerate, i.e., its induced metric is non-degenerate unless otherwise stated. The map G of a surface M into a semi-Riemannian space form $Q^2(\epsilon)$ by parallel translation of a unit normal vector of M to the origin is called the Gauss map of M, where ϵ $(= \pm 1)$ denotes the sign of the vector field G. A helicoid or a right cone in $\mathbb{E}^3$ has the unique form of Gauss map G, which looks like the 1-type Gauss map in the usual sense [5,6]. However, it is quite different from the 1-type Gauss map, and thus, the authors defined the following definition.

Definition 1. ([7]) *The Gauss map G of a surface M in $\mathbb{E}^3_1$ is of pointwise 1-type if the Gauss map G of M satisfies:*

$$\Delta G = f(G + \mathbf{C})$$

for some non-zero smooth function f and a constant vector **C**. *Especially, the Gauss map G is called pointwise 1-type of the first kind if* **C** *is a zero vector. Otherwise, it is said to be of pointwise 1-type of the second kind.*

Some other surfaces of $\mathbb{E}^3$ such as conical surfaces have an interesting type of Gauss map. A surface in $\mathbb{E}^3_1$ parameterized by:

$$x(s,t) = p + t\beta(s),$$

where p is a point and $\beta(s)$ a unit speed curve is called a conical surface. The typical conical surfaces are a right (circular) cone and a plane.

Example 1. ([8]) Let *M be a surface in $\mathbb{E}^3$ parameterized by:*

$$x(s,t) = (t\cos^2 s, t\sin s\cos s, t\sin s).$$

Then, the Gauss map G can be obtained by:

$$G = \frac{1}{\sqrt{1+\cos^2 s}}(-\sin^3 s, (2-\cos^2 s)\cos s, -\cos^2 s).$$

Its Laplacian turns out to be:

$$\Delta G = fG + g\mathbf{C}$$

for some non-zero smooth functions f, g and a constant vector **C**. *The surface M is a kind of conical surface generated by a spherical curve $\beta(s) = (\cos^2 s, \sin s\cos s, \sin s)$ on the unit sphere $\mathbb{S}^2(1)$ centered at the origin.*

Based on such an example, by generalizing the notion of the pointwise 1-type Gauss map, the so-called generalized 1-type Gauss map was introduced.

Definition 2. ([8]) *The Gauss map G of a surface M in $\mathbb{E}^3_1$ is said to be of generalized 1-type if the Gauss map G satisfies:*

$$\Delta G = fG + g\mathbf{C} \tag{1}$$

for some non-zero smooth functions f, g and a constant vector **C**. *If $f \neq g$, G is said to be of proper generalized 1-type.*

Definition 3. *A conical surface with the generalized 1-type Gauss map is called a conical surface of G-type.*

Remark 1. ([8]) *We can construct a conical surface of G-type with the functions f, g and the vector* **C** *if we solve the differential Equation* (1).

Here, we provide an example of a cylindrical ruled surface in the 3-dimensional Minkowski space $\mathbb{E}^3_1$ with the generalized 1-type Gauss map.

Example 2. *Let M be a ruled surface in the Minkowski 3-space $\mathbb{E}^3_1$ parameterized by:*

$$x(s,t) = \left(\frac{1}{2}\left(s\sqrt{s^2-1} - \ln(s+\sqrt{s^2-1})\right), \frac{1}{2}s^2, t\right), \quad s \geq 1.$$

Then, the Gauss map G is given by:

$$G = (-s, -\sqrt{s^2-1}, 0).$$

By a direct computation, we see that its Laplacian satisfies:

$$\Delta G = \frac{s - \sqrt{s^2 - 1}}{(s^2 - 1)^{\frac{3}{2}}} G + \frac{s(s - \sqrt{s^2 - 1})}{(s^2 - 1)^{\frac{3}{2}}} (1, -1, 0),$$

which indicates that M has the generalized 1-type Gauss map.

2. Preliminaries

Let M be a non-degenerate surface in the Minkowski 3-space $\mathbb{E}^3_1$ with the Lorentz metric $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) denotes the standard coordinate system in $\mathbb{E}^3_1$. From now on, a surface in $\mathbb{E}^3_1$ means non-degenerate unless otherwise stated. A curve in $\mathbb{E}^3_1$ is said to be space-like, time-like, or null if its tangent vector field is space-like, time-like, or null, respectively. Then, the Laplacian Δ is given by:

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial \bar{x}_i} \left(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial \bar{x}_j} \right),$$

where $(g^{ij}) = (g_{ij})^{-1}$, $\mathcal{G}$ is the determinant of the matrix (g_{ij}) consisting of the components of the first fundamental form and $\{\bar{x}_i\}$ are the local coordinate system of M.

A ruled surface M in the Minkowski 3-space $\mathbb{E}^3_1$ is defined as follows: Let I and J be some open intervals in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve in $\mathbb{E}^3_1$ defined on I and $\beta = \beta(s)$ a transversal vector field with $\alpha'(s)$ along α. From now on, $'$ denotes the differentiation with respect to the parameter s unless otherwise stated. The surface M with a parametrization given by:

$$x(s,t) = \alpha(s) + t\beta(s), \quad s \in I, \quad t \in J$$

is called a ruled surface. In this case, the curve $\alpha = \alpha(s)$ is called a base curve and $\beta = \beta(s)$ a director vector field or a ruling. A ruled surface M is said to be cylindrical if β is constant. Otherwise, it is said to be non-cylindrical.

If we consider the causal character of the base and director vector field, we can divide a few different types of ruled surfaces in $\mathbb{E}^3_1$: If the base curve α is space-like or time-like, the director vector field β can be chosen to be orthogonal to α. The ruled surface M is said to be of type M_+ or M_-, respectively, depending on α being space-like or time-like, respectively. Furthermore, the ruled surface of type M_+ can be divided into three types M^1_+, M^2_+, and M^3_+. If β is space-like, it is said to be of type M^1_+ or M^2_+ if β' is non-null or null, respectively. When β is time-like, β' must be space-like because of the character of the causal vectors, which we call M^3_+. On the other hand, when α is time-like, β is always space-like. Accordingly, it is also said to be of type M^1_- or M^2_- if β' is non-null or null, respectively. The ruled surface of type M^1_+ or M^2_+ (resp. M^3_+, M^1_- or M^2_-) is clearly space-like (resp. time-like).

If the base curve α is null, the ruling β along α must be null since M is non-degenerate. Such a ruled surface M is called a null scroll. Other cases, such as α is non-null and β is null, or α is null and β is non-null, are determined to be one of the types $M^1_\pm$, $M^2_\pm$, and M^3_+, or a null scroll by an appropriate change of the base curve [9].

Consider a null scroll: Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{E}^3_1$ with Cartan frame $\{A, B, C\}$, that is A, B, C are vector fields along α in $\mathbb{E}^3_1$ satisfying the following conditions:

$$\langle A, A \rangle = \langle B, B \rangle = 0, \quad \langle A, B \rangle = 1, \quad \langle A, C \rangle = \langle B, C \rangle = 0, \quad \langle C, C \rangle = 1,$$

$$\alpha' = A, \quad C' = -aA - k(s)B,$$

where a is a constant and $k(s)$ a nowhere vanishing function. A null scroll parameterized by $x = x(s,t) = \alpha(s) + tB(s)$ is called a B-scroll, which has constant mean curvature $H = a$ and constant Gaussian curvature $K = a^2$. Furthermore, its Laplacian ΔG of the Gauss map G is given by:

$$\Delta G = -2a^2 G,$$

from which we see that a B-scroll is minimal if and only if it is flat [2,10].

Throughout the paper, all surfaces in $\mathbb{E}^3_1$ are smooth and connected unless otherwise stated.

3. Cylindrical Ruled Surfaces in $\mathbb{E}^3_1$ with the Generalized 1-Type Gauss Map

Let M be a cylindrical ruled surface of type M^1_+, M^1_- or M^3_+ in $\mathbb{E}^3_1$. Then, M is parameterized by a base curve α and a unit constant vector β such that:

$$x(s,t) = \alpha(s) + t\beta$$

satisfying $\langle \alpha', \alpha' \rangle = \varepsilon_1 \ (= \pm 1)$, $\langle \alpha', \beta \rangle = 0$, and $\langle \beta, \beta \rangle = \varepsilon_2 \ (= \pm 1)$.

We now suppose that M has generalized 1-type Gauss map G. Then, the Gauss map G satisfies Condition (1). We put the constant vector $\mathbf{C} = (c_1, c_2, c_3)$ in (1) for some constants c_1, c_2, and c_3.

Suppose that $f = g$. In this case, the Gauss map G is of pointwise 1-type. A classification of cylindrical ruled surfaces with the pointwise 1-type Gauss map in $\mathbb{E}^3_1$ was described in [11].

If M is of type M^1_+, then M is an open part of a Euclidean plane or a cylinder over a curve of infinite-type satisfying:

$$c^2 f^{-\frac{1}{3}} - \ln |c^2 f^{-\frac{1}{3}} + 1| = \pm c^3 (s + k) \tag{2}$$

if $\mathbf{C}$ is null, or

$$\begin{aligned} &\sqrt{\left(c^2 f^{-\frac{1}{3}} + 1\right)^2 + (-c_1^2 + c_2^2)} - \ln \left(c^2 f^{-\frac{1}{3}} + 1 + \sqrt{\left(c^2 f^{-\frac{1}{3}} + 1\right)^2 + (-c_1^2 + c_2^2)} \right) \\ &\quad + \ln \sqrt{|-c_1^2 + c_2^2|} = \pm c^3 (s + k) \end{aligned} \tag{3}$$

if $\mathbf{C}$ is non-null, where c is some non-zero constant and k is a constant.

If M is of type M^1_-, M is an open part of a Minkowski plane or a cylinder over a curve of infinite-type satisfying:

$$c^2 f^{-\frac{1}{3}} + \ln |c^2 f^{-\frac{1}{3}} - 1| = \pm c^3 (s + k) \tag{4}$$

or:

$$\begin{aligned} &\sqrt{\left(c^2 f^{-\frac{1}{3}} - 1\right)^2 - (-c_1^2 + c_2^2)} + \ln \left(c^2 f^{-\frac{1}{3}} - 1 + \sqrt{\left(c^2 f^{-\frac{1}{3}} - 1\right)^2 + |-c_1^2 + c_2^2|} \right) \\ &\quad - \ln \sqrt{|-c_1^2 + c_2^2|} = \pm c^3 (s + k) \end{aligned} \tag{5}$$

depending on the constant vector, $\mathbf{C}$, being null or non-null, respectively, for some non-zero constant c and some constant k.

If M is of type M^3_+, M is an open part of either a Minkowski plane or a cylinder over a curve of infinite-type satisfying:

$$\sqrt{c_2^2 + c_3^2 - \left(c^2 f^{-\frac{1}{3}} - 1\right)^2} - \sin^{-1} \left(\frac{c^2 f^{-\frac{1}{3}} - 1}{\sqrt{c_2^2 + c_3^2}} \right) = \pm c^3 (s + k), \tag{6}$$

where c is a non-zero constant and k a constant.

We now assume that $f \neq g$. Here, we consider two cases.

Case 1. Let M be a cylindrical ruled surface of type M^1_+ or M^1_-, i.e., $\varepsilon_2 = 1$. Without loss of generality, the base curve α can be put as $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$ parameterized by arc length s and the director vector field β as a unit constant vector $\beta = (0,0,1)$. Then, the Gauss map G of M and the Laplacian ΔG of the Gauss map are respectively obtained by:

$$G = (-\alpha_2'(s), -\alpha_1'(s), 0) \quad \text{and} \quad \Delta G = (\varepsilon_1 \alpha_2'''(s), \varepsilon_1 \alpha_1'''(s), 0). \tag{7}$$

With the help of (1) and (7), it immediately follows:

$$\mathbf{C} = (c_1, c_2, 0)$$

for some constants c_1 and c_2. We also have:

$$\begin{aligned} \varepsilon_1 \alpha_2''' &= -f\alpha_2' + gc_1, \\ \varepsilon_1 \alpha_1''' &= -f\alpha_1' + gc_2. \end{aligned} \tag{8}$$

Firstly, we consider the case that M is of type M^1_+. Since α is space-like, we may put:

$$\alpha_1'(s) = \sinh\phi(s) \quad \text{and} \quad \alpha_2'(s) = \cosh\phi(s)$$

for some function $\phi(s)$ of s. Then, (8) can be written in the form:

$$\begin{aligned} (\phi')^2 \cosh\phi + \phi'' \sinh\phi &= -f\cosh\phi + gc_1, \\ (\phi')^2 \sinh\phi + \phi'' \cosh\phi &= -f\sinh\phi + gc_2. \end{aligned}$$

This implies that:

$$(\phi')^2 = -f + g(c_1\cosh\phi - c_2\sinh\phi) \tag{9}$$

and:

$$\phi'' = g(-c_1\sinh\phi + c_2\cosh\phi). \tag{10}$$

In fact, ϕ' is the signed curvature of the base curve $\alpha = \alpha(s)$.

Suppose ϕ is a constant, i.e., $\phi' = 0$. Then, α is part of a straight line. In this case, M is an open part of a Euclidean plane.

Now, we suppose that $\phi' \neq 0$. From (8), we see that the functions f and g depend only on the parameter s, i.e., $f(s,t) = f(s)$ and $g(s,t) = g(s)$. Taking the derivative of Equation (9) and using (10), we get:

$$3\phi'\phi'' = -f' + g'(c_1\cosh\phi - c_2\sinh\phi).$$

With the help of (9), it follows that:

$$\frac{3}{2}\left((\phi')^2\right)' = -f' + \frac{g'}{g}\left((\phi')^2 + f\right).$$

Solving the above differential equation, we have:

$$\phi'(s)^2 = k_1 g^{\frac{2}{3}} + \frac{2}{3} g^{\frac{2}{3}} \int g^{-\frac{2}{3}} f\left(-\frac{f'}{f} + \frac{g'}{g}\right) ds, \quad k_1\ (\neq 0) \in \mathbb{R}. \tag{11}$$

We put:

$$\phi'(s) = \pm\sqrt{p(s)},$$

where $p(s) = |k_1 g^{\frac{2}{3}} + \frac{2}{3} g^{\frac{2}{3}} \int g^{-\frac{2}{3}} f \left(-\frac{f'}{f} + \frac{g'}{g}\right) ds|$. This means that the function ϕ is determined by the functions f, g and a constant vector satisfying (1). Therefore, the cylindrical ruled surface M satisfying (1) is determined by a base curve α such that:

$$\alpha(s) = \left(\int \sinh\phi(s)ds, \int \cosh\phi(s)ds, 0\right)$$

and the director vector field $\beta(s) = (0,0,1)$.

In this case, if f and g are constant, the signed curvature ϕ' of a base curve α is non-zero constant, and the Gauss map G is of the usual 1-type. Hence, M is an open part of a hyperbolic cylinder or a circular cylinder [12].

Suppose that one of the functions f and g is not constant. Then, M is an open part of a cylinder over the base curve of infinite-type satisfying (11). For a curve of finite-type in a plane of $\mathbb{E}^3_1$, see [12] for the details.

Next, we consider the case that M is of type M^1_-. Since α is time-like, we may put:

$$\alpha_1'(s) = \cosh\phi(s) \quad \text{and} \quad \alpha_2'(s) = \sinh\phi(s)$$

for some function $\phi(s)$ of s.

As was given in the previous case of type M^1_+, if the signed curvature ϕ' of the base curve α is zero, M is part of a Minkowski plane.

We now assume that $\phi' \neq 0$. Quite similarly as above, we have:

$$\phi'(s)^2 = k_2 g^{\frac{2}{3}} + \frac{2}{3} g^{\frac{2}{3}} \int g^{-\frac{2}{3}} f \left(\frac{f'}{f} - \frac{g'}{g}\right) ds, \quad k_2\ (\neq 0) \in \mathbb{R}, \tag{12}$$

or, we put:

$$\phi'(s) = \pm\sqrt{q(s)},$$

where $q(s) = |k_2 g^{\frac{2}{3}} + \frac{2}{3} g^{\frac{2}{3}} \int g^{-\frac{2}{3}} f \left(\frac{f'}{f} - \frac{g'}{g}\right) ds|$.

Case 2. Let M be a cylindrical ruled surface of type M^3_+. In this case, without loss of generality, we may choose the base curve α to be $\alpha(s) = (0, \alpha_2(s), \alpha_3(s))$ parameterized by arc length s and the director vector field β as $\beta = (1,0,0)$. Then, the Gauss map G of M and the Laplacian ΔG of the Gauss map are obtained respectively by:

$$G = (0, \alpha_3', -\alpha_2') \quad \text{and} \quad \Delta G = (0, -\alpha_3''', \alpha_2'''). \tag{13}$$

The relationship (13) and the condition (1) imply that the constant vector $\mathbf{C}$ has the form:

$$\mathbf{C} = (0, c_2, c_3)$$

for some constants c_2 and c_3.

If f and g are both constant, the Gauss map is of 1-type in the usual sense, and thus, M is an open part of a circular cylinder [1].

We now assume that the functions f and g are not both constant. Then, with the help of (1) and (13), we get:

$$\begin{aligned} -\alpha_3''' &= f\alpha_3' + g c_2, \\ \alpha_2''' &= -f\alpha_2' + g c_3. \end{aligned} \tag{14}$$

Since α is parameterized by the arc length s, we may put:

$$\alpha_2'(s) = \cos\phi(s) \quad \text{and} \quad \alpha_3'(s) = \sin\phi(s)$$

for some function $\phi(s)$ of s. Hence, (14) can be expressed as:

$$(\phi')^2 \sin\phi - \phi'' \cos\phi = f \sin\phi + gc_2,$$
$$(\phi')^2 \cos\phi + \phi'' \sin\phi = f \cos\phi - gc_3.$$

It follows:

$$(\phi')^2 = f + g(c_2 \sin\phi - c_3 \cos\phi). \tag{15}$$

Thus, M is a cylinder over the base curve α given by:

$$\alpha(s) = \left(0, \int \cos\left(\int \sqrt{r(s)}ds\right) ds, \int \sin\left(\int \sqrt{r(s)}ds\right) ds\right)$$

and the ruling $\beta(s) = (1, 0, 0)$, where $r(s) = |f(s) + g(s)\,(c_2 \sin\phi(s) - c_3 \cos\phi(s))\,|$.

Consequently, we have:

Theorem 1. (Classification of cylindrical ruled surfaces in $\mathbb{E}^3_1$) *Let M be a cylindrical ruled surface with the generalized 1-type Gauss map in the Minkowski 3-space $\mathbb{E}^3_1$. Then, M is an open part of a Euclidean plane, a Minkowski plane, a circular cylinder, a hyperbolic cylinder, or a cylinder over a base curve of infinite-type satisfying* (2)–(6)*,* (11)*,* (12)*, or* (15)*.*

4. Non-Cylindrical Ruled Surfaces with the Generalized 1-Type Gauss Map

In this section, we classify all non-cylindrical ruled surfaces with the generalized 1-type Gauss map in $\mathbb{E}^3_1$.

We start with the case that the surface M is non-cylindrical of type M^1_+, M^3_+, or M^1_-. Then, M is parameterized by, up to a rigid motion,

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \alpha', \beta \rangle = 0$, $\langle \beta, \beta \rangle = \varepsilon_2\ (= \pm 1)$, and $\langle \beta', \beta' \rangle = \varepsilon_3\ (= \pm 1)$. Then, $\{\beta, \beta', \beta \times \beta'\}$ is an orthonormal frame along the base curve α. For later use, we define the smooth functions q, u, Q, and R as follows:

$$q = \|x_s\|^2 = \varepsilon_4 \langle x_s, x_s \rangle, \quad u = \langle \alpha', \beta' \rangle, \quad Q = \langle \alpha', \beta \times \beta' \rangle, \quad R = \langle \beta'', \beta \times \beta' \rangle,$$

where ε_4 is the sign of the coordinate vector field $x_s = \partial x / \partial s$. The vector fields α', β'', $\alpha' \times \beta$, and $\beta \times \beta''$ are represented in terms of the orthonormal frame $\{\beta, \beta', \beta \times \beta'\}$ along the base curve α as:

$$\begin{aligned} \alpha' &= \varepsilon_3 u \beta' - \varepsilon_2 \varepsilon_3 Q \beta \times \beta', \\ \beta'' &= -\varepsilon_2 \varepsilon_3 \beta - \varepsilon_2 \varepsilon_3 R \beta \times \beta', \\ \alpha' \times \beta &= \varepsilon_3 Q \beta' - \varepsilon_3 u \beta \times \beta', \\ \beta \times \beta'' &= -\varepsilon_3 R \beta'. \end{aligned} \tag{16}$$

Therefore, the smooth function q is given by:

$$q = \varepsilon_4(\varepsilon_3 t^2 + 2ut + \varepsilon_3 u^2 - \varepsilon_2 \varepsilon_3 Q^2).$$

Note that t is chosen so that q takes positive values.
Furthermore, the Gauss map G of M is given by:

$$G = q^{-1/2}\left(\varepsilon_3 Q \beta' - (\varepsilon_3 u + t) \beta \times \beta'\right). \tag{17}$$

By using the determinants of the first fundamental form and the second fundamental form, the mean curvature H and the Gaussian curvature K of M are obtained by, respectively,

$$\begin{aligned} H &= \frac{1}{2}\varepsilon_2 q^{-3/2}\left(Rt^2 + (2\varepsilon_3 uR + Q')t + u^2R + \varepsilon_3 uQ' - \varepsilon_3 u'Q - \varepsilon_2 Q^2 R\right), \\ K &= q^{-2}Q^2. \end{aligned} \tag{18}$$

Applying the Gauss and Weingarten formulas, the Laplacian of the Gauss map G of M in $\mathbb{E}^3_1$ is represented by:

$$\Delta G = 2\mathrm{grad}H + \langle G, G\rangle(\mathrm{tr}A_G^2)G, \tag{19}$$

where A_G denotes the shape operator of the surface M in $\mathbb{E}^3_1$ and $\mathrm{grad}H$ is the gradient of H. Using (18), we get:

$$\begin{aligned} 2\mathrm{grad}H &= 2\langle e_1, e_1\rangle e_1(H)e_1 + 2\langle e_2, e_2\rangle e_2(H)e_2 \\ &= 2\varepsilon_4 e_1(H)e_1 + 2\varepsilon_2 e_2(H)e_2 \\ &= q^{-7/2}\{-\varepsilon_2(\varepsilon_3 u + t)A_1\beta' - \varepsilon_4 qB_1\beta + \varepsilon_3 QA_1\beta \times \beta'\}, \end{aligned}$$

where $e_1 = \frac{x_s}{||x_s||}$, $e_2 = \frac{x_t}{||x_t||}$,

$$\begin{aligned} A_1 =& 3(u't + \varepsilon_3 uu' - \varepsilon_2\varepsilon_3 QQ')\{Rt^2 + (2\varepsilon_3 uR + Q')t + u^2R + \varepsilon_3 uQ' - \varepsilon_3 u'Q - \varepsilon_2 Q^2R\} \\ & - (\varepsilon_3 t^2 + 2ut + \varepsilon_3 u^2 - \varepsilon_2\varepsilon_3 Q^2)\{R't^2 + (2\varepsilon_3 u'R + 2\varepsilon_3 uR' + Q'')t + 2uu'R + u^2R' \\ & + \varepsilon_3 uQ'' - \varepsilon_3 u''Q - 2\varepsilon_2 QQ'R - \varepsilon_2 Q^2R'\}, \\ B_1 =& \varepsilon_3 Rt^3 + (3uR + 2\varepsilon_3 Q')t^2 + (3\varepsilon_3 u^2R + 4uQ' - 3u'Q - \varepsilon_2\varepsilon_3 Q^2R)t + u^3R + 2\varepsilon_3 u^2Q' \\ & - \varepsilon_2 uQ^2R - 3\varepsilon_3 uu'Q + \varepsilon_2\varepsilon_3 Q^2Q'. \end{aligned}$$

The straightforward computation gives:

$$\mathrm{tr}A_G^2 = -\varepsilon_2\varepsilon_4 q^{-3}D_1,$$

where:

$$D_1 = -\varepsilon_4(u't + \varepsilon_3 uu' - \varepsilon_2\varepsilon_3 QQ')^2 + \varepsilon_3 q\{(\varepsilon_2 QR + \varepsilon_3 u')^2 - \varepsilon_2(Q' + \varepsilon_3 uR + Rt)^2 - 2\varepsilon_3 Q^2\}.$$

Thus, the Laplacian ΔG of the Gauss map G of M is obtained by:

$$\Delta G = q^{-7/2}[-\varepsilon_4 qB_1\beta + \{-\varepsilon_2(\varepsilon_3 u + t)A_1 + \varepsilon_3 QD_1\}\beta' + \{\varepsilon_3 QA_1 - (\varepsilon_3 u + t)D_1\}\beta \times \beta']. \tag{20}$$

Now, suppose that the Gauss map G of M is of generalized 1-type. Hence, from (1), (17) and (20), we get:

$$\begin{aligned} & q^{-7/2}[-\varepsilon_4 qB_1\beta + \{-\varepsilon_2(\varepsilon_3 u + t)A_1 + \varepsilon_3 QD_1\}\beta' + \{(\varepsilon_3 QA_1 - (\varepsilon_3 u + t)D_1\}\beta \times \beta'] \\ &= fq^{-1/2}\left(\varepsilon_3 Q\beta' - (\varepsilon_3 u + t)\beta \times \beta'\right) + g\mathbf{C}. \end{aligned} \tag{21}$$

If we take the indefinite scalar product to Equation (21) with β, β' and $\beta \times \beta'$, respectively, then we obtain respectively,

$$-\varepsilon_2\varepsilon_4 q^{-5/2}B_1 = g\,\langle \mathbf{C}, \beta\rangle, \tag{22}$$

$$q^{-7/2}\{-\varepsilon_2\varepsilon_3(\varepsilon_3 u + t)A_1 + QD_1\} = fq^{-1/2}Q + g\,\langle \mathbf{C}, \beta'\rangle, \tag{23}$$

$$q^{-7/2}\{-\varepsilon_2 QA_1 + \varepsilon_2\varepsilon_3(\varepsilon_3 u + t)D_1\} = fq^{-1/2}\varepsilon_2\varepsilon_3(\varepsilon_3 u + t) + g\,\langle \mathbf{C}, \beta \times \beta'\rangle. \tag{24}$$

On the other hand, the constant vector $\mathbf{C}$ can be written as;

$$\mathbf{C} = c_1\beta + c_2\beta' + c_3\beta \times \beta',$$

where $c_1 = \varepsilon_2\langle \mathbf{C}, \beta\rangle$, $c_2 = \varepsilon_3\langle \mathbf{C}, \beta'\rangle$, and $c_3 = -\varepsilon_2\varepsilon_3\langle \mathbf{C}, \beta \times \beta'\rangle$. Differentiating the functions c_1, c_2, and c_3 with respect to s, we have:

$$\begin{aligned} &c_1' - \varepsilon_2\varepsilon_3 c_2 = 0, \\ &c_1 + c_2' - \varepsilon_3 R c_3 = 0, \\ &\varepsilon_2\varepsilon_3 R c_2 - c_3' = 0. \end{aligned} \tag{25}$$

Furthermore, Equations (22)–(24) are expressed as follows:

$$-\varepsilon_4 q^{-5/2} B_1 = g c_1, \tag{26}$$

$$q^{-7/2}\{-\varepsilon_2(\varepsilon_3 u + t)A_1 + \varepsilon_3 Q D_1\} = f q^{-1/2}\varepsilon_3 Q + g c_2, \tag{27}$$

$$q^{-7/2}\{-\varepsilon_3 Q A_1 + (\varepsilon_3 u + t) D_1\} = f q^{-1/2}(\varepsilon_3 u + t) - g c_3. \tag{28}$$

Combining Equations (26)–(28), we have:

$$\{-\varepsilon_2(\varepsilon_3 u + t)A_1 + \varepsilon_3 Q D_1\}c_1 + q\varepsilon_4 B_1 c_2 = q^3 f \varepsilon_3 Q c_1, \tag{29}$$

$$\{-\varepsilon_3 Q A_1 + (\varepsilon_3 u + t) D_1\}c_1 - q\varepsilon_4 B_1 c_3 = q^3 f(\varepsilon_3 u + t)c_1. \tag{30}$$

Hence, Equations (29) and (30) yield that:

$$-\varepsilon_2\varepsilon_3 A_1 c_1 + B_1\{c_2(\varepsilon_3 u + t) + \varepsilon_3 Q c_3\} = 0. \tag{31}$$

First of all, we prove:

Theorem 2. *Let M be a non-cylindrical ruled surface of type M_+^1, M_+^3, or M_-^1 parameterized by the base curve α and the director vector field β in $\mathbb{E}_1^3$ with the generalized 1-type Gauss map. If β, β', and β'' are coplanar along α, then M is an open part of a plane, the helicoid of the first kind, the helicoid of the second kind or the helicoid of the third kind.*

Proof. If the constant vector $\mathbf{C}$ is zero, then we can pass this case to that of the pointwise 1-type Gauss map of the first kind. Thus, according to the classification theorem in [4], M is an open part of the helicoid of the first kind, the helicoid of the second kind, or the helicoid of the third kind.

Now, we assume that the constant vector $\mathbf{C}$ is non-zero. If the function Q is identically zero on M, then M is an open part of a plane because of (18).

We now consider the case of the function Q being not identically zero. Consider a non-empty open subset $U = \{s \in \mathrm{dom}(\alpha) | Q(s) \neq 0\}$ of $\mathrm{dom}(\alpha)$. Since β, β', and β'' are coplanar along α, R vanishes. Thus, c_3 is a constant, and $c_1'' = -\varepsilon_2\varepsilon_3 c_1$ from (25). Since the left-hand side of (31) is a polynomial in t with functions of s as the coefficients, all of the coefficients that are functions of s must be zero. From the leading coefficient, we have:

$$\varepsilon_2\varepsilon_3 c_1 Q'' + 2c_2 Q' = 0. \tag{32}$$

Observing the coefficient of the term involving t^2 of (31), with the help of (32), we get:

$$\varepsilon_2\varepsilon_3 c_1(3u'Q' + u''Q) + 3c_2 u'Q - 2c_3 QQ' = 0. \tag{33}$$

Examining the coefficient of the linear term in t of (31) and using (32) and (33), we also get:

$$Q\{c_1\left(\varepsilon_2(u')^2+(Q')^2\right)+\varepsilon_2\varepsilon_3c_2QQ'-\varepsilon_3c_3u'Q\}=0.$$

On U,

$$c_1\left(\varepsilon_2(u')^2+(Q')^2\right)+\varepsilon_2\varepsilon_3c_2QQ'-\varepsilon_3c_3u'Q=0. \tag{34}$$

Similarly, from the constant term with respect to t of (31), we have:

$$\varepsilon_3c_1(-3u'Q'+u''Q)+\varepsilon_2c_3QQ'=0 \tag{35}$$

by using (32)–(34). Combining (33) and (35), we obtain:

$$2\varepsilon_3c_1u'Q'+\varepsilon_2c_2u'Q-\varepsilon_2c_3QQ'=0. \tag{36}$$

Now, suppose that $u'(s)\neq 0$ at some point $s\in U$ and then $u'\neq 0$ on an open interval $U_1\subset U$. Equation (34) yields:

$$\varepsilon_3c_3Q=\frac{1}{u'}\{c_1\left(\varepsilon_2(u')^2+(Q')^2\right)+\varepsilon_2\varepsilon_3c_2QQ'\}. \tag{37}$$

Substituting (37) into (36), we get:

$$\{(u')^2-\varepsilon_2(Q')^2\}(\varepsilon_3c_1Q'+\varepsilon_2c_2Q)=0,$$

or, using $c_2=\varepsilon_2\varepsilon_3c_1'$ in (25),

$$\{(u')^2-\varepsilon_2(Q')^2\}(c_1Q)'=0.$$

Suppose that $\left((u')^2-\varepsilon_2(Q')^2\right)(s_0)\neq 0$ for some $s_0\in U_1$. Then, c_1Q is constant on a component U_2 containing s_0 of U_1.

If $c_1=0$ on U_2, we easily see that $c_2=0$ by (25). Hence, (34) yields that $c_3u'Q=0$, and so, $c_3=0$. Since $\mathbf{C}$ is a constant vector, $\mathbf{C}$ is zero on M. This contradicts our assumption. Thus, $c_1\neq 0$ on U_2. From the equation $c_1''+\varepsilon_2\varepsilon_3c_1=0$, we get:

$$c_1=k_1\cos(s+s_1)\quad\text{or}\quad c_1=k_2\cosh(s+s_2)$$

for some non-zero constants k_i and $s_i\in\mathbb{R}$ $(i=1,\,2)$. Since c_1Q is constant, k_1 and k_2 must be zero. Hence, $c_1=0$, a contradiction. Thus, $(u')^2-\varepsilon_2(Q')^2=0$ on U_1, from which we get $\varepsilon_2=1$ and $u'=\pm Q'$. If $u'\neq -Q'$, then $u'=Q'$ on an open subset U_3 in U_1. Hence, (34) implies that $Q'(2\varepsilon_3c_1Q'+c_2Q-c_3Q)=0$. On U_3, we get $c_3Q=2\varepsilon_3c_1Q'+c_2Q$. Putting it into (35), we have:

$$\varepsilon_3c_1(Q')^2-\varepsilon_3c_1QQ''-c_2QQ'=0. \tag{38}$$

Combining (32) and (38), c_1Q is constant on U_3. Similarly as above, we can derive that $\mathbf{C}$ is zero on M, which is a contradiction. Therefore, we have $u'=-Q'$ on U_1. Similarly, as we just did to the case under the assumption $u'\neq -Q'$, it is also proven that the constant vector $\mathbf{C}$ becomes zero. It is also a contradiction, and so, $U_1=\emptyset$. Thus, $u'=0$ and $Q'=0$. From (18), the mean curvature H vanishes. In this case, the Gauss map G is of pointwise 1-type of the first kind. Hence, the open set U is empty. Therefore, we see that if the director vector field β, β', and β'' are coplanar, the function Q vanishes on M. Hence, M is an open part of a plane because of (18). □

From now on, we assume that R is non-vanishing, i.e., $\beta\wedge\beta'\wedge\beta''\neq 0$ everywhere on M.

If $f=g$, the Gauss map of the non-cylindrical ruled surface of type M^1_+, M^1_- or M^3_+ in $\mathbb{E}^3_1$ is of pointwise 1-type. According to the classification theorem given in [5,13], M is part of a circular cone or a hyperbolic cone.

Now, we suppose that $f \neq g$ and the constant vector $\mathbf{C}$ is non-zero unless otherwise stated. Similarly as before, we develop our argument with (31). The left-hand side of (31) is a polynomial in t with functions of s as the coefficients, and thus, they are zero. From the leading coefficient of the left-hand side of (31), we obtain:

$$\varepsilon_2 c_1 R' + \varepsilon_3 c_2 R = 0. \tag{39}$$

With the help of (25), $c_1 R$ is constant. If we examine the coefficient of the term of t^3 of the left-hand side of (31), we get:

$$c_1(-\varepsilon_2\varepsilon_3 u' R + \varepsilon_2 Q'') + 2c_2\varepsilon_3 Q' + c_3 QR = 0. \tag{40}$$

From the coefficient of the term involving t^2 in (31), using (25) and (40), we also get:

$$c_1(-3\varepsilon_2\varepsilon_3 u'Q' + QQ'R - \varepsilon_2\varepsilon_3 u''Q - Q^2R') - 3c_2 u'Q + 2c_3 QQ' = 0. \tag{41}$$

Furthermore, considering the coefficient of the linear term in t of (31) and making use of Equations (25), (40), and (41), we obtain:

$$Q\{c_1(\varepsilon_2(u')^2 + (Q')^2) + c_2\varepsilon_2\varepsilon_3 QQ' - c_3\varepsilon_3 u'Q\} = 0. \tag{42}$$

Now, we consider the open set $V = \{s \in \mathrm{dom}(\alpha) | Q(s) \neq 0\}$. Suppose $V \neq \emptyset$. From (42),

$$c_1(\varepsilon_2(u')^2 + (Q')^2) + c_2\varepsilon_2\varepsilon_3 QQ' - c_3\varepsilon_3 u'Q = 0. \tag{43}$$

Similarly as above, observing the constant term in t of the left-hand side of (31) with the help of (25) and (39), and using (40), (41) and (43), we have:

$$Q^2(2c_1\varepsilon_3 u'Q' + c_2\varepsilon_2 u'Q - c_3\varepsilon_2 QQ') = 0.$$

Since $Q \neq 0$ on V, one can have:

$$2c_1\varepsilon_3 u'Q' + c_2\varepsilon_2 u'Q - c_3\varepsilon_2 QQ' = 0. \tag{44}$$

Our making use of the first and the second equations in (25), (40) reduces to:

$$c_1\varepsilon_2 u'R - \varepsilon_2\varepsilon_3(c_1 Q)'' - c_1 Q = 0. \tag{45}$$

Suppose that $u'(s) \neq 0$ for some $s \in V$. Then, $u' \neq 0$ on an open subset $V_1 \subset V$. From (43), on V_1:

$$c_3 Q = \frac{1}{u'}\{\varepsilon_2\varepsilon_3 c_1(u')^2 + \varepsilon_3 c_1(Q')^2 + \varepsilon_2 c_2 QQ'\}. \tag{46}$$

Putting (46) into (44), we have $\{(u')^2 - \varepsilon_2(Q')^2\}(\varepsilon_3 c_1 Q' + \varepsilon_2 c_2 Q) = 0$. With the help of $c_1' = \varepsilon_2\varepsilon_3 c_2$, it becomes:

$$\{(u')^2 - \varepsilon_2(Q')^2\}(c_1 Q)' = 0.$$

Suppose that $\left((u')^2 - \varepsilon_2(Q')^2\right)(s) \neq 0$ on V_1. Then, $c_1 Q$ is constant on a component V_2 of V_1. Hence, (45) yields that:

$$c_1 Q = \varepsilon_2 c_1 u' R. \tag{47}$$

If $c_1 \equiv 0$ on V_2, (25) gives that $c_2 = 0$ and $c_3 R = 0$. Since $R \neq 0$, $c_3 = 0$. Hence, the constant vector $\mathbf{C}$ is zero, a contradiction. Therefore, $c_1 \neq 0$ on V_2. From (47), $Q = \varepsilon_2 u' R$. Moreover, u' is a non-zero constant because $c_1 Q$ and $c_1 R$ are constants. Thus, (41) and (44) can be reduced to as follows:

$$c_1 Q'R - c_1 QR' + 2c_3 Q' = 0, \tag{48}$$

$$\varepsilon_3 c_1 u' Q' - \varepsilon_2 c_3 Q Q' = 0. \tag{49}$$

Upon our putting $Q = \varepsilon_2 u' R$ into (48), $c_3 Q' = 0$ is derived. By (49), $c_1 u' Q' = 0$. Hence, $Q' = 0$. It follows that Q and R are non-zero constants on V_2.

On the other hand, since the torsion of the director vector field β viewed as a curve in $\mathbb{E}^3_1$ is zero, β is part of a plane curve. Moreover, β has constant curvature $\sqrt{\varepsilon_2 - \varepsilon_2\varepsilon_3 R^2}$. Hence, β is a circle or a hyperbola on the unit pseudo-sphere or the hyperbolic space of radius 1 in $\mathbb{E}^3_1$. Without loss of generality, we may put:

$$\beta(s) = \frac{1}{p}(R, \cos ps, \sin ps) \quad \text{or} \quad \beta(s) = \frac{1}{p}(\sinh ps, \cosh ps, R),$$

where $p^2 = \varepsilon_2(1 - \varepsilon_3 R^2)$ and $p > 0$. Then, the function $u = \langle \alpha', \beta' \rangle$ is given by:

$$u = -\alpha_2'(s) \sin ps + \alpha_3'(s) \cos ps \quad \text{or} \quad u = -\alpha_1'(s) \cosh ps + \alpha_2'(s) \sinh ps,$$

where $\alpha'(s) = (\alpha_1'(s), \alpha_2'(s), \alpha_3'(s))$. Therefore, we have:

$$u' = -(\alpha_2'' + p\alpha_3') \sin ps - (p\alpha_2' - \alpha_3'') \cos ps \quad \text{or} \quad u' = (-\alpha_1'' + p\alpha_2') \cosh ps - (p\alpha_1' - \alpha_2'') \sinh ps.$$

Since u' is a constant, u' must be zero. It is a contradiction on V_1, and so:

$$(u')^2 = \varepsilon_2 (Q')^2$$

on V_1. It immediately follows that:

$$\varepsilon_2 = 1$$

on V_1. Therefore, we get $u' = \pm Q'$. Suppose $u' \neq -Q'$ on V_1. Then, $u' = Q'$ and (43) can be written as:

$$Q'(2\varepsilon_3 c_1 Q' + c_2 Q - c_3 Q) = 0.$$

Since $Q' \neq 0$ on V,

$$c_3 Q = 2\varepsilon_3 c_1 Q' + c_2 Q. \tag{50}$$

Putting (50) into (40) and (41), respectively, we obtain:

$$\varepsilon_3 c_1 Q' R + c_2 Q R + 2\varepsilon_3 c_2 Q' + c_1 Q'' = 0, \tag{51}$$

$$\varepsilon_3 c_1 (Q')^2 + c_1 Q Q' R - \varepsilon_3 c_1 Q Q'' - c_1 Q^2 R' - c_2 Q Q' = 0. \tag{52}$$

Putting together Equations (51) and (52) with the help of (39), we get:

$$(\varepsilon_3 c_1 Q' + c_2 Q)(Q' + 2\varepsilon_3 Q R) = 0.$$

Suppose $(\varepsilon_3 c_1 Q' + c_2 Q)(s) \neq 0$ on V_1. Then, $Q' = -2\varepsilon_3 Q R$. If we make use of it, we can derive $R(\varepsilon_3 c_1 Q' + c_2 Q) = 0$ from (51). Since R is non-vanishing, $\varepsilon_3 c_1 Q' + c_2 Q = 0$, a contradiction. Thus:

$$\varepsilon_3 c_1 Q' + c_2 Q = 0, \tag{53}$$

that is, $c_1 Q$ is constant on each component of V_1. From (45), $c_1 Q = c_1 u' R$. Similarly as before, it is seen that $c_1 \neq 0$ and u' is a non-zero constant. Hence, $Q = u' R$. If we use the fact that $c_1 Q$ and Q' are constant, $c_2 Q' = 0$ is derived from (51). Therefore, $c_2 = 0$ on each component of V_1. By (53), $c_1 = 0$ on each component of V_1. Hence, (50) implies that $c_3 = 0$ on each component of V_1. The vector $\mathbf{C}$ is constant and thus zero on M, a contradiction. Thus, we obtain $u' = -Q'$ on V_1. Equation (43) with $u' = -Q'$ gives that:

$$c_3 Q = -2\varepsilon_3 c_1 Q' - c_2 Q. \tag{54}$$

Putting (54) together with $u' = -Q'$ into (40), we have:

$$c_1 Q'' = \varepsilon_3 c_1 Q' R + c_2 QR - 2\varepsilon_3 c_2 Q'. \tag{55}$$

Furthermore, Equations (39), (41), (54) and (55) give:

$$(\varepsilon_3 c_1 Q' + c_2 Q)(Q' - 2\varepsilon_3 QR) = 0$$

on V_1. Suppose $\varepsilon_3 c_1 Q' + c_2 Q \neq 0$. Then, $Q' = 2\varepsilon_3 QR$, and thus, $Q'' = 2\varepsilon_3 Q' R + 2\varepsilon_3 QR'$. Putting it into (55) with the help of (39), we get:

$$R(\varepsilon_3 c_1 Q' + c_2 Q) = 0,$$

from which $\varepsilon_3 c_1 Q' + c_2 Q = 0$, a contradiction. Therefore, we get:

$$\varepsilon_3 c_1 Q' + c_2 Q = 0$$

on V_1. Thus, $c_1 Q$ is constant on each component of V_1. Similarly developing the argument as before, we see that the constant vector $\mathbf{C}$ is zero, which contradicts our assumption. Consequently, the open subset V_1 is empty, i.e., the functions u and Q are constant on each component of V. Since $Q = u'R$, Q vanishes on V. Thus, the open subset V is empty, and hence, Q vanishes on M. Thus, (18) shows that the Gaussian curvature K automatically vanishes on M.

Thus, we obtain:

Theorem 3. *Let M be a non-cylindrical ruled surface of type M_+^1, M_+^3, or M_-^1 parameterized by the non-null base curve α and the director vector field β in $\mathbb{E}_1^3$ with the generalized 1-type Gauss map. If β, β', and β'' are not coplanar along α, then M is flat.*

Combining Definition 3, Theorems 2 and 3, and the classification theorem of flat surfaces with the generalized 1-type Gauss map in Minkowski 3-space in [8], we have the following:

Theorem 4. *Let M be a non-cylindrical ruled surface of type M_+^1, M_+^3, or M_-^1 in $\mathbb{E}_1^3$ with the generalized 1-type Gauss map. Then, M is locally part of a plane, the helicoid of the first kind, the helicoid of the second kind, the helicoid of the third kind, a circular cone, a hyperbolic cone, or a conical surface of G-type.*

We now consider the case that the ruled surface M is non-cylindrical of type M_+^2, M_-^2. Then, up to a rigid motion, a parametrization of M is given by:

$$x(s,t) = \alpha(s) + t\beta(s)$$

satisfying $\langle \alpha', \beta \rangle = 0$, $\langle \alpha', \alpha' \rangle = \varepsilon_1 (= \pm 1)$, $\langle \beta, \beta \rangle = 1$, and $\langle \beta', \beta' \rangle = 0$ with $\beta' \neq 0$.

Again, we put the smooth functions q and u as follows:

$$q = \|x_s\|^2 = |\langle x_s, x_s \rangle|, \quad u = \langle \alpha', \beta' \rangle.$$

We see that the null vector fields β' and $\beta \times \beta'$ are orthogonal, and they are parallel. It is easily derived as $\beta' = \beta \times \beta'$. Moreover, we may assume that $\beta(0) = (0,0,1)$ and β can be taken by:

$$\beta(s) = (as, as, 1)$$

for a non-zero constant a. Then, $\{\alpha', \beta, \alpha' \times \beta\}$ forms an orthonormal frame along the base curve α. With respect to this frame, we can put:

$$\beta' = \varepsilon_1 u(\alpha' - \alpha' \times \beta) \quad \text{and} \quad \alpha'' = -u\beta + \frac{u'}{u}\alpha' \times \beta. \tag{56}$$

Note that the function u is non-vanishing.

On the other hand, we can compute the Gauss map G of M such as:

$$G = q^{-1/2}(\alpha' \times \beta - t\beta'). \tag{57}$$

We also easily get the mean curvature H and the Gaussian curvature K of M by the usual procedure, respectively,

$$H = \frac{1}{2}q^{-3/2}\left(u't - \varepsilon_1 \frac{u'}{u}\right) \quad \text{and} \quad K = q^{-2}u^2. \tag{58}$$

Upon our using (19), the Laplacian of the Gauss map G of M is expressed as:

$$\Delta G = q^{-7/2}\left(A_2\alpha' + B_2\beta + D_2\alpha' \times \beta\right) \tag{59}$$

with respect to the orthonormal frame $\{\alpha', \beta, \alpha' \times \beta\}$, where we put:

$$\begin{aligned}
A_2 =& 3\varepsilon_1 \frac{(u')^2}{u}t + \varepsilon_4\varepsilon_1 q\left(-\frac{u''}{u} + \frac{(u')^2}{u^2} + uu''t^2 + \varepsilon_1\frac{(u')^2}{u}t\right) + q\frac{(u')^2}{u}t - 3\varepsilon_1 u(u')^2t^3 \\
&+ \varepsilon_4\varepsilon_1 u(u')^2t^3 + 2\varepsilon_4\varepsilon_1 qu^3 t, \\
B_2 =& \varepsilon_4 qu'(4\varepsilon_1 - ut), \\
D_2 =& 3\varepsilon_1 u(u')^2t^3 - 3(u')^2t^2 - \varepsilon_4 q\left(\varepsilon_1 uu''t^2 - u''t + \frac{(u')^2}{u}t\right) - \varepsilon_1 q\frac{(u')^2}{u^2} - q\frac{(u')^2}{u}t \\
&- \varepsilon_4(u')^2t^2 - 2\varepsilon_4 qu^2 - \varepsilon_4\varepsilon_1 u(u')^2t^3 - 2\varepsilon_4\varepsilon_1 qu^3 t.
\end{aligned}$$

We now suppose that the Gauss map G of M is of generalized 1-type satisfying Condition (1). Then, from (56), (57), and (59), we get:

$$q^{-7/2}\left(A_2\alpha' + B_2\beta + D_2\alpha' \times \beta\right) = fq^{-1/2}\{(1 + \varepsilon_1 ut)\alpha' \times \beta - \varepsilon_1 ut\alpha'\} + g\mathbf{C}. \tag{60}$$

If the constant vector $\mathbf{C}$ is zero, the Gauss map G is nothing but of pointwise 1-type of the first kind. By the result of [4], M is part of the conjugate of Enneper's surface of the second kind.

From now on, for a while, we assume that $\mathbf{C}$ is a non-zero constant vector. Taking the indefinite scalar product to Equation (60) with the orthonormal vector fields α', β, and $\alpha' \times \beta$, respectively, we obtain:

$$\varepsilon_1 q^{-7/2}A_2 = -fq^{-1/2}ut + g\,\langle \mathbf{C}, \alpha'\rangle, \tag{61}$$

$$q^{-7/2}B_2 = g\,\langle \mathbf{C}, \beta\rangle, \tag{62}$$

$$\varepsilon_1 q^{-7/2}D_2 = fq^{-1/2}(\varepsilon_1 + ut) - g\,\langle \mathbf{C}, \alpha' \times \beta\rangle. \tag{63}$$

In terms of the orthonormal frame $\{\alpha', \beta, \alpha' \times \beta\}$, the constant vector $\mathbf{C}$ can be written as:

$$\mathbf{C} = c_1\alpha' + c_2\beta + c_3\alpha' \times \beta,$$

where we have put $c_1 = \varepsilon_1\langle \mathbf{C}, \alpha'\rangle$, $c_2 = \langle \mathbf{C}, \beta\rangle$, and $c_3 = -\varepsilon_1\langle \mathbf{C}, \alpha' \times \beta\rangle$. Then, Equations (61)–(63) are expressed as follows:

$$\varepsilon_1 q^{-7/2}A_2 = -fq^{-1/2}ut + \varepsilon_1 gc_1, \tag{64}$$

$$q^{-7/2}B_2 = g\,c_2, \tag{65}$$

$$\varepsilon_1 q^{-7/2}D_2 = fq^{-1/2}(\varepsilon_1 + ut) + \varepsilon_1 g c_3. \tag{66}$$

Differentiating the functions $c_1,\ c_2$, and c_3 with respect to the parameter s, we get:

$$\begin{aligned} c_1' &= -\varepsilon_1 u c_2 - \frac{u'}{u}c_3, \\ c_2' &= uc_1 + uc_3, \\ c_3' &= -\frac{u'}{u}c_1 + \varepsilon_1 u c_2. \end{aligned} \tag{67}$$

Combining Equations (64)–(66), we obtain:

$$c_2(\varepsilon_1 + ut)A_2 - \{\varepsilon_1 c_1 + (c_1 + c_3)ut\}B_2 + c_2 utD_2 = 0. \tag{68}$$

As before, from (68), we obtain the following:

$$c_2(2uu'' - 3(u')^2) + (c_1 + c_3)u^2u' = 0, \tag{69}$$

$$7c_2(u')^2 - 5c_1u^2u' - 7c_3u^2u' = 0, \tag{70}$$

$$c_2(7(u')^2 - 3uu'') - 11c_1u^2u' - 4c_3u^2u' = 0, \tag{71}$$

$$c_2(uu'' - (u')^2) + 4c_1u^2u' = 0. \tag{72}$$

Combining Equations (69) and (71), we get:

$$5c_2(uu'' - (u')^2) - 7c_1u^2u' = 0. \tag{73}$$

From (72) and (73), we get $c_1u' = 0$. Hence, Equations (70) and (72) become:

$$u'(c_2u' - c_3u^2) = 0, \tag{74}$$

$$c_2(uu'' - (u')^2) = 0. \tag{75}$$

Now, suppose that $u'(s_0) \neq 0$ at some point $s_0 \in \mathrm{dom}(\alpha)$. Then, there exists an open interval J such that $u' \neq 0$ on J. Then, $c_1 = 0$ on J. Hence, (67) reduces to:

$$\begin{aligned} \varepsilon_1 u^2 c_2 + u'c_3 &= 0, \\ c_2' &= uc_3, \\ c_3' &= \varepsilon_1 u c_2. \end{aligned} \tag{76}$$

From the above relationships, we see that c_2' is constant on J. In this case, if $c_2 = 0$, then $c_3 = 0$. Hence, $\mathbf{C}$ is zero on J. Thus, the constant vector $\mathbf{C}$ is zero on M. This contradicts our assumption. Therefore, c_2 is non-zero. Solving the differential Equation (74) with the help of $c_2' = uc_3$ in (76), we get $u = kc_2$ for some non-zero constant k. Moreover, since c_2' is constant, $u'' = 0$. Thus, Equation (75) implies that $u' = 0$, which is a contradiction. Therefore, there does not exist such a point $s_0 \in \mathrm{dom}(\alpha)$ such that $u'(s_0) \neq 0$. Hence, u is constant on M. With the help of (58), the mean curvature H of M vanishes on M. It is easily seen from (19) that the Gauss map G of M is of pointwise 1-type of the first kind, which means (1) is satisfied with $\mathbf{C} = 0$. Thus, this case does not occur.

As a consequence, we give the following classification:

Theorem 5. *Let M be a non-cylindrical ruled surface of type M^2_+ or M^2_- in $\mathbb{E}^3_1$ with the generalized 1-type Gauss map G. Then, the Gauss map G is of pointwise 1-type of the first kind and M is an open part of the conjugate of Enneper's surface of the second kind.*

Remark 2. *There do not exist non-cylindrical ruled surfaces of type M^2_+ or M^2_- in $\mathbb{E}^3_1$ with the proper generalized 1-type Gauss map G.*

5. Null Scrolls in the Minkowski 3-Space $\mathbb{E}^3_1$

In this section, we examine the null scrolls with the generalized 1-type Gauss map in the Minkowski 3-space $\mathbb{E}^3_1$. In particular, we focus on proving the following theorem.

Theorem 6. *Let M be a null scroll in the Minkowski 3-space $\mathbb{E}^3_1$. Then, M has generalized 1-type Gauss map G if and only if M is part of a Minkowski plane or a B-scroll.*

Proof. Suppose that a null scroll M has the generalized 1-type Gauss map. Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{E}^3_1$ and $\beta = \beta(s)$ a null vector field along α such that $\langle \alpha', \beta \rangle = 1$. Then, the null scroll M is parameterized by:

$$x(s,t) = \alpha(s) + t\beta(s)$$

and we have the natural coordinate frame $\{x_s, x_t\}$ given by:

$$x_s = \alpha' + t\beta' \quad \text{and} \quad x_t = \beta.$$

We put the smooth functions u, v, Q, and R by:

$$u = \langle \alpha', \beta' \rangle, \quad v = \langle \beta', \beta' \rangle, \quad Q = \langle \alpha', \beta' \times \beta \rangle, \quad R = \langle \alpha', \beta'' \times \beta \rangle. \tag{77}$$

Then, $\{\alpha', \beta, \alpha' \times \beta\}$ is a pseudo-orthonormal frame along α.
Straightforward computation gives the Gauss map G of M and the Laplacian ΔG of G by:

$$G = \alpha' \times \beta + t\beta' \times \beta \quad \text{and} \quad \Delta G = -2\beta'' \times \beta + 2(u + tv)\beta' \times \beta.$$

With respect to the pseudo-orthonormal frame $\{\alpha', \beta, \alpha' \times \beta\}$, the vector fields β', $\beta' \times \beta$, and $\beta'' \times \beta$ are represented as:

$$\beta' = u\beta - Q\alpha' \times \beta, \quad \beta' \times \beta = Q\beta \quad \text{and} \quad \beta'' \times \beta = R\beta - v\alpha' \times \beta. \tag{78}$$

Thus, the Gauss map G and its Laplacian ΔG are expressed by:

$$G = \alpha' \times \beta + tQ\beta \quad \text{and} \quad \Delta G = -2(R - uQ - tvQ)\beta + 2v\alpha' \times \beta. \tag{79}$$

Since M has the generalized 1-type Gauss map, the Gauss map G satisfies:

$$\Delta G = fG + g\mathbf{C} \tag{80}$$

for some non-zero smooth functions f, g and a constant vector $\mathbf{C}$. From (79), we get:

$$-2(R - uQ - tvQ)\beta + 2v\alpha' \times \beta = f(\alpha' \times \beta + tQ\beta) + g\mathbf{C}. \tag{81}$$

If the constant vector $\mathbf{C}$ is zero, M is an open part of a Minkowski plane or a B-scroll according to the classification theorem in [4].

We now consider the case that the constant vector **C** is non-zero. If we take the indefinite inner product to Equation (81) with α', β, and $\alpha' \times \beta$, respectively, we get:

$$-2(R - uQ - tvQ) = ftQ + gc_2, \quad gc_1 = 0, \quad 2v = f + gc_3, \tag{82}$$

where we have put

$$c_1 = \langle \mathbf{C}, \beta \rangle, \; c_2 = \langle \mathbf{C}, \alpha' \rangle \quad \text{and} \quad c_3 = \langle \mathbf{C}, \alpha' \times \beta \rangle.$$

Since $g \neq 0$, Equation (82) gives $\langle \mathbf{C}, \beta' \rangle = 0$. Together with (78), we see that $c_3 Q = 0$. Suppose that $Q(s) \neq 0$ on an open interval $\tilde{I} \subset \mathrm{dom}(\alpha)$. Then, $c_3 = 0$ on $\tilde{I}$. Therefore, the constant vector **C** can be written as $\mathbf{C} = c_2\beta$ on $\tilde{I}$. If we differentiate $\mathbf{C} = c_2\beta$ with respect to s, $c_2'\beta + c_2\beta' = 0$, and thus, $c_2 v = 0$. On the other hand, from (77) and (78), we have $v = Q^2$. Hence, v is non-zero on $\tilde{I}$, and so, $c_2 = 0$. It contradicts that **C** is a non-zero vector. In the sequel, Q vanishes identically. Then, $\beta' = u\beta$, which implies $R = 0$. Thus, the Gauss map G is reduced to $G = \alpha' \times \beta$, which depends only on the parameter s, from which the shape operator S of M is easily derived as:

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad S = \begin{pmatrix} 0 & 0 \\ k(s) & 0 \end{pmatrix}$$

for some non-vanishing function k. Therefore, the null scroll M is part of a Minkowski plane or a flat B-scroll described in Section 2 determined by $A = \alpha'$, $B = \beta$, $C = G$ satisfying $C' = -k(s)B$. Thus, null scrolls in $\mathbb{E}^3_1$ with the generalized 1-type Gauss map satisfying (80) are part of Minkowski planes or B-scrolls whether **C** is zero or not.

The converse is obvious. This completes the proof. □

Corollary 1. *There do not exist null scrolls in $\mathbb{E}^3_1$ with the proper generalized 1-type Gauss map.*

Open problem: Classify ruled submanifolds with the generalized 1-type Gauss map in Minkowski space.

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