A Refinement of Schwarz–Pick Lemma for Higher Derivatives

Ern Gun Kwon 1, Jinkee Lee 2,*, Gun Kwon 3 and Mi Hui Kim 4

1 Department of Mathematics Education, Andong National University, Andong 36729, Korea; egkwon@anu.ac.kr
2 Department of Mathematics, Pusan National University, Busan 46241, Korea
3 Department of Mechanical Engineering, Graduate School, Yeungnam University, Gyeongsan 38541, Korea; gunbbang80kg@gmail.com
4 Department of Mathematics, Graduate School, Andong National University, Andong 36729, Korea; mihezzang@naver.com
* Correspondence: jklee235@pusan.ac.kr

Received: 23 November 2018; Accepted: 10 January 2019; Published: 13 January 2019

Abstract: In this paper, a Schwarz–Pick estimate of a holomorphic self map $f$ of the unit disc $D$ having the expansion $f(w) = c_0 + c_n(w - z)^n + ...$ in a neighborhood of some $z$ in $D$ is given. This result is a refinement of the Schwarz–Pick lemma, which improves a previous result of Shinji Yamashita.

Keywords: Schwarz Lemma; maximum principle; Littlewood inequality

MSC: 30C80

1. Introduction

For the open unit disc $D$ of the complex plane and the boundary $\partial D$ of $D$, the following Schwarz–Pick lemma (see [1], Lemma 1.2) is well-known.

**Theorem 1.** Let $f : D \rightarrow D$ be holomorphic and $z_0 \in D$. Then,

$$\left|\frac{f(z) - f(z_0)}{1 - f(z_0)f(z)}\right| \leq \left|\frac{z - z_0}{1 - z_0^2}\right|, \quad z \in D,$$

and

$$\frac{|f'(z_0)|}{1 - |f(z_0)|^2} \leq \frac{1}{1 - |z_0|^2}. \quad (2)$$

Equality in (1) holds at some point $z \neq z_0$ or equality in (2) holds if and only if

$$f(z) = c\frac{z - a}{1 - az}, \quad z \in D \quad (3)$$

for some $c \in \partial D$ and $a \in D$.

Among those interesting extensions of (2), there is a result of Shinji Yamashita (see [2], Theorem 1):

**Theorem 2.** Let $f$ be a function holomorphic and bounded, $|f| < 1$, in $D$, and let $z \in D$. Suppose that

$$f(w) = c_0 + c_n(w - z)^n + c_{n+1}(w - z)^{n+1} + ...$$
in a neighborhood of \( z \), where \( n \geq 1 \) depends on \( z \) and \( c_n = 0 \) is possible. Then,

\[
\frac{(1 - |z|^2)^n |f^{(n)}(z)|}{n!(1 - |f(z)|^2)} \leq 1. \tag{4}
\]

The inequality (4) is sharp in the sense that equality holds for the function

\[
f(w) = e^{ia} \left( \frac{w - z}{1 - \bar{z}w} \right)^n \quad (a \; \text{a real constant})
\]
of \( w \).

For \( f \) holomorphic in \( D \), \( 0 \leq r < 1 \), and \( 0 \leq p \leq \infty \), as it is commonly used we denote \( M_p(r, f) \) by the \( p \)-mean of \( f \) on \( \partial D \), that is,

\[
M_p(r, f) = \begin{cases} 
\exp \left( \int_{-\pi}^{\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \right) & \text{if } p = 0, \\
\left( \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} & \text{if } 0 < p < \infty, \\
\sup |f(z)| & \text{if } p = \infty.
\end{cases}
\]

If \( f \) is holomorphic, then \( M_p(r, f) \) is an increasing function of \( p : 0 \leq p \leq \infty \) as well as an increasing function of \( r : 0 \leq r < 1 \) (see [3]).

For \( a \in D \), let \( \phi_a \) be defined by

\[
\phi_a(z) = \frac{z + a}{1 + \bar{a}z}, \quad z \in D.
\]

\( \phi_a \) satisfies \( \phi_a(\phi_{-a}(z)) = z \) for all \( z \in D \). It is well-known that \( \phi_a(\partial D) = \partial D \) and that the set of automorphisms, i.e., bijective biholomorphic mappings, of \( D \) consists of the mappings of the form \( \alpha \phi_a(z) \), where \( a \in D \) and \( |\alpha| = 1 \).

Extending (2) in terms of \( M_p(r, f) \), there is another result of Shinji Yamashita(see [4], Theorem 2):

**Theorem 3.** Let \( f \) be a function holomorphic and bounded, \( |f| < 1 \), in \( D \) and let \( 0 \leq p \leq \infty \). Then

\[
(1 - |w|^2) |f'(w)| \leq 1 \quad \text{for all } w \in D \text{ and } 0 < r < 1,
\]

where

\[
f_w(z) = \frac{f(\frac{z + w}{1 + \bar{z}w}) - f(w)}{1 - f(w)f(\frac{z + w}{1 + \bar{z}w})}, \quad z \in D.
\]

If the equality \( r^{-1} M_p(r, f_w) = 1 \) holds in (5) for \( w \in D \) and \( 0 < r < 1 \), then \( f \) is of the form (3).

Note that \( n = 1 \) in (4) reduces to (2) and that (5) refines (2). As the same manner, it is expected that there might be a refinement of Theorem 2 which reduces to Theorem 3 when \( n = 1 \). This is our objective of this note.

**2. Result**

The following is our corresponding result:
Theorem 4. Let $f$ be a function holomorphic and bounded, $|f| < 1$, in $D$ and let $z \in D$. If

$$f(w) = c_0 + c_n(w-z)^n + c_{n+1}(w-z)^{n+1} + \ldots.$$  \hfill (6)

in a neighborhood of $z$, then

$$\left| \frac{f(w) - f(z)}{1 - f(z)f(w)} \right| \leq \left| \frac{w-z}{1 - zw} \right|^n, \quad w \in D,$$  \hfill (7)

and

$$\frac{(1 - |z|^2)^n|f^{(n)}(z)|}{n!(1 - |f(z)|^2)} \leq \frac{1}{p^2} M_p(r, f_z) \leq 1$$  \hfill (8)

for all $r : 0 < r < 1$, and for all $p : 0 < p \leq \infty$, where

$$f_z(w) = \frac{f(z) - f(w)}{1 - f(z)f(w)}, \quad w \in D.$$  \hfill (9)

Equality in (7) holds at some point $w \in D$, $w \neq z$ if and only if

$$f(w) = \frac{\alpha (\varphi_{-z}(w))^n + c_0}{1 + \alpha c_0 (\varphi_{-z}(w))^n}, \quad w \in D$$  \hfill (10)

with $|\alpha| = 1$.

Equality in the first inequality or in the second inequality of (8) holds for some $r : 0 < r < 1$ and $p : 0 < p \leq \infty$ if and only if $f$ is of the form (10) (with $|\alpha| \leq 1$ or $|\alpha| = 1$, respectively).

Remark 1. (1) The case $n = 1$ of Theorem 4 should reduce to Theorem 1. Comparing (3) and (10), there should exist $z' \in D$ and $\beta : |\beta| = 1$ for which

$$\frac{\alpha \varphi_{-z}(w) + c_0}{1 + \alpha c_0 \varphi_{-z}(w)} = \beta \frac{w - z'}{1 - z'w}$$  \hfill (11)

for all $w \in D$. This can be verified as follows:

Since any automorphism, i.e., bijective holomorphic mapping, of $D$ is of the form of the right-hand side of (11), it suffices to show that the left-hand side of (11), denote $\Phi(w)$, is an automorphism of $D$. That $\Phi(w)$ is holomorphic and into $D$ is obvious. We show $\Phi(w)$ is bijective: If $\Phi(w_1) = \Phi(w_2)$, then $\varphi_{-z}(w_1) = \varphi_{-z}(w_2)$, and the injectivity of $\varphi_{-z}$ shows $w_1 = w_2$. Thus, $\Phi(w)$ is injective. Next, for any $\zeta \in D$, by the surjectivity of $\varphi_{c_0}$, there exists $\eta \in D$ such that

$$\frac{\alpha \eta + c_0}{1 + \alpha c_0 \eta} = \zeta.$$  

For this $\eta$, there is $\xi \in D$ such that $\eta = \varphi_{-z}(\xi)$, whence $\Phi(w)$ is surjective.

(2) Fix $z \in D$ and self-map $f$ of $D$. Then, applying Littlewood’s inequality (see [3,5,6]), it follows that

$$\left| \frac{f(w) - f(z)}{1 - f(z)f(w)} \right| \leq \prod_{j} \left| f \left( \frac{w + z_j}{1 + z_jw} \right) - f(z) \right| = \prod_{f(z_j) = f(z)} \left| \frac{w - z_j}{1 - z_jw} \right|,$$  \hfill (12)

with equality holding only if $f$ is an inner function. Equation (7) follows directly from (12).

In addition, the inequality

$$\frac{(1 - |z|^2)^n|f^{(n)}(z)|}{n!(1 - |f(z)|^2)} \leq 1$$
of (8) can be obtained as a one stroke limit from (7):
\[
1 \geq \left| \frac{f(w) - f(z)}{1 - f(z)f(w)} \right| |w - z|^n = \left| \frac{f(w) - f(z)}{(w - z)^n} \right| \left| \frac{1 - \bar{z}w}{f(z)f(w)} \right| \\
\rightarrow \frac{|f^{(n)}(z)(1 - |z|^2)^n}{n!(1 - |f(z)|^2)}
\]
as \(w \to z\) (by applying L’Hospital’s rule).

The point of Theorem 4 lies in its connection with \(M_\nu(r, \cdot)\) and in clarifying the condition of equality to make Yamashita type theorem complete.

After proving Theorem 4 in Section 3, applications of Theorem 4 to some coefficient problems will be given in Section 4.

3. Proof of Theorem 4

We may assume \(c_n \neq 0\). (7) can be expressed as
\[
|\varphi - f(z)| \leq |\varphi - z(w)|^n, \quad w \in D.
\]

By (6), \(f(w) - f(z)\) has a zero of order \(n\) at \(w = z\) so that
\[
\frac{\varphi - f(z) \circ f(w)}{\varphi - z(w)^n}, \quad w \in D
\]
is holomorphic in \(D\) whose modulus at \(w \in \partial D\) is not greater than 1, so that the maximum principle gives (7).

Next, to verify inequality (8), take \(\delta > 0\) such that (6) holds for \(w\):
\[
|w - z| < \delta.
\]

Then, by (6),
\[
f \circ \varphi_z(w) - f(z) = c_n(\varphi_z(w) - z)^n + \ldots
\]
\[
= c_n \left( \frac{w}{1 + \bar{z}w} \right)^n (1 - |z|^2)^n + O(w^{n+1})
\]
for \(w : |w| < \frac{\delta}{1 + |z|}\). This is because
\[
|w| < \frac{\delta}{1 + |z|} \quad \Rightarrow \quad |w| < \frac{\delta(1 + \bar{z}w)}{1 - |z|^2} \quad \Rightarrow \quad |\varphi_z(w) - z| < \delta.
\]

Thus, \(f_z(w)\) defined by (9) has a zero of order \(n\) at \(w = 0\). Hence,
\[
h(w) = \frac{1}{w^n}f_z(w), \quad w \in D,
\]
is holomorphic in \(D\). Since \(h(0) \neq 0\) in a neighborhood of 0, \(\log |h|\) is harmonic in the neighborhood, hence there exists \(r_0\) such that
\[
|h(0)| = \exp \left( \int_{-\pi}^{\pi} \log |h(re^{i\theta})| \frac{d\theta}{2\pi} \right)
\]
for \(r : r < r_0\).

On the other hand, by (15),
\[
nh(0) = \frac{d^n}{dw^n}(w^n h(w)) \bigg|_{w=0} = \frac{d^n}{dw^n} f_z(w) \bigg|_{w=0},
\]
In order to calculate the final term of (17), let’s put \( F(w) = f \circ \varphi_z(w) - f(z) \) and \( G(w) = 1 - f(z) f \circ \varphi_z(w) \). Then,

\[
\frac{d^n}{dw^n}f_z(w)\bigg|_{w=0} = \sum_{j=0}^{n} \binom{n}{j} F^{(j)}(0) (G^{-1})^{(n-j)}(0).
\]

By (14),

\[
F^{(j)}(0) = \begin{cases} 0, & \text{if } j < n, \\ c_n n! (1 - |z|^2)^n, & \text{if } j = n, \end{cases}
\]

so that

\[
\frac{d^n}{dw^n}f_z(w)\bigg|_{w=0} = \frac{F(n)(0)}{G(0)} = \frac{c_n n! (1 - |z|^2)^n}{1 - |f(z)|^2},
\]

whence

\[
h(0) = \frac{c_n (1 - |z|^2)^n}{1 - |f(z)|^2}.
\]  

(18)

Noting from (6) that \( c_n = \frac{\ell^{(n)}}{n!} \), we have, by (15), (16) and (18),

\[
\frac{(1 - |z|^2)^n |f^{(n)}(z)|}{n!(1 - |f(z)|^2)} = \exp \left( \int_{-\pi}^{\pi} \log |z| \frac{1}{r^n} f_z(re^{i\theta}) \frac{d\theta}{2\pi} \right) = \frac{1}{r^n} M_0(r, f_z)
\]

(19)

for \( r < r_0 \).

Now, the first inequality of (8) follows from the fact that \( M_p(r, h) \) is an increasing function of \( p \) for \( 0 < p \leq \infty \) and also an increasing function of \( r \) for \( 0 < r < 1 \).

In addition, since \( M_p(r, h) \leq M_\infty(r, h) \) and \( |h| < 1 \) by the maximum principle, the second inequality of (8) follows.

We next check the conditions of equality. Elementary calculation shows that

\[
\frac{\varphi_z f_z(w)}{\varphi_z(w)^n} = \alpha \iff f(w) = \frac{\alpha (\varphi_z(w))^n + c_0}{1 + \alpha c_0 (\varphi_z(w))^n} \iff f_z(w) = \alpha w^n.
\]

(20)

Thus, if equality in (7) holds, at some point \( w \in D \), \( w \neq z \); then, (13) is a constant function of modulus 1 by virtue of the maximum principle, which gives (10) with \( |a| = 1 \) by (20). To see that \( f(w) \) of (10) with \( |a| = 1 \) gives the equality in (7) is straightforward also by (20).

If, for some \( p, 0 < p \leq \infty \), and for some \( r : 0 < r < 1 \) the first inequality of (8) becomes equality, then, by (19), \( M_0(r, h) = M_q(r, h) \) for \( 0 \leq q \leq p \), so that \( |h(r\zeta)|=\text{constant} \), a.e. \( \zeta \in \partial D \). Since \( h \) is holomorphic and \( |h(0)| = |h(r\zeta)|=\text{constant} \), a.e. \( \zeta \in \partial D \) for \( \rho \leq r \), it follows that \( h \) is a constant function. Letting \( h = \alpha \) with \( |\alpha| \leq 1 \) and solving this, as in (20), gives (10).

Finally, suppose the second inequality of (8) becomes equal for some \( \rho_0 : 0 < \rho_0 < 1 \) so that \( M_p(\rho_0, h) = \frac{1}{\rho} M_p(\rho_0, f_z) = 1 \). Then, \( M_p(\rho, h) = 1 \) for \( \rho : \rho_0 \leq \rho < 1 \). Since \( \log M_p(\rho, h) \) is a convex function of \( \rho \) (see [3]) and \( \log M_p(\rho, h) = 0 \) for \( \rho_0 \leq \rho < 1 \), it follows that \( \log M_p(\rho, h) \geq 0 \) for \( \rho \leq \rho_0 \) and \( \log M_p(\rho, h) = 0 \) for all \( \rho : 0 < \rho < \rho_0 \). Thus,

\[
h(0) = \lim_{\rho \to 0} M_p(\rho, f) = \lim_{\rho \to 0} e^{\log M_p(\rho, h)} = 1.
\]

Since \( h \) maps \( D \) into \( D \), this forces, by the maximum principle, that \( h(w) \) is a constant, \( h(w) = \alpha \), with \( |\alpha| = 1 \). Hence, (20) gives (10).

Conversely, by (20), \( f(w) \) of (10) with \( |\alpha| \leq 1 \) makes \( h \) in (15) constant, so that the two inequalities in (8) become equalities.
4. Applications

Theorem 4 immediately gives the following estimate:

**Corollary 1.** Let $f$ be a function holomorphic and bounded, $|f| < 1$, in $D$ and let $z \in D$. If

$$f(w) = c_0 + c_n(w - z)^n + c_{n+1}(w - z)^{n+1} + \ldots.$$ 

in a neighborhood of $z$, then

$$|c_n| \leq \frac{1 - |c_0|^2}{r^n(1 - |z|^2)^n} M_p(r, f_z) \leq \frac{1 - |c_0|^2}{(1 - |z|^2)^n} \quad (21)$$

for all $r : 0 < r < 1$, and for all $p : 0 \leq p \leq \infty$, where $f_z$ is defined by (9).

Equality in the first inequality or in the second inequality of (21) holds for some $r : 0 < r < 1$ and $p : 0 < p \leq \infty$ if and only if $f$ is of the form (10) (with $|\alpha| \leq 1$ or $|\alpha| = 1$, respectively).

For the case $z = 0$, we can also obtain

**Corollary 2.** Let $f$ be a function holomorphic and bounded, $|f| < 1$, in $D$ and let $z \in D$. If

$$f(w) = c_0 + c_1 w + c_2 w^2 + \ldots + c_n w^n + \ldots,$$ 

then

$$|c_n| \leq (1 - |c_0|^2) \frac{M_p(r, g_0)}{r^n} \leq 1 - |c_0|^2 \quad (22)$$

for all $r : 0 < r < 1$, and for all $p : 0 \leq p \leq \infty$, where

$$g_0(w) = \frac{g(w) - g(0)}{1 - g(0)g(w)}, \quad w \in D$$

with

$$g(w) = \frac{1}{n} \sum_{k=1}^{n} f(e^{2\pi k/n}w), \quad w \in D.$$

Equality in the first inequality or in the second inequality of (22) holds for some $r : 0 < r < 1$ and $p : 0 < p \leq \infty$ if and only if $g$ is of the form

$$g(w) = \frac{\alpha w^n + c_0}{1 + \alpha c_0 w^n}, \quad w \in D \quad (23)$$

(with $|\alpha| \leq 1$ or $|\alpha| = 1$, respectively).

**Proof of Corollary 2.** As was frequently used (see [7] for example), we make use of the facts that

$$\sum_{k=1}^{n} e^{2\pi j k/n} = n \text{ if } j \text{ is a multiple of } n, \text{ and } 0 \text{ if } j \text{ is otherwise. Noting that}$$

$$g(w) = \frac{1}{n} \sum_{k=1}^{n} f(e^{2\pi k/n}w) = c_0 + c_n w^n + c_{2n} w^{2n} + \ldots, \quad w \in D$$

is holomorphic and $|g| < 1$ in $D$, by Corollary 1 with $z = 0$, we have

$$|c_n| \leq \frac{1 - |c_0|^2}{r^n} M_p(r, g_0) \leq 1 - |c_0|^2 \quad (24)$$

for all $r : 0 < r < 1$, and for all $p : 0 \leq p \leq \infty$. 


In addition, by Corollary 1, equality in the first inequality or in the second inequality of (22) holds for some $r : 0 < r < 1$ and $p : 0 < p \leq \infty$ if and only if $g$ is of the form (23) (with $|\alpha| \leq 1$ or $|\alpha| = 1$, respectively). □

5. Conclusions

With imperative applications to particular situations, various forms of Schwarz Lemma have been called for. In this paper, we presented Schwarz-Pick Lemma for higher derivatives in connection with $p$-mean $M_p(r, f)$ (see Theorem 4). It refined a previous result of Shinji Yamashita and clarified the condition of equality. As an immediate consequence, the result could be applied to refine well-known estimates for $n$-th Taylor coefficient of holomorphic self maps of $D$ (see Corollary 1 and 2). We are expecting its further extensions and applications.

Author Contributions: Supervision, E.G.K.; funding acquisition, J.L. All authors contributed to each section. All the authors read and approved the final manuscript.

Funding: This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2017R1E1A1A03070738).

Acknowledgments: The authors would like to thank the anonymous reviewers for their helpful comments.

Conflicts of Interest: The authors declare no conflict of interest.

References


© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).