Some New Applications of Weakly $\mathcal{H}$-Embedded Subgroups of Finite Groups

Li Zhang 1, Li-Jun Huo 2 and Jia-Bao Liu 1,*

1 School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China; zhang12@mail.ustc.edu.cn
2 School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, China; huoj@cqut.edu.cn
* Correspondence: liujiabaoad@163.com

Received: 14 January 2019; Accepted: 3 February 2019; Published: 10 February 2019

Abstract: A subgroup $H$ of a finite group $G$ is said to be weakly $\mathcal{H}$-embedded in $G$ if there exists a normal subgroup $T$ of $G$ such that $H^G = HT$ and $H \cap T \in \mathcal{H}(G)$, where $H^G$ is the normal closure of $H$ in $G$, and $\mathcal{H}(G)$ is the set of all $\mathcal{H}$-subgroups of $G$. In the recent research, Asaad, Ramadan and Heliel gave new characterization of $p$-nilpotent: Let $p$ be the smallest prime dividing $|G|$, and $P$ a non-cyclic Sylow $p$-subgroup of $G$. Then $G$ is $p$-nilpotent if and only if there exists a $p$-power $d$ with $1 < d < |P|$ such that all subgroups of $P$ of order $d$ and $pd$ are weakly $\mathcal{H}$-embedded in $G$. As new applications of weakly $\mathcal{H}$-embedded subgroups, in this paper, (1) we generalize this result for general prime $p$ and get a new criterion for $p$-supersolubility; (2) adding the condition “$N_G(P)$ is $p$-nilpotent”, here $N_G(P) = \{g \in G | P^g = P\}$ is the normalizer of $P$ in $G$, we obtain $p$-nilpotence for general prime $p$. Moreover, our tool is the weakly $\mathcal{H}$-embedded subgroup. However, instead of the normality of $H^G = HT$, we just need $HT$ is $S$-quasinormal in $G$, which means that $HT$ permutes with every Sylow subgroup of $G$.

Keywords: finite groups; weakly $\mathcal{H}$-embedded subgroups; $p$-supersolubility; $p$-nilpotence

1. Introduction

Throughout this paper, “$G$ is a group” always means that “$G$ is a finite group”. For convenience, one can refer to [1–4] for the definitions and notions in the paper.

The $T$-groups are defined as the groups $G$ in which normality is a transitive relation, that is, if $H \lhd K \lhd G$, then $H \lhd G$. In 2000, Bianchi Gillio Berta Mauri, Herzog and Verardi [5] proved a characterization of soluble $T$-groups by means of $\mathcal{H}$-subgroup: a subgroup $H$ of a group $G$ is called an $\mathcal{H}$-subgroup in $G$ if $N_G(H) \cap H^G \lhd H$, for every element $g \in G$, where $N_G(H) = \{x \in G | H^x = H\}$ is the normalizer of $H$ in $G$. They proved that a group $G$ is a supersolvable $T$-group if and only if every subgroup of $G$ is an $\mathcal{H}$-subgroup of $G$. Later, except for the exploration of $T$-groups, $\mathcal{H}$-subgroups were widely used to character finite groups. Csörgő and Herzog [6] obtained that a group $G$ is supersolvable if every cyclic subgroup of $G$ of prime order or order 4 is an $\mathcal{H}$-subgroup. Asaad [7] proved that a group $G$ is supersolvable if every maximal subgroup of every Sylow subgroup of $G$ is an $\mathcal{H}$-subgroup. The set of all $\mathcal{H}$-subgroups of a group $G$ is denoted by $\mathcal{H}(G)$. Moreover, Guo and Wei [8] gave new characterization of $p$-nilpotent or supersolvable by assuming some subgroups of $G$ of the same order all belong to $\mathcal{H}(G)$, which provide a unified version of the results mentioned above if the order of $G$ is odd. Moreover, Li, Zhao and Xu [9] considered the case when $G$ is of even order.

Recently, Asaad et al. [10] introduced a new subgroup embedding property called weakly $\mathcal{H}$-subgroup, which generalizes both $c$-normality and $\mathcal{H}$-subgroup, called weakly $\mathcal{H}$-subgroup. Soon after, Asaad and Ramadan [11] gave the definition of weakly $\mathcal{H}$-embedded subgroup. Please note
that a subgroup \( H \) of \( G \) is said to be a weakly \( \mathcal{H} \)-embedded subgroup (weakly \( \mathcal{H} \)-subgroup) of \( G \) if there exists a normal subgroup \( T \) of \( G \) such that \( H^G = HT \) (\( G = HT \)) and \( H \cap T \in \mathcal{H}(G) \), where \( H^G \) is the normal closure of \( H \) in \( G \). Clearly, \( \mathcal{C} \)-normal subgroups, \( \mathcal{H} \)-subgroups and weakly \( \mathcal{H} \)-subgroups imply weakly \( \mathcal{H} \)-embedded subgroups. However, the converse does not hold in general, see [11] (Examples 1.3, 1.4 and 1.5).

In fact, these subgroups were widely used to investigate the structure of finite groups. As a result, many interesting results have been subsequently obtained, such as [7,10–13].

In the recent research about \( \mathcal{H} \)-subgroups, Asaad, Ramadan, and Heliel gave a new characterization of \( \mathcal{P} \)-nilpotency.

**Theorem 1.** ([12] Theorem A) Let \( p \) be the smallest prime dividing \( |G| \), and \( P \) a non-cyclic Sylow \( p \)-subgroup of \( G \). Then \( G \) is \( \mathcal{P} \)-nilpotent if and only if there exists a \( p \)-power \( d \) with \( 1 < d < |P| \) such that all subgroups of \( P \) of order \( d \) and \( pd \) are weakly \( \mathcal{H} \)-embedded in \( G \).

However, according to this result, some natural questions arise:

**Problem 1.**

1. If delete the condition “\( p \) is the smallest prime dividing \( |G| \)”, can we claim that \( G \) is \( \mathcal{P} \)-supersoluble?
2. Does there exist another condition to obtain \( \mathcal{P} \)-nilpotence rather than “\( p \) is the smallest prime dividing \( |G| \)”?
3. As we know, the condition that \( HT \) is the smallest normal subgroup of \( G \) containing \( H \), is too strict. Can we replace it by a weaker embedding subgroup property?

In this paper, we further explore weakly \( \mathcal{H} \)-embedded subgroups and pay attention to Problem 1. However, instead of the normality of \( HT \), we just consider \( HT \) is \( S \)-quasinormal in \( G \). As we know, a subgroup \( K \) is \( S \)-quasinormal in \( G \), means that \( K \) permutes with every Sylow subgroup \( P \) of \( G \), that is \( KP = PK \). However, for convenience, we also called it a weakly \( \mathcal{H} \)-embedded subgroup, that is:

**Definition 1.** A subgroup \( H \) of a group \( G \) is said to be weakly \( \mathcal{H} \)-embedded in \( G \) if there exists a normal subgroup \( T \) of \( G \) such that \( HT \) is \( S \)-quasinormal in \( G \) and \( H \cap T \in \mathcal{H}(G) \).

As an application of these subgroups, we give a positive answer to Problem 1 in the class of \( \mathcal{P} \)-soluble groups, for detail:

**Theorem 2.** Let \( E \) be a \( \mathcal{P} \)-soluble normal subgroup of a group \( G \) such that \( G/E \) is \( \mathcal{P} \)-supersoluble, where \( p \) is a prime divisor of \( |E| \). Let \( P \) be a Sylow \( p \)-subgroup of \( E \). Suppose that \( P \) has a subgroup \( D \) with \( 1 \leq |D| < |P| \) such that all subgroups of \( P \) of order \( |D| \) and \( p|D| \) are weakly \( \mathcal{H} \)-embedded in \( G \). When \( |D| = 1 \) and \( P \) is a non-abelian 2-group, we further assume that all cyclic subgroups of \( P \) of order 4 are weakly \( \mathcal{H} \)-embedded in \( G \). Then \( G \) is \( \mathcal{P} \)-supersoluble.

Moreover, to avoid the condition “\( p \) is the smallest prime dividing \( |G| \)” of Theorem 1, we further prove that the conclusion holds if this condition is replaced by “\( N_G(P) \) is \( \mathcal{P} \)-nilpotent”. Consequently, we give an answer to Problem 1.

**Theorem 3.** Let \( E \) be a normal subgroup of \( G \) such that \( G/E \) is \( \mathcal{P} \)-nilpotent, and \( P \) be a non-cyclic Sylow \( p \)-subgroup of \( E \), where \( p \) is a prime dividing \( |E| \). Assume that \( N_G(P) \) is \( \mathcal{P} \)-nilpotent and \( P \) has a subgroup \( D \) with order \( 1 < |D| < |P| \) such that all subgroups of \( P \) of order \( |D| \) and \( p|D| \) are weakly \( \mathcal{H} \)-embedded in \( G \). Then \( G \) is \( \mathcal{P} \)-nilpotent.

In the second section, we list some lemmas which will be useful for the proofs of the above results. The proofs of Theorems 2 and 3 are put in the third section. Some previously known results are generalized by our theorems, and we list some in the fourth section.
2. Preliminaries

Lemma 1. (see ([1], Chapter 1) or ([3], Chapter 1, Lemmas 5.34 and 5.35)) Assume that $H, E$ are subgroups of $G$ and $N \leq G$.

1. If $H$ is S-quasinormal in $G$, then $H \cap E$ is S-quasinormal in $E$, and $HN/N$ is S-quasinormal in $G/N$.
2. Assume that $H$ is a $p$-group. Then $H$ is S-quasinormal in $G$ if and only if $O^p(G) \leq N_G(H)$.
3. The set of S-quasinormal subgroups of $G$ is a sublattice of the subnormal subgroup lattice of $G$.
4. If $H$ is a $p$-group and $H$ is subnormal in $G$, then $H \leq O_p(G)$.

Lemma 2. ([11] Lemma 2.1) Let $H, N$ be subgroups of $G$ satisfying $H \subseteq \mathcal{H}(G)$ and $N \leq G$. Then:

1. If $E$ is a subgroup of $G$ containing $H$, then $H \subseteq \mathcal{H}(E)$.
2. If $H$ is subnormal in $G$, then $H$ is normal in $G$.
3. Assume that $N \leq N_G(H)$. Then $NH \in \mathcal{H}(G)$.
4. If $E$ is a subgroup of $G$ satisfying $N \leq E$, then $E \in \mathcal{H}(G)$ if and only if $E/N \in \mathcal{H}(G/N)$.
5. If $H$ is a $p$-group and $p \nmid |N|$, then $NH \in \mathcal{H}(G)$ and $HN/N \in \mathcal{H}(G/N)$.

Lemma 3. Let $H$ be a weakly $\mathcal{H}$-embedded subgroup of a group $G$.

1. Assume that $E$ is a subgroup of $G$ containing $H$. Then $H$ is weakly $\mathcal{H}$-embedded in $E$.
2. If $N$ is a normal subgroup of $G$ satisfying $N \leq H$, then $HN/N$ is weakly $\mathcal{H}$-embedded in $G/N$.
3. Assume that $H$ is a $p$-group and $N$ is a normal $p'$-subgroup of $G$. Then $HN/N$ is weakly $\mathcal{H}$-embedded in $G/N$.

Proof. By the hypothesis, $G$ has a normal subgroup $T$ such that $HT$ is S-quasinormal in $G$ and $H \cap T \in \mathcal{H}(G)$.

1. Clearly, $T \cap E$ is a normal subgroup of $E$ such that $H(T \cap E) = HT \cap E$ is S-quasinormal in $E$ and $H \cap (T \cap E) = H \cap T \in \mathcal{H}(E)$ (see Lemmas 1(1) and 2(1)). This shows that $H$ is weakly $\mathcal{H}$-embedded in $E$.

2. Consider the normal subgroup $TN/N$ of $G/N$. Please note that $N \leq N_G(H \cap T)$, so $(H \cap T)N \in \mathcal{H}(G)$ by Lemma 2(3). Furthermore, we have that $(H/N)(TN/N) = HT/N$ is S-quasinormal in $G/N$ and

$$(H/N) \cap (TN/N) = (H \cap T)N/N \in \mathcal{H}(G/N)$$

(see Lemmas 1(1) and 2(4)). By the definition, $HN/N$ is weakly $\mathcal{H}$-embedded in $G/N$.

3. By Lemma 1(1), the normal subgroup $TN/N$ of $G/N$ such that $(HN/N)(TN/N) = HTN/N$ is S-quasinormal in $G/N$. Please note that

$$(|HN \cap T : H \cap T|, |HN \cap T : N \cap T|) = (|N \cap HT : H \cap NT|) \leq 1,$$

so $HN \cap T = (H \cap T)(N \cap T)$. Combining with Lemma 2(5),

$$(HN/N) \cap (TN/N) = (HN \cap T)N/N = (H \cap T)N/N \in \mathcal{H}(G/N).$$

Hence $HN/N$ is weakly $\mathcal{H}$-embedded in $G/N$. □

Recall that a class of groups $\mathfrak{F}$ is called a formation if for every group $G$, every homomorphic image of $G/G^\mathfrak{F}$ belongs to $\mathfrak{F}$, where $G^\mathfrak{F} = \bigcap \{N \leq G | G/N \in \mathfrak{F}\}$. Furthermore, a formation $\mathfrak{F}$ is said to be saturated if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. The intersection of all formations containing the set $\{G/O^p_{p'}(G) | G \in \mathfrak{F}\}$ is denoted by $\mathfrak{F}(p)$, and $F(p)$ denotes the class of all groups $G$ such that $G^{\mathfrak{F}(p)}$ is a $p$-group. Associated with a saturated formation $\mathfrak{F}$, there is a function $f$ of the form $f : \mathbb{P} \rightarrow \{\text{group formations}\}$, where $f(p) = F(p)$ for any prime $p$, which divides $|G|$ for some $G \in \mathfrak{F}$, and $f(\mathfrak{F}) = \emptyset$ otherwise. The function $f$ is called the canonical local satellite of $\mathfrak{F}$. For more detail, please turn to ([3] P. 3) or ([2] Chap. IV, Theorem 3.7 and Definitions 3.9). Now we recall the subgroup $Z_\mathfrak{F}(G)$
of $G$, which is called the $\mathfrak{F}$-hypercenter of $G$. In fact, $Z_{\mathfrak{F}}(G)$ the product of all such normal subgroups $N$ of $G$ whose $G$-chief factors $H/K$ satisfying $(H/K) \times (G/C_G(H/K)) \in \mathfrak{F}$.

Lemma 4. Let $\mathfrak{F}$ be a saturated formation and $f$ the canonical local satellite of $\mathfrak{F}$. Let $P$ be a normal $p$-subgroup of $G$. Then $P \leq Z_{\mathfrak{F}}(G)$ if and only if one of the following holds:

1. $G/C_G(P) \in f(p)$ ([13] Chap. 1, Lemma 2.26) or ([14] Lemma 2.14);
2. $P/\Phi(P) \leq Z_{\mathfrak{F}}(G/\Phi(P))$ ([15] Lemma 2.8).

Lemma 5. ([1] Lemma 2.1.6) If $G$ is $p$-supersoluble and $O_p'(G) = 1$, then $G$ has the unique Sylow $p$-subgroup.

Lemma 6. ([2] Chap. A, Lemma 8.4) Let $N$ be a nilpotent normal subgroup of $G$ and $M$ a maximal subgroup of $G$ such that $N \nsubseteq M$. Then $N \cap M$ is a normal subgroup of $G$.

3. Proofs of Main Results

The following proposition plays an important role in the proof of Theorem 2.

Proposition 1. Let $P$ be a normal $p$-subgroup of a group $G$. Assume that $P$ has a subgroup $D$ satisfying $1 \leq |D| < |P|$, such that all subgroups of $P$ of order $|D|$ and $|p|D|$ are weakly $\mathcal{H}$-embedded in $G$. When $|D| = 1$ and $P$ is a non-abelian $2$-group, we further assume that all cyclic subgroups of $P$ of order $4$ are weakly $\mathcal{H}$-embedded in $G$. Then $P \leq Z_{\mathcal{H}}(G)$.

Proof. Assume by contradiction that $(G, P)$ is a counterexample of minimal order $|G| + |P|$. We proceed via the following steps.

1. $P$ is not a minimal normal subgroup of $G$.

Assume that $P$ is minimal normal in $G$. Let $H$ be a subgroup of $P$ of order $|D|$ or $|p|D|$, which is normal in some Sylow subgroup of $G$. By the hypothesis, $H$ is weakly $\mathcal{H}$-embedded in $G$. So $G$ has a normal subgroup $T$ such that $HT$ is $S$-quasinormal in $G$ and $H \cap T \in \mathcal{H}(G)$. Please note that $P \cap T$ is normal in $G$, so $P \cap T = 1$ or $P \cap T = P$ by the minimality of $P$. If $P \cap T = 1$, then $H = H(P \cap T) = P \cap HT$ is $S$-quasinormal in $G$. However, by the choice of $H$ and Lemma 1(2), $H \leq G$, a contradiction. So $P \leq T$. In this case, $H = H \cap T \in \mathcal{H}(G)$ and then $H \leq G$ by the relationship $H \leq P \leq G$ and Lemma 2(2), which is impossible. Thus, $P$ is not a minimal normal of $G$.

2. If every maximal subgroup of $P$ is weakly $\mathcal{H}$-embedded in $G$, then $P \leq Z_{\mathcal{H}}(G)$.

Let $N$ be a minimal normal subgroup of $G$ contained in $P$. By Lemma 3(2), $(G/N, P/N)$ satisfies the hypothesis. So, the choice of $(G, P)$ implies that: (i) $P/N \leq Z_{\mathcal{H}}(G/N)$; (ii) $N$ is non-cyclic; (iii) $N$ is the unique minimal normal subgroup of $G$ contained in $P$. Now assume that $\Phi(P) = 1$. In this case, $P$ is elementary abelian and $P = N \times B$, where $B$ is a complement of $N$. Let $N_1$ be a maximal subgroup of $N$ such that $N_1$ is normal in some Sylow $p$-subgroup $G_p$ of $G$. Then $P_1 = N_1B$ is a maximal subgroup of $P$. By the hypothesis, $G$ has a normal subgroup $T$ such that $P_1T$ is $S$-quasinormal in $G$ and $P_1 \cap T \in \mathcal{H}(G)$. Please note that $P \cap T$ is a normal subgroup of $G$ contained in $P$, so $N \leq P \cap T$ or $P \cap T = 1$ by (iii). First, assume that $N \leq T$. Then $1 < N_1 \leq P_1 \cap T$. However, $P_1 \cap T \leq G$ by the relationship $P_1 \cap T \leq P \leq G$ and Lemma 2(2). Thus, the uniqueness of $N$ deduces that $N \leq P_1 \cap T \leq P_1$, a contradiction. Secondly, if $P \cap T = 1$, then $P_1 = P_1(P \cap T) = P \cap P_1T$ is $S$-quasinormal in $G$, moreover $P_1 \cap N = N_1B$ is $S$-quasinormal in $G$ by Lemma 1(3). Hence Lemma 1(2) and the choice of $N_1$ imply that $N_1 \leq G$, a contradiction. The above shows that $\Phi(P) \neq 1$ and consequently, $N \leq \Phi(P)$. Furthermore, $P/\Phi(P) \leq Z_{\mathcal{H}}(G/\Phi(P))$. However, we have $P \leq Z_{\mathcal{H}}(G)$ by Lemma 4. This contradiction shows that (2) holds.

(3) If every cyclic subgroup of $P$ of order $p$ or $4$ (when $P$ is a non-abelian $2$-group) is weakly $\mathcal{H}$-embedded in $G$, then $P \leq Z_{\mathcal{H}}(G)$.

If $P$ is not a non-abelian $2$-group, then we use $\Omega$ to denote the subgroup $\Omega_1(P)$ of $P$. Otherwise, $\Omega = \Omega_2(P)$. 
Let $R$ be a normal subgroup of $G$ such that $P/R$ is a $G$-chief factor. Obviously, $R$ satisfies the hypothesis. So $R \leq Z \Upsilon(G)$ and $P/R$ is non-cyclic by the choice of $(G, P)$. Moreover, for any normal subgroup $L$ of $G$ satisfying $L < P$, we have $L \leq R$. In fact, if $L \not\leq R$, then similarly $L \leq Z \Upsilon(G)$, and $P = RL \leq Z \Upsilon(G)$, a contradiction. Now, assume that $\Omega \leq R$. Then $\Omega \leq Z \Upsilon(G)$. From Lemma 4 and ([16] Lemma 2.4), it follows that $G/C_G(\Omega) \in F(p)$ and $C_G(\Omega)/C_G(P) \in \mathfrak{N}_p$, where $F$ is the canonical local satellite of $\Upsilon$ and $\mathfrak{N}_p$ is the class of $p$-groups. Consequently, $G/C_G(P) \in \mathfrak{N}_PF(P) = F(p)$, and thereby $P \leq Z \Upsilon(G)$ by Lemma 4 again. This contradiction shows that $\Omega = P$.

Let $L/R$ be a minimal subgroup of $Z(GP/R) \cap P/R$ and $x \in L \setminus R$, where $G_P$ is a Sylow $p$-subgroup of $G$. Then $H = \langle x \rangle$ has order $p$ or $4$ and $L = HR$. By the hypothesis, $H$ is weakly $\mathcal{H}$-embedded in $G$, so $G$ has a normal subgroup $T$ such that $HT$ is $S$-quasinormal in $G$ and $H \cap T \in \mathcal{H}(G)$. Please note that $P \cap T \leq G$. Combining with the above result, we have $P \cap T = P$ or $P \cap T \leq R$. If $P \cap T = P$, that is, $P \leq T$, then $H = H \cap T \in \mathcal{H}(G)$. Moreover, the relationship $H \leq \mathcal{S}P \leq G$ and Lemma 2(2) deduce $H \leq G$. By the choice of $H$, we have $P/R = L/R$ is cyclic, which is a contradiction. Now assume that $P \cap T \leq R$. Then

$$L/R = HR/R = H(P \cap T)R/R = P/R \cap HTR/R$$

is $S$-quasinormal in $G/R$ by Lemma 1(3). From Lemma 1(2) and the choice of $L/R$, it follows that $L/R \leq G/R$, which also shows that $P/R = L/R$, a contradiction. This completes the proof of (3).

4. $p < |D| < |P|/p$ (it follows directly from (2) and (3)).

5. $\Phi(P) = 1$.

Suppose that $\Phi(P) > 1$. We compare the order of $\Phi(P)$ with $|D|$. First, assume that $|\Phi(P)| > |D|$. In this case, we have $\Phi(P) \leq Z \Upsilon(G)$ by the hypothesis and the choice of $P$. Let $N$ be a minimal normal subgroup of $G$ contained in $\Phi(P)$. Clearly, $|N| = p$ and by (4), $P/N$ satisfies the hypothesis. Thus, $P/N \leq Z \Upsilon(G/N)$ and consequently $P \leq Z \Upsilon(G)$, a contradiction. So $|\Phi(P)| \leq |D|$. Please note that $P/\Phi(P)$ is elementary abelian, so we can easily prove that $P/\Phi(P)$ satisfies the hypothesis. Therefore, $P/\Phi(P) \leq Z \Upsilon(G/\Phi(P))$ and by Lemma 4, we further have $P \leq Z \Upsilon(G)$. This contradiction shows that $\Phi(P) = 1$.

6. Final contradiction.

Let $N$ be a minimal normal subgroup of $G$ contained in $P$. Clearly, $N < P$. Compare the order of $N$ with $|D|$. If $|D| < |N|$, then $N$ satisfies the hypothesis and the choice of $P$ implies that $N \leq Z \Upsilon(G)$. Consequently, $|N| = p$ and then $|D| = 1$, which contradicts (4). Thus, $|D| \geq |N|$. By (5), $P$ is elementary abelian, and all subgroups of $P/N$ of order $|D|/|N|$ and $p|D|/|N|$ are weakly $\mathcal{H}$-embedded in $G$ (see Lemma 3(2)). Therefore $P/N \leq Z \Upsilon(G/N)$ by the choice of $P$. Please note that $|P/N| \geq |P|/|D| > p^2$. So there exists a normal subgroup $E$ of $G$ contained in $P$ satisfying $N \leq E \leq P$ and $|P/E| = p$. Consider the subgroup $E$. Then $E \leq Z \Upsilon(G)$ by the hypothesis and the choice of $P$, which implies $|N| = p$. Combining with $P/N \leq Z \Upsilon(G/N)$, we finally obtain $P \leq Z \Upsilon(G)$, which is a contradiction. The final contradiction completes the proof of the proposition.}

Now we give the proof of Theorem 2:

**Proof.** Suppose that the assertion is false and consider a counterexample $(G, E)$ with minimal $|G| + |E|$. We proceed via the following steps.

1. $O_{p'}(E) = 1$.

   Clearly, $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis by Lemma 3(3). If $O_{p'}(E) > 1$, then the choice of $G$ implies that $G/O_{p'}(E)$ is $p$-supersoluble. Furthermore, $G$ is $p$-supersoluble, which is a contradiction. Thus, $O_{p'}(E) = 1$.

2. $E = G$.

   Suppose that $E < G$. Please note that Lemma 3(1) shows that $(E, E)$ satisfies the hypothesis, so $E$ is $p$-supersoluble. Combining (1) with Lemma 5, we have $P \leq E$ and consequently, $P \leq G$. From the
hypothesis and Proposition 1, it follows that \( P \leq Z_{H}(G) \). This result implies \( E \leq Z_{pH}(G) \) and then \( G \) is \( p \)-supersoluble, which is a contradiction. Thus, \( E = G \).

(3) If every maximal subgroup of \( P \) is weakly \( H \)-embedded in \( G \), then \( G \) is \( p \)-supersoluble.

Let \( N \) be a minimal normal subgroup of \( G \). Since \( G \) is \( p \)-soluble and \( O_{P}(G) = 1, N \leq O_{P}(G) \). By Lemma 3(2), \( G/N \) satisfies the hypothesis, so: (i) \( G/N \) is \( p \)-supersoluble; (ii) \( |N| > p \); (iii) \( N \) is the unique minimal normal subgroup of \( G \). Obviously, \( N \not\in \Phi(G) \), so there exists a maximal subgroup \( M \) of \( G \) such that \( G = N \times M \). By Lemma 6, \( O_{P}(G) \cap M \leq G \). So \( O_{P}(G) \cap M = 1 \) by the uniqueness of \( N \), and then

\[
O_{P}(G) = N(O_{P}(G) \cap M) = N.
\]

On one hand, \( O_{P}(G) \leq C_{G}(O_{P}(G)) \) by the minimality of \( O_{P}(G) \). On the other hand, since \( G \) is \( p \)-soluble and \( O_{P}(G) = 1 \),

\[
C_{G}(O_{P}(G)) = C_{G}(F(G)) \leq F(G) = O_{P}(G).
\]

In general, \( C_{G}(O_{P}(G)) = O_{P}(G) \). Now we show that \( O_{P}(G) < P \). In fact, if \( P \leq G \), then \( P \leq Z_{H}(G) \) by Proposition 1. Similar to step (2), it is impossible.

Using the above symbol, \( G = O_{P}(G) \times M \) and then \( P = O_{P}(G) \times (P \cap M) \). Let \( P_{1} \) be a maximal subgroup of \( P \) containing \( P \cap M \). Then \( P_{1} \cap O_{P}(G) > 1 \) and it is not normal in \( G \). In fact, if \( P_{1} \cap O_{P}(G) \leq G \), then \( O_{P}(G) \leq P_{1} \cap O_{P}(G) \leq P_{1} \) by the minimality of \( O_{P}(G) \) and consequently, \( P = P_{1} \), a contradiction. By the hypothesis, \( P_{1} \) is weakly \( H \)-embedded in \( G \). So \( G \) has a normal subgroup \( T \) such that \( P_{1}T \) is \( S \)-quasinormal in \( G \) and \( P_{1} \cap T \in \mathcal{H}(G) \). If \( T = 1 \), then \( P_{1} \) is \( S \)-quasinormal in \( G \), which implies that \( P_{1} \leq O_{P}(G) \) by Lemma 1(3)(4) and then \( O_{P}(G) = P \). However, it contradicts the above result. So, the uniqueness of \( O_{P}(G) \) implies that \( O_{P}(G) \leq T \). Next, we prove that

\[
P_{1} \cap O_{P}(G) \in \mathcal{H}(G).
\]

First, we show that \( N_{G}(P_{1} \cap O_{P}(G)) = N_{G}(P_{1} \cap T) \). On one hand, note that

\[
P_{1} \cap O_{P}(G) = (P_{1} \cap T) \cap O_{P}(G),
\]

so

\[
P \leq N_{G}(P_{1} \cap T) \leq N_{G}(P_{1} \cap O_{P}(G)) < G.
\]

On the other hand, \( N_{G}(P_{1} \cap O_{P}(G)) \) is \( p \)-supersoluble by Lemma 3(1) and the relation

\[
N_{G}(P_{1} \cap O_{P}(G)) < G.
\]

Please note that \( C_{G}(O_{P}(G)) = O_{P}(G) \), so it is rather clear that \( O_{P'}(N_{G}(P_{1} \cap O_{P}(G))) = 1 \). Thus, \( P \) is normal in \( N_{G}(P_{1} \cap O_{P}(G)) \) by Lemma 5. At this moment, we have

\[
P_{1} \cap T \leq P \leq N_{G}(P_{1} \cap O_{P}(G)),
\]

and by Lemma 2(1),

\[
P_{1} \cap T \in \mathcal{H}(N_{G}(P_{1} \cap O_{P}(G))).
\]

Consequently, \( P_{1} \cap T \leq N_{G}(P_{1} \cap O_{P}(G)) \) by Lemma 2(2), that is, \( N_{G}(P_{1} \cap O_{P}(G)) \leq N_{G}(P_{1} \cap T) \).

Together with the above proof, we finally obtain \( N_{G}(P_{1} \cap O_{P}(G)) = N_{G}(P_{1} \cap T) \). Please note that \( P_{1} \cap T \in \mathcal{H}(G) \). So, for any element \( g \in G \),

\[
(P_{1} \cap O_{P}(G))^{g} \cap N_{G}(P_{1} \cap O_{P}(G)) = (P_{1} \cap T)^{g} \cap O_{P}(G) \cap N_{G}(P_{1} \cap T) \leq P_{1} \cap T \cap O_{P}(G) = P_{1} \cap O_{P}(G).
\]

This shows that \( P_{1} \cap O_{P}(G) \in \mathcal{H}(G) \). By Lemma 2(2), we further have \( P_{1} \cap O_{P}(G) \not\subseteq G \), a contradiction. This completes the proof of (3).
According to Theorem 1, we only need to consider that $p$ is odd. We proceed via the following steps.

(1) $O_p'(E) = 1$.

If $O_p'(E) > 1$, then it is normal in $G$. Consider $\overline{G} = G/O_p'(E)$. Please note that $\overline{P}$ a Sylow $p$-subgroup of $\overline{E}$ and $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ is $p$-nilpotent. Moreover, by hypothesis and Lemma 3(3), all subgroups of $\overline{P}$ of order $|D|$ and order $p|D|$ are weakly $H$-embedded in $\overline{G}$, that is $\overline{G}$ satisfies the hypothesis for $\overline{G}$. Thus, the choice of $G$ implies that $\overline{G}$ is $p$-nilpotent. Consequently, $G$ is $p$-nilpotent, a contradiction. So $O_p'(E) = 1$.

(2) $E = G$.

By Lemma 3(1), all subgroups of $P$ of order $|D|$ and order $p|D|$ are weakly $H$-embedded in $E$. Since $N_E(P) = N_G(P) \cap E$, $N_E(P)$ is $p$-nilpotent. Then $E$ satisfies the hypothesis. If $E < G$, then $E$ is $p$-nilpotent by the choice of $G$. Let $E_{p'}$ be the normal $p'$-Hall subgroup of $E$. Clearly, $E_{p'} \leq G$. So, by (1), $E_{p'} = 1$, that is, $E = P$. In this case, $G = N_G(P)$ is $p$-nilpotent. This contradiction shows that $E = G$.

(3) $O_p'(G) > 1$.

Next we give the proof of Theorem 3:

Proof. Suppose that the assertion is false and consider a counterexample $G$ of minimal order. According to Theorem 1, we only need to consider that $p$ is odd. We proceed via the following steps.

(1) $O_p'(E) = 1$.

If $O_p'(E) > 1$, then it is normal in $G$. Consider $\overline{G} = G/O_p'(E)$. Please note that $\overline{P}$ a Sylow $p$-subgroup of $\overline{E}$ and $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ is $p$-nilpotent. Moreover, by hypothesis and Lemma 3(3), all subgroups of $\overline{P}$ of order $|D|$ and order $p|D|$ are weakly $H$-embedded in $\overline{G}$, that is $\overline{G}$ satisfies the hypothesis for $\overline{G}$. Thus, the choice of $G$ implies that $\overline{G}$ is $p$-nilpotent. Consequently, $G$ is $p$-nilpotent, a contradiction. So $O_p'(E) = 1$.

(2) $E = G$.

By Lemma 3(1), all subgroups of $P$ of order $|D|$ and order $p|D|$ are weakly $H$-embedded in $E$. Since $N_E(P) = N_G(P) \cap E$, $N_E(P)$ is $p$-nilpotent. Then $E$ satisfies the hypothesis. If $E < G$, then $E$ is $p$-nilpotent by the choice of $G$. Let $E_{p'}$ be the normal $p'$-Hall subgroup of $E$. Clearly, $E_{p'} \leq G$. So, by (1), $E_{p'} = 1$, that is, $E = P$. In this case, $G = N_G(P)$ is $p$-nilpotent. This contradiction shows that $E = G$.

(3) $O_p'(G) > 1$. 


Let $J(P)$ be the Thompson subgroup of $P$. Then clearly, $Z(J(P)) > 1$, $P \leq N_G(Z(J(P)))$ and $N_{N_G(Z(J(P)))}(P)$ is $p$-nilpotent. Assume that $N_G(Z(J(P))) < G$. Please note that $N_G(Z(J(P)))$ satisfies the hypothesis by Lemma 3(1). So, the choice of $G$ implies that $N_G(Z(J(P)))$ is $p$-nilpotent. However, it contradicts ([19] Theorem 8.3.1). Thus, $N_G(Z(J(P))) = G$, that is $Z(J(P)) \leq G$, which shows that (3) holds.

(4) $G$ is not $p$-soluble.

Suppose that $G$ is $p$-soluble. Then $G$ is $p$-supersoluble by the Theorem 2. Please note that $O_{p'}(G) = 1$. So $P \leq G$ by Lemma 5, which shows that $N_G(P) = G$ is $p$-nilpotent, a contradiction. Thus, (4) holds.

(5) Let $N$ be a minimal normal subgroup of $G$ contained in $O_p(G)$. Then $|N| > |D|$.

If $|N| = |D|$, then every subgroup of $P/N$ of order $p$ is weakly $H$-embedded in $G/N$ by Lemma 3(2). Denote $\overline{G} = G/N$. Let $\overline{M}$ be a proper subgroup of $\overline{G}$ and $\overline{M}_p$ a Sylow $p$-subgroup of $\overline{M}$. Clearly, $\overline{M}_P^\varphi \leq \overline{P}$ for some $\varphi \in \overline{G}$. Now consider $\overline{M}_\varphi$, which has a Sylow $p$-subgroup $\overline{M}_P^\varphi$ contained in $\overline{P}$. Without loss of generality, we can assume that the Sylow $p$-subgroup $\overline{M}_P^\varphi$ of $\overline{M}$ contains in $\overline{P}$. By Lemma 3(1), every cyclic subgroup of $\overline{M}_P^\varphi$ of order $p$ is weakly $H$-embedded in $\overline{M}$. Moreover, $N_{\overline{M}_P^\varphi}(P) = N_{\overline{M}}(P)$ is $p$-nilpotent. So $\overline{M}$ satisfies the hypothesis, and the choice of $G$ implies that $\overline{M}$ is $p$-nilpotent. Consequently, $G$ is a minimal non-$p$-nilpotent group. However, in this case, $G$ is soluble, which contradicts (4). Suppose that $|N| < |D|$. Then all subgroups of $P/N$ of order $|D|/|N|$ and $p|D|/|N|$ are weakly $H$-embedded in $G/N$ by Lemma 3(2), that is $G/N$ satisfies the hypothesis for $G$. So, from the choice of $G$, we deduce that $G/N$ is $p$-nilpotent. Similarly, $G$ is $p$-soluble in this case, a contradiction. Thus, $|N| > |D|$.

(6) Final contradiction.

By (5), all subgroups of $N$ of order $|D|$ and $p|D|$ are weakly $H$-embedded in $G$. Then $N \leq Z_H(G)$ by Proposition 1. From this result, we deduce that $|N| = p$ and $|D| = 1$, that is, every subgroup of $P$ of order $p$ is weakly $H$-embedded in $G$. Similarly, as the proof of (5), we can prove that in this case $G$ is soluble, a contradiction. The final contradiction completes the proof.

4. Some Applications

In this section, we list some applications of our results.

**Corollary 1.** Let $E$ be a normal subgroup of $G$. For every non-cyclic Sylow subgroup $P$ of $E$, assume that $P$ has a subgroup $D$ such that $1 < |D| < |P|$ and all subgroups of $P$ of order $|D|$ and $p|D|$ are weakly $H$-embedded in $G$. Then $E \leq Z_H(G)$.

**Proof.** Assume that $p$ is the smallest prime divisor of $|E|$ and $P$ is a Sylow $p$-subgroup of $E$. If $P$ is cyclic, then $E$ is $p$-nilpotent by the famous Burnside Theorem. Otherwise, by Lemma 3(1) and the hypothesis, all subgroups of $P$ of order $|D|$ and $p|D|$ are weakly $H$-embedded in $E$. So $E$ is $p$-nilpotent by Theorem 1, and then $E$ is soluble. By Lemma 3(1) again, we have that for any prime $p$ dividing $|E|$, $E$ satisfies the hypothesis of Theorem 2. So $E$ is supersoluble. Let $q$ be the maximal prime dividing $|E|$ and $Q$ the unique Sylow $q$-subgroup of $E$. Clearly, $Q \leq G$. Note that $Q$ satisfies the hypothesis of Proposition 1, so $Q \leq Z_H(G)$. Now consider $E/Q$. By Lemma 3(3), $E/Q$ satisfies the hypothesis of corollary. So $E/Q \leq Z_H(E/G)$ by induction. Therefore, $E \leq Z_H(G)$.

**Corollary 2.** ([12]) Assume that the Sylow subgroups of $G$ are non-cyclic for all primes $p$ dividing $|G|$. Assume further that for each such $p$ there is a $p$-power $d$ with $1 < d < |G|_p$ such that all subgroups of $P$ of order $d$ and $pd$ are weakly $H$-embedded in $G$, then $G$ is supersoluble.

**Proof.** Let $p$ be the smallest prime dividing $|G|$. By Theorem 1, $G$ is $p$-nilpotent. Consequently, $G$ is soluble. From the Theorem 2, it follows that $G$ is $q$-supersoluble, for any prime divisor $q$ of $|G|$, that is, $G$ is supersoluble.
Corollary 3. ([10]) Let $P$ be a normal $p$-subgroup of a group $G$. If all maximal subgroups of $P$ are weakly $H$-subgroups in $G$, then $P \leq Z(G)$.

Corollary 4. ([10]) Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$. A group $G$ lies in $\mathfrak{F}$ if and only if it has a normal subgroup $H$ such that $G/H \in \mathfrak{F}$ and all maximal subgroups of every Sylow subgroup of $H$ (or $F^*(H)$) are weakly $H$-subgroups in $G$.

Corollary 5. $G$ is supersolvable, if one of the following holds:
1. $G$ has a normal subgroup $H$ such that $G/H$ is supersolvable and all maximal subgroups of every Sylow subgroup of $H$ belong to $\mathcal{H}(G)$ [7];
2. all maximal subgroups of every Sylow subgroup of $F^*(G)$ belong to $\mathcal{H}(G)$ [7];
3. all maximal subgroups of every Sylow subgroup of a group $G$ are weakly $H$-subgroups in $G$ [10].

5. Conclusions
In this paper, we further explore weakly $H$-embedded subgroups. As new applications, we generalize the characterization of $p$-nilpotent given by Asaad, Ramadan and Heliel and get a new criterion for $p$-supersolubility for general prime $p$. Moreover, adding condition “$N_G(P)$ is $p$-nilpotent”, we obtain $p$-nilpotence for general prime $p$.

Author Contributions: Funding Acquisition, L.Z., L.-J.H. and J.-B.L.; Methodology, L.Z. and L.-J.H.; Supervision, J.-B.L.; Writing—Original Draft, L.Z.; all authors read and approved the final manuscript.

Funding: This work was supported by the Start-Up Scientific Research Foundation of Anhui Jianzhu University (2017QD20), the National Natural Science Foundation of China (11626049 and 11601006), the China Postdoctoral Science Foundation (2017M621579), the Postdoctoral Science Foundation of Jiangsu Province (1701081B), the Project of Anhui Jianzhu University (2016QD116 and 2017dc03) and the Anhui Province Key Laboratory of Intelligent Building and Building Energy Saving.

Conflicts of Interest: The authors declare no conflict of interest.

References
5. Bianchi, M.; MAURI, A.G.; Herzog, M.; Verardi, L. On finite solvable groups in which normality is a transitive relation. J. Group Theory 2000, 3, 147–156. [CrossRef]


