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Some Results on the Cohomology of Line Bundles on the Three Dimensional Flag Variety

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Abstract: Let k be an algebraically closed field of prime characteristic and let G be a semisimple, simply connected, linear algebraic group. It is an open problem to find the cohomology of line bundles on the flag variety G/B, where B is a Borel subgroup of G. In this paper we consider this problem in the case of $G = SL_3(k)$ and compute the cohomology for the case when $\langle \lambda, \alpha^\vee \rangle = -p^n a - 1$, $(1 \leq a \leq p,\ n > 0)$ or $\langle \lambda, \alpha^\vee \rangle = -p^n - r$, $(r \geq 2, n \geq 0)$. We also give the corresponding results for the two dimensional modules $N_\alpha(\lambda)$. These results will help us understand the representations of $SL_3(k)$ in the given cases.

Keywords: representation theory; algebraic groups; cohomology; line bundles

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1. Introduction

Let k denote an algebraically closed field and let G denote a semisimple, simply connected, linear algebraic group over k. Let B denote a Borel subgroup of G. It is an open problem to compute the cohomology group $H^i(G/B, k_\lambda)$, where k_λ denotes the line bundle with highest weight λ on G/B. For a field of characteristic zero the result is completely known by the famous Borel–Weil–Bott theorem [1]. Moreover, the character of these cohomology groups is given by the Weyl character formula [2]. If k is a field of prime characteristic then the problem is trivially known for $G = \mathrm{SL}_2(k)$ (the group of invertible 2×2 matrices with determinant 1). For $G = SL_3(k)$ (the group of invertible 3×3 matrices with determinant 1), Donkin proved some formulas for characters of G [3]. These formulas recursively describe the characters of G. Moreover, these formulas also involve recursion on characters of certain two dimensional modules $N_\alpha(\lambda)$. A considerable amount of labour is required to compute characters using these formulas (example computations are given in [4]). It is an extremely important problem to find simpler (non-recursive) results for G. Several attempts have been made to find a general result [2,4–7]. A non-recursive description for the characters may also lead to a general description for the cohomology.

In this paper we present some general results for $H^i(SL_3/B, k_\lambda)$. These results, along with the results presented in [5], can significantly simplify the recursion given [3]. We also given some general results for $H^i(SL_3/B, N_\alpha(\lambda))$. We first present the general setup of the problem and some known results from literature.

We choose a maximal torus T of G and for an algebraic group H we denote by $\mathrm{mod}(H)$ the category of finite dimensional (rational) H-modules over k. The group of characters (multiplicative) of T is denoted by $X(T)$. Let V be a T-module and $\lambda \in X(T)$ then we write V^λ for the corresponding weight space of V. If V^λ is non-zero then λ is called a weight of V. We denote the Weyl group of G as W and we take the usual action of W on T and $X(T)$. Suppose Φ is the set of non-zero weights for T, then we denote by $(\mathbb{R} \otimes_\mathbb{Z} X(T), \Phi)$ a root system. Let Φ^+ denote the set of positive roots. We denote by S the set of simple roots. For $\alpha \in \Phi$ the corresponding coroot $\alpha^\vee$ is defined by $\frac{2\alpha}{(\alpha,\alpha)}$. We denote by $X^+(T)$ the set of dominant weights. The element ρ is defined as the half sum of the positive roots.

The action of Weyl group W on $X(T)$ is defined as $w \cdot \lambda = w(\lambda+\rho)-\rho$. For $\alpha \in S$, we denote by P_α the parabolic subgroup containing B which has α as its only positive root. For an algebraic group K and a closed subgroup J we have the induction functor $\mathrm{Ind}_J^K : \mathrm{mod}(J) \to \mathrm{mod}(K)$. If $J \leq K \leq H$ and V is a J-module there is a spectral sequence given by $R^*\mathrm{Ind}_J^H V$, with its E_2 page $R^i\mathrm{Ind}_K^H R^j\mathrm{Ind}_J^K V$. This is called the Grothendieck Spectral sequence. We will also be using its special case when $B \leq P_\alpha \leq G$ given by $R^i\mathrm{Ind}_{P_\alpha}^G R^j\mathrm{Ind}_B^{P_\alpha} V$ and V a B-module. The dual of V will be denoted by V^*. For further details see e.g., [2,8–10].

If $\lambda \in X(T)$ then k_λ denotes the one dimensional B-module on which T acts via λ. For the rest of this paper we will also denote k_λ simply by λ. For a dominant weight, we denote by $\nabla_\alpha(\lambda)$ the induced module $\mathrm{Ind}_B^{P_\alpha}\lambda$. We will denote by $\nabla(\lambda)$, the induced module $\mathrm{Ind}_B^G\lambda$ and define $\Delta(\lambda) = \nabla(-w_0\lambda)^*$. We will also use $H^i(M)$ for $R^i\mathrm{Ind}_B^G M$. The P_α-module on which the unipotent radical $R_u(P_\alpha)$ acts trivially will be denoted by $P_\alpha/R_u(P_\alpha)$.

We will denote by $F : G \to G$ the Frobenius morphism of G. We know that there exists a unique two dimensional B-module (indecomposable) with character $e(0) + e(-\alpha)$ [2]. We denote this module by $N(\alpha)$. We will write $N_\alpha(\lambda)$ for the B-module $\lambda \otimes N(\alpha)$, $\lambda \in X(T)$. It is clear that $N_\alpha(\lambda) = \nabla_\alpha(\rho) \otimes (\lambda - \rho)$.

For $G = \mathrm{SL}_3(k)$ we have $\lambda = (a,b)$, $a, b \in \mathbb{Z}$. Also it has two simple roots $\alpha = (2,-1)$ and $\beta = (-1,2)$. If k is a field of prime characteristic then the following results hold.

Theorem 1. *For $\lambda \in X^+(T)$ we have $H^i(\lambda) = 0$ for all $i > 0$ [2].*

The above theorem is known as the Kempf's vanishing theorem.

Theorem 2. *Let $n = \dim(G/B)$ then $H^i(G/B, \mathcal{L}_M) = 0$, for all $i > n$ [2].*

Theorem 3. *If $n = \dim(G/B)$ [2], then*

$$H^i(G/B, \mathcal{L}_M)^* \simeq H^{n-i}(G/B, \mathcal{L}_{(M^* \otimes k_{-2\rho})}).$$

The following result is due to H. H. Andersen and it describes the complete vanishing behaviour of the first cohomology modules. Moreover, in the case of non-vanishing cohomology modules it is their highest weight (see e.g., [2]).

Proposition 1. *Suppose k is a field of characteristic p, $\alpha \in S$ and $\lambda \in X(T)$ with $\langle \lambda, \alpha^\vee \rangle \geq 0$.*

1. *Let $\langle \lambda, \alpha^\vee \rangle = bp^n - 1$ for some $b, n \in \mathbb{Z}^+$ with $0 < b < p$. Then*

$$H^1(s_\alpha \cdot \lambda) \neq 0 \Longleftrightarrow \lambda \in X^+(T).$$

2. *Let $\langle \lambda, \alpha^\vee \rangle = \sum_{j=0}^{n} a_j p^j$ with $0 \leq a_j < p$ and $a_j \neq 0$. Suppose there is some $j < n$ with $a_j < p-1$. Then*

$$H^1(s_\alpha . \lambda) \neq 0 \Longleftrightarrow s_\alpha . \lambda + a_n p^n \alpha \in X^+(T).$$

If λ is dominant then λ is the highest weight of $H^1(s_\alpha.\lambda)$. Suppose λ is not dominant and t be smallest integer such that $a_t < p-1$. Let $t' \geq t$ be minimal for $\mu = s_\alpha.\lambda + \sum_{j=t'}^{n} a_j p^j \alpha \in X^+(T)$. Then μ is the highest weight, with multiplicity 1, of $H^1(s_\alpha.\lambda)$.

The following result describes the vanishing of $R^i\mathrm{Ind}_B^{P_\alpha}\lambda$ [2].

Proposition 2. *Let $\alpha \in S$ and $\lambda \in X(T)$.*

1. *If* $\langle \lambda, \alpha^\vee \rangle = -1$ *then* $R^i\mathrm{Ind}_B^{P_\alpha}\lambda = 0$ *for all* i.
2. *If* $\langle \lambda, \alpha^\vee \rangle \geq 0$ *then* $R^i\mathrm{Ind}_B^{P_\alpha}\lambda = 0$ *for all* $i \neq 0$.
3. *If* $\langle \lambda, \alpha^\vee \rangle \leq -2$ *then* $R^i\mathrm{Ind}_B^{P_\alpha}\lambda = 0$ *for all* $i \neq 1$.

We define $\mathrm{Ind}_B^{P_\alpha}\lambda\nabla_\alpha(\lambda)$.

2. Results

In this section we present our main results. We first say a few words about the case $G = SL_2(k)$. In this case the dimension n of $SL_2(k)/B$ is 1. Therefore by using Theorem 2 we have that $H^i(SL_2(k)/B, \lambda) = 0$ whenever $i > 1$. We can now use the Serre duality to find $H^1(SL_2(k)/B, \lambda)$. This argument along with the Weyl character formula gives the complete result for $SL_2(k)$.

For $G = SL_3(k)$ we let $a, b \in \mathbb{Z}$ and the collection of dominant weights for SL_3 is then given by $X^+(T) = \{(a, b)|\ a, b \geq 0\}$. In case of dominant weights the cohomology is known to be zero by Theorem 2. Without loss of generality we can take $b < 0$ (For $a < 0$ the result follows from duality). In [3], Donkin also proved that $H^1(k_{-p^m\beta}) = k$, for $m \geq 0$ and a non-isolated simple root β. We present some more general results in the following propositions. The following result describes the cohomology for the case when $b = -p^n(r+1) - 1$.

Proposition 3. *If* $n > 0$ *then for* $0 \leq r \leq p-1$ *we have*

$$H^i(a, -p^n(r+1)-1) = \begin{cases} H^0(a - p^n(r+1), p^n(r+1)-1), & a \geq p^n(r+1) \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We use the second page of the spectral sequence to get

$$H^i(a, -p^n(r+1)-1) = R^i\mathrm{Ind}_B^G(a, -p^n(r+1)-1) \tag{1}$$

$$= \mathrm{Ind}_{P_\beta}^G R^i\mathrm{Ind}_B^{P_\beta}(a, -p^n(r+1)-1) \tag{2}$$

Since $R^i\mathrm{Ind}_B^{P_\beta}(a, -p^n(r+1)-1) = 0$ for each $i \neq 1$. We use Equation (4) to get

$$H^i(a, -p^n(r+1)-1) = \mathrm{Ind}_{P_\beta}^G R\mathrm{Ind}_B^{P_\beta}(a, -p^n(r+1)-1) \tag{3}$$

We use the Serre duality to get

$$R\mathrm{Ind}_B^{P_\beta}(a, -p^n(r+1)-1) = \nabla_\beta(-a+1, p^n(r+1)-1)^*$$

and $\nabla_\beta(-a+1, p^n(m+1)-1)^* = \nabla_\beta(a - p^n(r+1), p^n(r+1)-1)$ (From the SL_2 case). We plug these values back in Equation (6) to have

$$H^i(a, -p^n(r+1)-1) = \mathrm{Ind}_{P_\beta}^G \nabla_\beta(a - p^n(r+1), p^n(r+1)-1)$$

$$= \mathrm{Ind}_B^G(a - p^n(r+1), p^n(r+1)-1)$$

Finally $\mathrm{Ind}_B^G(a - p^n(r+1), p^n(r+1)-1) \neq 0$ if and only if $a \geq p^n(m+1)$. This completes the proof. □

The following result describes the cohomology when a, b are powers of p.

Proposition 4. *For each positive integer* n *we have*

$$H^i(p^n, -p^n) = H^0(1, p^n - 2).$$

Proof. As in the proof of the above proposition we have

$$H^i(p^n, -p^n) = R^i\mathrm{Ind}_B^G(p^n, -p^n) \tag{4}$$
$$= \mathrm{Ind}_{P_\beta}^G R^i\mathrm{Ind}_B^{P_\beta}(p^n, -p^n) \tag{5}$$

Since $R^i\mathrm{Ind}_B^{P_\beta}(p^n, -p^n) = 0$ for each $i \neq 1$. We use Equation (4) to get

$$H^i(p^n, -p^n) = \mathrm{Ind}_{P_\beta}^G R\mathrm{Ind}_B^{P_\beta}(p^n, -p^n) \tag{6}$$

We use the Serre duality to get

$$R\mathrm{Ind}_B^{P_\beta}(p^n, -p^n) = \nabla_\beta(-p^n + 1, p^n - 2)^*$$

and $\nabla_\beta(-p^n + 1, p^n - 2)^* = \nabla_\beta(1, p^n - 2)$ (From the SL_2 case). We plug these values back in Equation (6) to have

$$H^i(p^n, -p^n) = \mathrm{Ind}_B^G(1, p^n - 2).$$

This completes the proof. □

The following results describe the vanishing of first cohomology group for the given weights.

Proposition 5. *Let $n \geq 0$ and $m \geq 2$. Then*

$$H^1(p^n - 1, -p^n - r) = 0.$$

Proof. By assuming $\lambda = s_\beta.(p^n - 1, -p^n - m) = (-m, p^n + m - 2)$ we have $\langle \lambda, \beta^\vee \rangle = p^n + m - 2$. If $m = 1$ then the above result is true by Proposition 3. If $m > 1$ we write (base p-expansion) $p^n + m - 2 = \sum_{j=0}^{r} a_j p^j$. So by Proposition 1 case 2, we have

$$H^1(p^n - 1, -p^n - m) \neq 0 \Longleftrightarrow (p^n - 1, -p^n - m) + a_r p^r(-1, 2) \in X^+(T),$$

which is true if and only if $p^n \geq a_r p^r + 1$ and $2a_r p^r \geq p^n + m$. The first inequality gives us $n > r$ but from $n > r$ we have $2a_r p^r \geq p^n + m$ is never true. Hence the result. □

The following two results describe the cohomology for the module $N_\beta(\lambda)$, where $\beta = (-1, 2) \in S$

Proposition 6. *Let n be a non-negative integer and $m \geq 2$. Then*

$$H^1(N_\beta(p^n - 1, -p^n - m)) = H^1(p^n, -p^n - m - 2).$$

Proof. The module $N_\beta(\lambda)$ gives the following short exact sequence

$$0 \to (p^n, -p^n - m - 2) \to N_\beta(p^n - 1, -p^n - m) \to (p^n - 1, -p^n - m) \to 0.$$

Moreover $H^0(p^n - 1, -p^n - m) = 0$ and $H^1(p^n - 1, -p^n - m) = 0$ from Proposition 5. Using the long exact sequence of induction we get

$$0 \to H^1(p^n, -p^n - m - 2) \to H^1(N_\beta(p^n - 1, -p^n - m)) \to 0.$$

and hence the result. □

Proposition 7. *Let n be a non-negative integer and m be a positive integer. We have*

$$H^i(N_\beta(p^n - m, -p^n - 1)) = H^i(p^n - m + 1, -p^n - 3).$$

Proof. The short exact sequence for $N_\beta(\lambda)$ gives

$$0 \to (p^n - m + 1, -p^n - 3) \to N_\beta(p^n - m, -p^n - 1) \to (p^n - m, -p^n - 1) \to 0.$$

Moreover $H^i(p^n - m, -p^n - 1) = 0$ for each i (using proposition 3). Therefore from the long exact sequence of induction we get

$$0 \to H^i(p^n - m + 1, -p^n - 3) \to H^i(N_\beta(p^n - m, -p^n - 1)) \to 0.$$

and hence the result. □

3. Conclusions

In this paper we computed the cohomology of line bundles on the flag variety G/B for $G = SL_3(k)$ for certain weights. The problem of computing the computing this cohomology for the weights not mentioned here is still open. The problem is also open for every linear algebraic group except $SL_2(k)$.

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