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Remarks on the Generalized Fractional Laplacian Operator

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Abstract: The fractional Laplacian, also known as the Riesz fractional derivative operator, describes an unusual diffusion process due to random displacements executed by jumpers that are able to walk to neighbouring or nearby sites, as well as perform excursions to remote sites by way of Lévy flights. The fractional Laplacian has many applications in the boundary behaviours of solutions to differential equations. The goal of this paper is to investigate the half-order Laplacian operator $(-\Delta)^{\frac{1}{2}}$ in the distributional sense, based on the generalized convolution and Temple's delta sequence. Several interesting examples related to the fractional Laplacian operator of order 1/2 are presented with applications to differential equations, some of which cannot be obtained in the classical sense by the standard definition of the fractional Laplacian via Fourier transform.

Keywords: distribution; fractional Laplacian; Riesz fractional derivative; delta sequence; convolution

MSC: 46F10; 26A33

In recent years, the fractional Laplacian operator has gained considerable attention due to its applications in many disciplines, such as partial differential equations, long-range interactions, anomalous diffusions and non-local quantum theories. There is also the physical meaning of the fractional Laplacian operator in bounded domains through its associated stochastic processes. However, the half-order Laplacian operator $(-\Delta)^{\frac{1}{2}}$, often appearing in various literature works and applications, needs to be studied carefully as the first-order Riesz derivative is undefined in the classical sense. The goal of this work is to use a new distributional approach to defining operator $(-\Delta)^{\frac{1}{2}}$ in the generalized sense by Temple's delta sequence, as well as present fresh techniques in computing examples of the fractional Laplacian operator of order 1/2 and applications to solving partial differential equations related to this operator.

1. Introduction

Let $s \in (0, 1)$ and $\Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_n^2$. The fractional Laplacian of a function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as:

$$(-\Delta)^s u(x) = C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(\zeta)}{|x - \zeta|^{n+2s}} d\zeta, \quad (1)$$

where P.V. stands for the Cauchy principal value, and the constant $C_{n,s}$ is given by:

$$C_{n,s} = \left(\int_{\mathbb{R}^n} \frac{1 - \cos y_1}{|y|^{n+2s}} dy \right)^{-1} = \pi^{-n/2} 2^{2s} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)} s,$$

and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

On the other hand, the fractional Laplacian in R^n can be written by the Fourier transform:

$$(-\Delta)^s u(x) = \frac{1}{(2\pi)^n} \int_{R^n} |\zeta|^{2s} (u, e^{-i\zeta x})_{L^2} e^{i\zeta x} d\zeta = \mathcal{F}^{-1}\{|\zeta|^{2s} \mathcal{F}(u)(\zeta)\}(x)$$

where:

$$(u, e^{-i\zeta x})_{L^2} = \int_{R^n} u(x) e^{-i\zeta x} dx, \quad \hat{u}(\zeta) = \mathcal{F}\{u\}(\zeta) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} u(x) e^{-i\zeta x} dx,$$

$$\mathcal{F}^{-1}\{\hat{u}\}(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} \hat{u}(\zeta) e^{i\zeta x} d\zeta.$$

Hence, the fractional Laplacian is really a pseudo-differential operator with symbol $|\zeta|^{2s}$.

Let L^p be the Lebesgue space with $p \in [1, \infty)$, B_0 be the space of continuous functions vanishing at infinity, and B_{bu} be the space of bounded uniformly-continuous functions. M.Kwaśnicki recently presented ten equivalent definitions in [1] for defining $(-\Delta)^s$ over these three spaces, including the above Fourier definition.

There have been many studies, including numerical analysis approaches, on the fractional Laplacian with applications in solving certain differential equations on bounded domains and in the theory of stochastic processes and anomalous diffusion [2–10]. For example, the work in [11] used the fractional Laplacian for linear and nonlinear lossy media, as well as studying a linear integro-differential equation wave model. The work in [12] studied a finite difference method of solving parabolic partial integro-differential equations, with possibly singular kernels. These arise in option pricing theory when the random evolution of the underlying asset is driven by a Lévy process related to the fractional Laplacian or, more generally, a time-inhomogeneous jump-diffusion process. Using the fractional Laplacian operator, Araci et al. [13] investigated the following q -difference boundary value problem:

$$D_q^\gamma(\phi_p(D_q^\delta y(t))) + f(t, y(t)) = 0, \quad (0 < t < 1; \quad 3 < \delta < 4)$$

with the boundary conditions:

$$y(0) = (D_q y)(0) = (D_q^2 y)(0),$$

$$a_1(D_q y)(1) + a_2(D_q^2 y)(1) = 0, \quad \text{and} \quad D_{0+}^\gamma y(t)|_{t=0} = 0.$$

They proved the existence and uniqueness of a positive and nondecreasing solution for this problem by means of a fixed point theorem involving partially-ordered sets.

Let $\Omega \subset R^n$ denote a bounded, open domain. For $u(x): \Omega \rightarrow R$, D’Elia and Gunzburger [14] investigated the action of the nonlocal diffusion operator \mathcal{L} on the function $u(x)$ as:

$$\mathcal{L}u(x) = 2 \int_{R^n} (u(y) - u(x)) \gamma(x, y) dy \quad \forall x \in \Omega \subseteq R^n,$$

where the volume of Ω is non-zero and the kernel $\gamma(x, y): \Omega \times \Omega \rightarrow R$ is a non-negative symmetric mapping, as well as the nonlocal, steady-state diffusion equation:

$$-\mathcal{L}u = f \quad \text{on } \Omega,$$

$$u = 0 \quad \text{on } R^n \setminus \Omega.$$

An example of $\gamma(x, y)$ is given by:

$$\gamma(x, y) = \frac{\sigma(x, y)}{|y - x|^{n+2s}}$$

with $\sigma(x, y)$ bounded from above and below by positive constants. This nonlocal diffusion operator has, as special cases, the fractional Laplacian and fractional differential operators that arise in several applications. The corresponding evolution model was further studied in [15]. Recently, Hu et al. [16] studied the following high-dimensional Caputo-type parabolic equation with the fractional Laplacian by using the finite difference method:

$$\begin{aligned} {}_C D_{0,t}^\alpha u &= -(-\Delta)^s u + f, \quad x \in \Omega, \quad t > 0, \\ u(x, 0) &= u_0(x), \\ u(x, t) &= 0 \quad \text{on } x \in R^n \setminus \Omega, \end{aligned}$$

where $\alpha \in (0, 1), s \in (0, 1)$ and $\Omega \subset R^n$. In particular, this involves the half-order Laplacian operator $(-\Delta)^{\frac{1}{2}}$ when $s = 1/2$. The convergence and error estimate of the established finite difference scheme are shown with several examples.

On the other hand, the Riesz fractional derivative is generally given as:

$${}_{RZ} D_x^\alpha u(x) = -\frac{({}_{RL} D_{-\infty,x}^\alpha + {}_{RL} D_{\infty,x}^\alpha)u(x)}{2 \cos(\alpha\pi/2)} \tag{2}$$

where $0 < \alpha < 2$ and $\alpha \neq 1$, and:

$$\begin{aligned} {}_{RL} D_{-\infty,x}^\alpha u(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^x (x-\zeta)^{n-\alpha-1} u(\zeta) d\zeta, \\ {}_{RL} D_{\infty,x}^\alpha u(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{\infty}^x (x-\zeta)^{n-\alpha-1} u(\zeta) d\zeta \end{aligned}$$

for $n-1 < \alpha < n \in Z^+$.

Note that in space fractional quantum mechanics, the $\alpha = 2$ case corresponds to the Schrödinger equation for a massive non-relativistic particle, while the $\alpha = 1$ case needs to be examined carefully, both on physical and mathematical grounds, since Equation (2) is undefined for $\alpha = 1$.

It follows from [10,17–22] that:

$${}_{RL} D_{-\infty,x}^\alpha u(x) = \frac{d^2}{dx^2} [I_{-\infty,x}^{2-\alpha} u(x)], \quad 1 < \alpha < 2$$

and:

$$\begin{aligned} {}_{RL} D_{-\infty,x}^\alpha u(x) &= I_{-\infty,x}^{-\alpha} u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{-\infty}^x \frac{u(t)}{(x-t)^{\alpha-1}} dt \\ &= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_0^\infty \zeta^{-\alpha+1} u(x-\zeta) d\zeta \end{aligned}$$

by making the variable change $\zeta = x - t$.

Applying two identities:

$$\begin{aligned} \zeta^{-\alpha+1} &= (\alpha-1) \int_\zeta^\infty \frac{d\eta}{\eta^\alpha}, \\ \frac{\partial^2 u(x-\zeta)}{\partial x^2} &= \frac{\partial^2 u(x-\zeta)}{\partial \zeta^2} \end{aligned}$$

we get:

$$\begin{aligned}
 {}_{RL}D_{-\infty,x}^\alpha u(x) &= \frac{\alpha - 1}{\Gamma(2 - \alpha)} \int_0^\infty \frac{\partial^2 u(x - \zeta)}{\partial \zeta^2} \left[\int_\zeta^\infty \frac{d\eta}{\eta^\alpha} \right] d\zeta \\
 &= \frac{1 - \alpha}{\Gamma(2 - \alpha)} \int_0^\infty \left[\int_\zeta^\infty \frac{d\eta}{\eta^\alpha} \right] d \frac{\partial u(x - \zeta)}{\partial \zeta} \\
 &= \frac{1}{\Gamma(1 - \alpha)} \frac{\partial u(x - \zeta)}{\partial \zeta} \int_\zeta^\infty \frac{d\eta}{\eta^\alpha} \Big|_{\zeta=0}^\infty + \frac{1}{\Gamma(1 - \alpha)} \int_0^\infty \frac{\partial u(x - \zeta)}{\partial \zeta} \frac{1}{\zeta^\alpha} d\zeta \\
 &= -\frac{1}{\Gamma(1 - \alpha)} \int_0^\infty \frac{1}{\zeta^\alpha} du(x - \zeta) \\
 &= -\frac{1}{\Gamma(1 - \alpha)} \frac{u(x - \zeta)}{\zeta^\alpha} \Big|_{\zeta=0}^\infty - \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{u(x - \zeta)}{\zeta^{\alpha+1}} d\zeta \\
 &= \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{u(x)}{\zeta^{\alpha+1}} d\zeta - \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{u(x - \zeta)}{\zeta^{\alpha+1}} d\zeta \\
 &= -\frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{u(x - \zeta) - u(x)}{\zeta^{\alpha+1}} d\zeta, \quad 1 < \alpha < 2
 \end{aligned}$$

by removing the term:

$$\int_0^\infty \frac{d\eta}{\eta^\alpha},$$

due to the meaning of the finite part integral as the integral is divergent and the finite part:

$$\frac{\eta^{-\alpha+1}}{-\alpha + 1} \Big|_{\eta=\infty} = 0.$$

With the same argument, we come to:

$${}_{RL}D_{\infty,x}^\alpha u(x) = -\frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{u(x + \zeta) - u(x)}{\zeta^{\alpha+1}} d\zeta, \quad 1 < \alpha < 2.$$

Hence, we have another representation of the Riesz fractional derivative from Equation (2):

$${}_{RZ}D_x^\alpha u(x) = \frac{\Gamma(1 + \alpha) \sin \alpha\pi/2}{\pi} \int_0^\infty \frac{u(x + \zeta) - 2u(x) + u(x - \zeta)}{\zeta^{1+\alpha}} d\zeta, \quad 1 < \alpha < 2.$$

Similarly, we can claim that this representation still holds for the entire range $0 < \alpha \leq 2$ [23]. In particular,

$${}_{RZ}D_x^1 u(x) = \frac{1}{\pi} \int_0^\infty \frac{u(x + \zeta) - 2u(x) + u(x - \zeta)}{\zeta^2} d\zeta = \frac{1}{\pi} (\text{P.V.} \frac{1}{\zeta^2}, u(x + \zeta))$$

based on the formula:

$$(\text{P.V.} \frac{1}{x^2}, \phi(x)) = \int_0^\infty \frac{\phi(x) - 2\phi(0) + \phi(-x)}{x^2} dx.$$

Clearly, Equation (1) becomes:

$$\begin{aligned}
 (-\Delta)^{1/2}u(x) &= C_{1,1/2} \text{P.V.} \int_R \frac{u(x) - u(\zeta)}{|x - \zeta|^2} d\zeta \\
 &= \frac{1}{\pi} \text{P.V.} \int_R \frac{u(x) - u(\zeta)}{(x - \zeta)^2} d\zeta \\
 &= -\frac{1}{\pi} \int_0^\infty \frac{u(x + \zeta) - 2u(x) + u(x - \zeta)}{\zeta^2} d\zeta \tag{3}
 \end{aligned}$$

$$= -{}_{RZ}D_x^1 u(x) \tag{4}$$

for $s = 1/2$ and $n = 1$.

Therefore, investigations of the half-order Laplacian operator $(-\Delta)^{\frac{1}{2}}$ on R are equivalent to studies of the first-order Riesz derivative. We can define $-(-\Delta)^{\frac{1}{2}}u$ as the Riesz derivative ${}_{RZ}D_x^\alpha u(x)$ in the case of $\alpha = 1$, which is undefined in Equation (2) in the classical sense. The aim of this work is to study the operator $(-\Delta)^{\frac{1}{2}}$ on R in distribution explicitly and implicitly, using a particular delta sequence and the generalized convolution. We also present several interesting examples, such as $(-\Delta)^{\frac{1}{2}}\delta(x)$ and $(-\Delta)^{\frac{1}{2}}\theta(x)$, which are undefined in the classical sense. At the end of this work, we describe applications of such studies to solving the differential equations involving the half-order Laplacian operator.

2. The Explicit Approach to $(-\Delta)^{1/2}u$

In order to extend the fractional Laplacian $(-\Delta)^{1/2}$ distributionally, we briefly introduce the following basic concepts of distributions. Let $\mathcal{D}(R)$ be the Schwartz space (testing function space) [24,25] of infinitely-differentiable functions with compact support in R and $\mathcal{D}'(R)$ the (dual) space of distributions defined on $\mathcal{D}(R)$. A sequence $\phi_1, \phi_2, \dots, \phi_n, \dots$ goes to zero in $\mathcal{D}(R)$ if and only if these functions vanish outside a certain fixed bounded set and converge to zero uniformly together with their derivatives of any order.

The functional $\delta^{(n)}(x - x_0)$ is defined as:

$$(\delta^{(n)}(x - x_0), \phi(x)) = (-1)^n \phi^{(n)}(x_0)$$

where $\phi \in \mathcal{D}(R)$. Clearly, $\delta^{(n)}(x - x_0)$ is a linear and continuous functional on $\mathcal{D}(R)$, and hence, $\delta^{(n)}(x - x_0) \in \mathcal{D}'(R)$.

Define the unit step function $\theta(x)$ as:

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then,

$$(\theta(x), \phi(x)) = \int_0^\infty \phi(x) dx \quad \text{for } \phi \in \mathcal{D}(R),$$

which implies $\theta(x) \in \mathcal{D}'(R)$.

Let $f \in \mathcal{D}'(R)$. The distributional derivative of f , denoted by f' or df/dx , is defined as:

$$(f', \phi) = -(f, \phi')$$

for $\phi \in \mathcal{D}(R)$.

Clearly, $f' \in \mathcal{D}'(R)$ and every distribution has a derivative. As an example, we are going to show that $\theta'(x) = \delta(x)$ distributionally, although $\theta(x)$ is not defined at $x = 0$ in the classical sense. Indeed,

$$(\theta'(x), \phi(x)) = -(\theta(x), \phi'(x)) = -\int_0^\infty \phi'(x)dx = \phi(0) = (\delta(x), \phi(x)),$$

which claims:

$$\theta'(x) = \delta(x).$$

It can be shown that the ordinary rules of differentiation also apply to distributions. For instance, the derivative of a sum is the sum of the derivatives, and a constant can be commuted with the derivative operator.

Definition 1. Let f and g be distributions in $\mathcal{D}'(R)$ satisfying either of the following conditions:

- (a) either f or g has bounded support (set of all essential points).
- (b) the supports of f and g are bounded on the same side (either on the left or right).

Then, the convolution $f * g$ is defined by the equation:

$$((f * g)(x), \phi(x)) = (g(x), (f(y), \phi(x + y)))$$

for $\phi \in \mathcal{D}$. Clearly, we have:

$$f * g = g * f$$

from Definition 1.

It follows from the definition above that:

$$\delta^{(n)}(x - x_0) * f(x) = f^{(n)}(x - x_0)$$

for any distribution $f \in \mathcal{D}'(R)$.

Let $\delta_n(x) = n\rho(nx)$ be Temple's δ -sequence for $n = 1, 2, \dots$, where $\rho(x)$ is a fixed, infinitely-differentiable function on R , having the following properties [26,27]:

- (i) $\rho(x) \geq 0$,
- (ii) $\rho(x) = 0$ for $|x| \geq 1$,
- (iii) $\rho(x) = \rho(-x)$,
- (iv) $\int_{-\infty}^\infty \rho(x)dx = 1$.

An example of such a $\rho(x)$ function is given as:

$$\rho(x) = \begin{cases} ce^{-\frac{1}{1-x^2}} & \text{if } |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

where:

$$c^{-1} = \int_{-1}^1 e^{-\frac{1}{1-x^2}} dx.$$

This delta-sequence plays an important role in defining products of distributions [28,29]. Let f be a continuous function on R . Then,

$$(f * \delta_n)(x) = \int_{-\infty}^\infty f(x - t)\delta_n(t)dt = \int_{-\infty}^\infty f(t)\delta_n(x - t)dt$$

uniformly converges to f on any compact subset of R . Indeed, we assume that f is continuous on R , and L is any compact subset of R . Then, f is uniformly continuous on L , and for all $\epsilon > 0$, there exists $\delta_1 > 0$ such that:

$$|f(x - t) - f(x)| < \epsilon$$

for all $x \in L$ and $|t| < \delta_1$. This implies that:

$$|(f * \delta_n)(x) - f(x)| \leq \int_{-\infty}^{\infty} |f(x - t) - f(x)| \delta_n(t) dt < \epsilon$$

for all $x \in L$ and $1/n < \delta_1$ by noting that:

$$\int_{-\infty}^{\infty} \delta_n(t) dt = 1.$$

Furthermore, if $f \in \mathcal{D}'(R)$, then $(f * \delta_n)(x)$ converges to f in $\mathcal{D}'(R)$. Indeed,

$$\lim_{n \rightarrow \infty} ((f * \delta_n)(x), \phi(x)) = \lim_{n \rightarrow \infty} (f(x), (\delta_n(y), \phi(x + y))) = (f(x), \phi(x))$$

by noting that:

$$\lim_{n \rightarrow \infty} (\delta_n(y), \phi(x + y)) = \phi(x)$$

in $\mathcal{D}(R)$.

In order to study the half-order Laplacian operator in the distribution, we introduce an infinitely-differentiable function $\tau(x)$ satisfying the following conditions:

- (i) $\tau(x) = \tau(-x)$,
- (ii) $0 \leq \tau(x) \leq 1$,
- (iii) $\tau(x) = 1$ if $|x| \leq 1/2$,
- (iv) $\tau(x) = 0$ if $|x| \geq 1$.

Define:

$$\phi_m(x) = \begin{cases} 1 & \text{if } |x| \leq m, \\ \tau(m^m x - m^{m+1}) & \text{if } x > m, \\ \tau(m^m x + m^{m+1}) & \text{if } x < -m, \end{cases}$$

for $m = 1, 2, \dots$. Then, $\phi_m(x) \in \mathcal{D}(R)$ with the support $[-m - m^{-m}, m + m^{-m}]$.

From Equation (3), we have:

$$(-\Delta)^{1/2} u(x) = -\frac{1}{\pi} (\text{P.V.} \frac{1}{t^2}, u(x + t)) = -\frac{1}{\pi} (\text{P.V.} \frac{1}{t}, u'(x + t))$$

if $u \in \mathcal{D}(R)$ as:

$$\begin{aligned} (\text{P.V.} \frac{1}{t^2}, \phi(t)) &= \int_0^{\infty} \frac{\phi(t) - 2\phi(0) + \phi(-t)}{t^2} dt, \\ \frac{d}{dt} \text{P.V.} \frac{1}{t} &= -\text{P.V.} \frac{1}{t^2}. \end{aligned}$$

This suggests the following explicit definition for defining $(-\Delta)^{1/2} u(x)$. This explicit definition directly evaluates the half-order fractional Laplacian of $u(x)$ as a function of x , without relating to any testing function in the Schwartz space.

Definition 2. Let $u \in \mathcal{D}'(R)$ and $u'_n = u' * \delta_n = (u'(t), \delta_n(x - t))$ for $n = 1, 2, \dots$. We define the half-order Laplacian operator $(-\Delta)^{\frac{1}{2}}$ on $\mathcal{D}'(R)$ as:

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}u(x) &= -\frac{1}{\pi} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \left(P.V. \frac{1}{t}, \phi_m(t) u'_n(x + t) \right) \\ &= -\frac{1}{\pi} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_0^\infty \phi_m(t) \frac{u'_n(x + t) - u'_n(x - t)}{t} dt \end{aligned} \tag{5}$$

if it exists.

Clearly, the integral:

$$\int_0^\infty \phi_m(t) \frac{u'_n(x + t) - u'_n(x - t)}{t} dt$$

is well defined as $\phi_m(t)$ has a bounded support and:

$$\lim_{t \rightarrow 0^+} \frac{u'_n(x + t) - u'_n(x - t)}{t} = u''_n(x).$$

Theorem 1.

$$(-\Delta)^{\frac{1}{2}}\delta(x) = -\frac{1}{\pi} P.V. \frac{1}{x^2}.$$

Proof. Assuming $x > 0$, we choose a large n such that $1/n < x$. This infers that $\delta'_n(x + t) = 0$, and:

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}\delta(x) &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{\delta'_n(x + t) - \delta'_n(x - t)}{t} dt \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{\delta'_n(x - t)}{t} dt \end{aligned}$$

from Definition 2. Making the substitution $w = x - t$, we get:

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}\delta(x) &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^x \delta'_n(w) \frac{\phi_m(x - w)}{x - w} dw \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^\infty \delta'_n(w) \frac{\phi_m(x - w)}{x - w} dw \\ &= \lim_{m \rightarrow \infty} -\frac{1}{\pi} \frac{\partial}{\partial w} \left[\frac{\phi_m(x - w)}{x - w} \right] \Big|_{w=0} \\ &= \lim_{m \rightarrow \infty} \frac{\phi'_m(x)x - \phi_m(x)}{\pi x^2} \end{aligned}$$

by noting that $\delta_n(w)$ is a delta sequence, and:

$$\frac{\phi_m(x - w)}{x - w}$$

is a testing function of w if $w < x$, and:

$$\delta'_n(w) \frac{\phi_m(x - w)}{x - w}$$

is identical to zero if $w \geq x$ as $\delta'_n(w) = 0$. Choosing m such that $x < m$, we derive that:

$$\phi'_m(x) = 0 \quad \text{and} \quad \phi_m(x) = 1$$

as $\phi_m(x) = 1$ if $|x| < m$. Hence,

$$(-\Delta)^{\frac{1}{2}}\delta(x) = -\frac{1}{\pi} P.V. \frac{1}{x^2}.$$

If $x < 0$, we set $y = -x$ and:

$$\begin{aligned} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{\delta'_n(x+t) - \delta'_n(x-t)}{t} dt &= -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{\delta'_n(-y+t) - \delta'_n(-y-t)}{t} dt \\ &= -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{\delta'_n(y+t) - \delta'_n(y-t)}{t} dt, \end{aligned}$$

which implies that:

$$(-\Delta)^{\frac{1}{2}}\delta(x) = \lim_{m \rightarrow \infty} \frac{\phi'_m(y)y - \phi_m(y)}{\pi y^2} = -\frac{1}{\pi} P.V. \frac{1}{x^2}.$$

Finally, we have for $x = 0$ by making the variable change $u = nt$:

$$\begin{aligned} \int_0^\infty \phi_m(t) \frac{\delta'_n(t) - \delta'_n(-t)}{t} dt &= 2 \int_0^\infty \phi_m(t) \frac{\delta'_n(t)}{t} dt \\ &= 2n^2 \int_0^\infty \phi_m(t) \frac{\rho'(nt)}{t} dt \\ &= 2n^2 \int_0^1 \phi_m(u/n) \frac{\rho'(u)}{u} du = 2n^2 \int_0^1 \frac{\rho'(u)}{u} du \\ &= O(n^2) \end{aligned}$$

by noting that the term $\phi_m(u/n) = 1$, and:

$$\int_0^1 \frac{\rho'(u)}{u} du = \int_0^1 \frac{\rho'(u) - \rho'(0)}{u} du$$

is well defined as $\rho(u)$ is an even function. This completes the proof of Theorem 1. \square

From [24], we have the distributions $P.V.x^{-2m}$ for $m = 1, 2, \dots$ and $P.V.x^{-2m-1}$ for $m = 0, 1, \dots$ given as:

$$\begin{aligned} (P.V.x^{-2m}, \phi) &= \int_0^\infty x^{-2m} \{\phi(x) + \phi(-x) \\ &\quad - 2 \left[\phi(0) + \frac{x^2}{2!} + \dots + \frac{x^{2m-2}}{(2m-2)!} \phi^{(2m-2)}(0) \right]\} dx, \\ (P.V.x^{-2m-1}, \phi) &= \int_0^\infty x^{-2m-1} \{\phi(x) - \phi(-x) \\ &\quad - 2 \left[x\phi'(0) + \frac{x^3}{3!} + \dots + \frac{x^{2m-1}}{(2m-1)!} \phi^{(2m-1)}(0) \right]\} dx. \end{aligned}$$

Theorem 2.

$$(-\Delta)^{\frac{1}{2}}\delta^{(m)}(x) = \frac{(-1)^{m+1}(m+1)!}{\pi} P.V. \frac{1}{x^{m+2}}$$

for $m = -1, 0, 1, \dots$. In particular, we have for $m = -1$ that:

$$(-\Delta)^{\frac{1}{2}}\theta(x) = \frac{1}{\pi} P.V. \frac{1}{x}.$$

Proof. We start with the case $m = -1$. Then, by Definition 2,

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}\theta(x) &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{\delta_n(x+t) - \delta_n(x-t)}{t} dt \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{\delta_n(x-t)}{t} dt \end{aligned}$$

for $x > 0$ and a large n such that $1/n < x$. Following the proof of Theorem 1, we come to:

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}\theta(x) &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^x \delta_n(w) \frac{\phi_m(x-w)}{x-w} dw \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^\infty \delta_n(w) \frac{\phi_m(x-w)}{x-w} dw \\ &= \lim_{m \rightarrow \infty} \frac{\phi_m(x)}{\pi x} \\ &= \frac{1}{\pi} \text{P.V.} \frac{1}{x}. \end{aligned}$$

The case $x < 0$ follows similarly. To compute $(-\Delta)^{\frac{1}{2}}\delta^{(m)}(x)$, we note that for a large n :

$$(-\Delta)^{\frac{1}{2}}\delta^{(m)}(x) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\pi} \int_{-\infty}^\infty \delta_n^{(m+1)}(w) \frac{\phi_m(x-w)}{x-w} dw$$

from the proof of Theorem 1 and:

$$\begin{aligned} &\lim_{m \rightarrow \infty} \frac{(-1)^{m+1}}{\pi} \frac{\partial^{m+1}}{\partial w^{m+1}} \left[\frac{\phi_m(x-w)}{x-w} \right] \Big|_{w=0} \\ &= \lim_{m \rightarrow \infty} \frac{(-1)^{m+1}}{\pi} \sum_{k=0}^{m+1} \binom{m+1}{k} (-1)^k \phi_m^{(k)}(x-w) \left(\frac{1}{x-w} \right)^{(m+1-k)} \Big|_{w=0} \\ &= \lim_{m \rightarrow \infty} \frac{(-1)^{m+1}}{\pi} \binom{m+1}{0} \phi_m(x-w) \left(\frac{1}{x-w} \right)^{(m+1)} \Big|_{w=0} \\ &= \frac{(-1)^{m+1}(m+1)!}{\pi} \text{P.V.} \frac{1}{x^{m+2}} \end{aligned}$$

since $\lim_{m \rightarrow \infty} \phi_m(x) = 1$ and:

$$\lim_{m \rightarrow \infty} \frac{(-1)^{m+1}}{\pi} \sum_{k=1}^{m+1} \binom{m+1}{k} (-1)^k \phi_m^{(k)}(x-w) \left(\frac{1}{x-w} \right)^{(m+1-k)} \Big|_{w=0} = 0$$

from the definition of $\phi_m(x)$. This completes the proof of Theorem 2. \square

We should note that Theorem 2 cannot be derived by the standard definition of the fractional Laplacian via Fourier transform, and:

$$(-\Delta)^{\frac{1}{2}}\theta(x) = \frac{1}{\pi} \text{P.V.} \frac{1}{x}$$

holds in the distributional sense and:

$$\frac{d}{dx}\theta(x) = \delta(x),$$

although $\theta(x)$ is discontinuous at $x = 0$ in the classical sense.

Theorem 3.

$$(-\Delta)^{\frac{1}{2}} \sin x = \sin x, \tag{6}$$

$$(-\Delta)^{\frac{1}{2}} \cos x = \cos x, \tag{7}$$

$$(-\Delta)^{\frac{1}{2}}(ax + b) = 0. \tag{8}$$

where a and b are arbitrary constants.

Proof. Clearly, $\sin x$ is an infinitely-differentiable function on R . This claims that:

$$(\sin x)' * \delta_n = \cos x * \delta_n$$

uniformly converges to $\cos x$ on any compact subset of R . Therefore,

$$\begin{aligned} (-\Delta)^{\frac{1}{2}} \sin x &= \lim_{m \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{\cos(x+t) - \cos(x-t)}{t} dt \\ &= \lim_{m \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{-2 \sin x \sin t}{t} dt \\ &= \frac{2 \sin x}{\pi} \int_0^\infty \frac{\sin t}{t} dt = \sin x \end{aligned}$$

by using:

$$\int_0^\infty \frac{\sin t}{t} dt = \pi/2.$$

Similarly,

$$\begin{aligned} (-\Delta)^{\frac{1}{2}} \cos x &= \lim_{m \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{-\sin(x+t) + \sin(x-t)}{t} dt \\ &= \lim_{m \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{-2 \cos x \sin t}{t} dt \\ &= \frac{2 \cos x}{\pi} \int_0^\infty \frac{\sin t}{t} dt = \cos x. \end{aligned}$$

Finally,

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}(ax + b) &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{(b * \delta_n)(x+t) - (b * \delta_n)(x-t)}{t} dt \\ &= \lim_{m \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{b-b}{t} dt = 0. \end{aligned}$$

This completes the proof of Theorem 3. \square

We would like to mention that Theorem 3 extends several classical results obtained in [30] to distributions by Definition 2 with a new approach.

Theorem 4.

$$(-\Delta)^{\frac{1}{2}} \arctan x = \frac{x}{1+x^2}.$$

Proof. Clearly, $(\arctan x)' = \frac{1}{1+x^2}$ is continuous on R , and

$$(\arctan x)' * \delta_n = \left(\frac{1}{1+t^2}, \delta_n(x-t) \right)$$

uniformly converges to $1/(1 + x^2)$ on any compact subset of R . Therefore,

$$\begin{aligned} (-\Delta)^{\frac{1}{2}} \arctan x &= \lim_{m \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{\frac{1}{1+(x+t)^2} - \frac{1}{1+(x-t)^2}}{t} dt \\ &= \frac{4x}{\pi} \lim_{m \rightarrow \infty} \int_0^\infty \phi_m(t) \frac{dt}{(1+(x-t)^2)(1+(x+t)^2)} \\ &= \frac{4x}{\pi} \int_0^\infty \frac{dt}{(1+(x-t)^2)(1+(x+t)^2)} \end{aligned}$$

by noting that the integral:

$$\int_0^\infty \frac{dt}{(1+(x-t)^2)(1+(x+t)^2)}$$

is well defined for every point $x \in R$. It remains to show that:

$$\int_0^\infty \frac{dt}{(1+(x-t)^2)(1+(x+t)^2)} = \frac{\pi}{4+4x^2}$$

for all $x \in R$. First, we note that:

$$R(z) = \frac{1}{(1+(x-z)^2)(1+(x+z)^2)}$$

is even with respect to z and has two singular points $z_1 = x + i$ and $z_2 = i - x$ in the upper half-plane. Clearly, we have for $x \neq 0$ that:

$$\begin{aligned} \text{Res}\{R(z), x + i\} &= \lim_{z \rightarrow x+i} \frac{z - x - i}{(1+(x-z)^2)(1+(x+z)^2)} = \frac{1}{2i(1+(2x+i)^2)}, \\ \text{Res}\{R(z), i - x\} &= \lim_{z \rightarrow i-x} \frac{z + x - i}{(1+(x-z)^2)(1+(x+z)^2)} = \frac{1}{2i(1+(2x-i)^2)}. \end{aligned}$$

By Cauchy’s residue theorem, we get:

$$\begin{aligned} \int_{-\infty}^\infty \frac{dt}{(1+(x-t)^2)(1+(x+t)^2)} &= 2\pi i [\text{Res}\{R(z), x + i\} + \text{Res}\{R(z), i - x\}] \\ &= \frac{2\pi i}{2i(1+(2x+i)^2)} + \frac{2\pi i}{2i(1+(2x-i)^2)} \\ &= \frac{8\pi x^2}{(1+(2x+i)^2)(1+(2x-i)^2)} \\ &= \frac{\pi}{2+2x^2} \end{aligned}$$

using:

$$(1+(2x+i)^2)(1+(2x-i)^2) = 16x^2 + 16x^4.$$

This implies that:

$$\int_0^\infty \frac{dt}{(1+(x-t)^2)(1+(x+t)^2)} = \frac{\pi}{4+4x^2}$$

for all nonzero x . Furthermore, we derive for $x = 0$ that:

$$\int_0^\infty \frac{dt}{(1+t^2)^2} = \frac{\pi}{4}$$

by using the identity [31]:

$$\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^n} = \frac{(2n-3)(2n-5)\cdots 1}{(2n-2)(2n-4)\cdots 2} \pi.$$

This completes the proof of Theorem 4. \square

3. The Implicit Approach to $(-\Delta)^{\frac{1}{2}}u$

It seems infeasible to calculate directly the fractional Laplacian operator of some functions or distributions by Definition 2. For example,

$$\begin{aligned} (-\Delta)^{\frac{1}{2}}e^x &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{u'_n(x+t) - u'_n(x-t)}{t} dt \\ &= \lim_{m \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \phi_m(t) \frac{e^{x+t} - e^{x-t}}{t} dt \\ &= -\frac{e^x}{\pi} \lim_{m \rightarrow \infty} \int_0^\infty \phi_m(t) \frac{e^t - e^{-t}}{t} dt \end{aligned}$$

where:

$$u_n(x) = (e^t, \delta_n(x-t))$$

uniformly converges to e^x as it is continuous on R . Clearly, the right-hand side of the above integral is divergent as:

$$\lim_{t \rightarrow \infty} \frac{e^t - e^{-t}}{t} = \infty.$$

In this section, we are going to provide another definition for dealing with $(-\Delta)^{\frac{1}{2}}u(x)$ efficiently, based on a testing function with compact support. This definition is implicit and only used to define the meaning of:

$$((-\Delta)^{\frac{1}{2}}u(x), \phi(x)),$$

rather than finding an explicit function of x . It clearly makes sense in the distribution as we regard $(-\Delta)^{\frac{1}{2}}u(x)$ as a functional (not a function) on the Schwartz testing space $\mathcal{D}(R)$. We must point out that this implicit-definition, using a different generalization, is independent of the explicit one provided in Section 2.

Definition 3. Let $u \in \mathcal{D}'(R)$ and $u'_n = u' * \delta_n = (u'(t), \delta_n(x-t))$ for $n = 1, 2, \dots$. We define the half-order Laplacian operator $(-\Delta)^{\frac{1}{2}}_i$ on $\mathcal{D}'(R)$ (adding the index i to distinguish from $(-\Delta)^{\frac{1}{2}}$) for $\phi \in \mathcal{D}(R)$ as:

$$\begin{aligned} ((-\Delta)^{\frac{1}{2}}_i u(x), \phi(x)) &= -\frac{1}{\pi} \lim_{n \rightarrow \infty} \left(P.V. \frac{1}{t}, \phi(t) u'_n(x+t) \right) \\ &= -\frac{1}{\pi} \lim_{n \rightarrow \infty} \int_0^\infty \frac{u'_n(x+t)\phi(t) - u'_n(x-t)\phi(-t)}{t} dt \end{aligned} \tag{9}$$

if it exists.

Clearly, the integral:

$$\int_0^\infty \frac{u'_n(x+t)\phi(t) - u'_n(x-t)\phi(-t)}{t} dt$$

is well defined, as $\phi(t)$ has a bounded support and:

$$\lim_{t \rightarrow 0^+} \frac{u'_n(x+t)\phi(t) - u'_n(x-t)\phi(-t)}{t} = 2u''_n(x)\phi(0) + 2u'_n(x)\phi'(0).$$

It follows from Definition 3 that:

$$((-\Delta)_i^{\frac{1}{2}} e^x, \phi(x)) = -\frac{e^x}{\pi} (\text{P.V.} \frac{1}{x}, e^x \phi(x))$$

by noting that:

$$(\text{P.V.} \frac{1}{x}, e^x \phi(x)) = \int_0^\infty \frac{e^x \phi(x) - e^{-x} \phi(-x)}{x} dx$$

is well defined and is a number for each ϕ , since $e^x \phi(x)$ is also a testing function in the Schwartz space.

Furthermore,

$$((-\Delta)_i^{\frac{1}{2}} \cosh x, \phi(x)) = -\frac{e^x}{2\pi} (\text{P.V.} \frac{1}{x}, e^x \phi(x)) + \frac{e^{-x}}{2\pi} (\text{P.V.} \frac{1}{x}, e^{-x} \phi(x)).$$

Theorem 5. Let $u(x)$ be an infinitely-differentiable function satisfying:

$$u(x) = \sum_{k=0}^\infty \frac{u^{(k)}(0)}{k!} x^k.$$

Then,

$$((-\Delta)_i^{\frac{1}{2}} u(x), \phi(x)) = -\frac{u'(x)}{\pi} (\text{P.V.} \frac{1}{x}, \phi(x)) - \frac{1}{\pi} \left(\frac{u'(x+t) - u'(x)}{t}, \phi(t) \right). \tag{10}$$

Proof. Clearly,

$$u(x+t) = \sum_{k=0}^\infty \frac{u^{(k)}(x)}{k!} t^k,$$

$$u'(x+t) = \sum_{k=0}^\infty \frac{u^{(k+1)}(x)}{k!} t^k$$

and:

$$u'_n(x) = (u' * \delta_n)(x) = (u'(t), \delta_n(x-t))$$

uniformly converges to $u(x)$ on any compact subset of R . By Definition 3,

$$\begin{aligned} ((-\Delta)_i^{\frac{1}{2}} u(x), \phi(x)) &= \lim_{n \rightarrow \infty} -\frac{1}{\pi} \int_0^\infty \frac{u'_n(x+t)\phi(t) - u'_n(x-t)\phi(-t)}{t} dt \\ &= -\frac{1}{\pi} \int_0^\infty \frac{u'(x+t)\phi(t) - u'(x-t)\phi(-t)}{t} dt \\ &= -\frac{1}{\pi} \int_0^\infty \frac{\sum_{k=0}^\infty \frac{u^{(k+1)}(x)}{k!} t^k \phi(t) - \sum_{k=0}^\infty \frac{u^{(k+1)}(x)}{k!} (-t)^k \phi(-t)}{t} dt \\ &= -\frac{1}{\pi} \int_0^\infty \frac{\phi(t)u'(x) - \phi(-t)u'(x)}{t} dt \\ &\quad - \frac{1}{\pi} \int_0^\infty \frac{\sum_{k=1}^\infty \frac{u^{(k+1)}(x)}{k!} t^k \phi(t) - \sum_{k=1}^\infty \frac{u^{(k+1)}(x)}{k!} (-t)^k \phi(-t)}{t} dt \\ &= -\frac{u'(x)}{\pi} (\text{P.V.} \frac{1}{x}, \phi(x)) - \frac{1}{\pi} \left(\frac{u'(x+t) - u'(x)}{t}, \phi(t) \right) \end{aligned}$$

since

$$\sum_{k=1}^\infty \frac{u^{(k+1)}(x)}{k!} t^k = u'(x+t) - u'(x).$$

We should note that the term:

$$\left(\frac{u'(x+t) - u'(x)}{t}, \phi(t) \right)$$

is well defined for every point $x \in R$, and it is indeed not Cauchy's principal value as:

$$\lim_{t \rightarrow 0} \frac{u'(x+t) - u'(x)}{t} = u''(x).$$

This completes the proof of Theorem 5. \square

It follows from Theorem 5 that:

$$((-\Delta)_i^{\frac{1}{2}}(ax + b), \phi(x)) = -\frac{a}{\pi} (\text{P.V.} \frac{1}{x}, \phi(x))$$

which implies that:

$$(-\Delta)_i^{\frac{1}{2}}(ax + b) = -\frac{a}{\pi} \text{P.V.} \frac{1}{x} \neq (-\Delta)^{\frac{1}{2}}(ax + b) = 0.$$

Furthermore,

$$((-\Delta)_i^{\frac{1}{2}} \sin x, \phi(x)) = -\frac{\cos x}{\pi} \left(\text{P.V.} \frac{\cos t}{t}, \phi(t) \right) + \frac{\sin x}{\pi} \left(\frac{\sin t}{t}, \phi(t) \right).$$

Remark 1. To end this section, we must point out that if we replace $\phi(t)$ by $\phi_m(t)$ used in Section 2 and add the limit, then the two operators $(-\Delta)_i^{\frac{1}{2}}$ and $(-\Delta)^{\frac{1}{2}}$ are identical for some functions. For instance,

$$\lim_{m \rightarrow \infty} ((-\Delta)_i^{\frac{1}{2}} \sin x, \phi_m(x)) = \frac{\sin x}{\pi} \int_{-\infty}^{\infty} \frac{\sin t}{t} dt = \sin x$$

as:

$$\left(\text{P.V.} \frac{\cos t}{t}, \phi_m(t) \right) = 0.$$

4. Conclusions

This paper introduced two independent definitions for defining the fractional Laplacian of the half-order $(-\Delta)^{\frac{1}{2}}$ in the distribution, both explicitly and implicitly. We demonstrate several examples, such as $(-\Delta)^{\frac{1}{2}} \delta^{(m)}(x)$ and $(-\Delta)^{\frac{1}{2}} \arctan x$, some of which are undefined in the classical sense. The results obtained have potential applications in solving the differential equations involving the half-order Laplacian operator. For example, the differential equation:

$$(-\Delta)^{\frac{1}{2}} u(x) = \frac{x}{1+x^2}$$

has a solution:

$$u(x) = \arctan x + ax + b$$

on any non-empty subset of R , and the differential equation:

$$(-\Delta)^{\frac{1}{2}} u(x) = \text{P.V.} \frac{1}{x}$$

has a solution:

$$u(x) = \pi\theta(x) + ax + b$$

where a and b are arbitrary constants.

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