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Best Proximity Point Theorems for Two Weak Cyclic Contractions on Metric-Like Spaces

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Abstract: In this paper, we establish two best proximity point theorems in the setting of metric-like spaces that are based on cyclic contraction: Meir–Keeler–Kannan type cyclic contractions and a generalized Ćirić type cyclic φ-contraction via the $\mathcal{MT}$-function. We express some examples to indicate the validity of the presented results. Our results unify and generalize a number of best proximity point results on the topic in the corresponding recent literature.

Keywords: best proximity point theorem; cyclic Meir–Keeler–Kannan type contraction; generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction; metric-like space

MSC: 47H10; 54C60; 54H25; 55M20

1. Introduction and Preliminaries

Fixed point theory provided not only the traditional tools but also the most crucial tools to prove the existence of solutions for several distinct and interesting problems both in pure and applied mathematics. For a self-mapping F on a non-empty set S, the equation $Fx = x$ is named a fixed point equation. If a fixed point equation possesses a solution, we say that F has a fixed point. For example, if F is linear operator, then the fixed point equation $Fx = x$ has no solution, or infinite solutions, or a unique solution. In this paper, we shall focus on the case that the fixed point equation $Fx = x$ has no solution, which is also to say that F is fixed point free. In the setting of a metric space (S, d), if F is fixed point free, then we have $d(x, Fx) > 0$ for all $x \in S$. It is quite natural to ask the following: If $d(x, Fx) > 0$ for all $x \in S$ is there $x^* \in S$ such that $d(x^*, Fx^*) \leq d(x, Fx)$ for all $x \in S$, that is, is there any point $x^* \in S$ such that $d(x^*, Fx^*)$ is the minimum throughout the domain of F? Roughly speaking, if we have no exact solution of the fixed point equation $Fx = x$, we look for the approximative solution of the fixed point equation $Fx = x$. If the answer is affirmative, the point $x^* \in S$ is named the best proximity point of the domain and range of the mapping F. In the last decades, this topic has been discussed densely by several authors, see, e.g., [1–12].

In what follows, we simply describe the research backgrounds and preliminaries. As it is used commonly, we shall denote the set of all non-negative real numbers by $\mathbb{R}_0^+$. Through the paper, instead of considering the whole metric space (S, d), we shall restrict ourselves to two nonempty subsets, A and B, of it. Further, instead of considering self-mapping, we shall consider non-self mapping $F : A \to B$. We formalize our consideration with

$$d(x, Fx) = d(A, B) := inf\{d(a, b) : a \in A, b \in B\}.$$

In the case of the existence of such $x \in A$ we shall say that x is the best proximity point of F for the pair (A, B). Commonly, the pair (A, B) is not mentioned and we only say that x is the best proximity point of F. Hence, x is an approximative solution of the fixed point equation $Fx = x$. Note that in case of $A \cap B \neq \emptyset$, the best proximity point coincides with fixed point. On the other hand, even if $A \cap B = \emptyset$, the corresponding fixed point equation still may not possess a solution.

First, we recall the notion of cyclic contraction.

Definition 1. *Ref. [13] A mapping $F : A \cup B \to A \cup B$ is called cyclic if $F(A) \subset B$ and $F(B) \subset A$, where A, B are nonempty subsets of a metric space (S, d). In addition, if there exists $k \in [0, 1)$ such that*

$$d(Fx, Fy) \leq kd(x, y) + (1 - k)d(A, B),$$

for all $x \in A$ and $y \in B$, then a mapping F is called cyclic contraction.

Now, we shall mention the result of Eldred and Veeramani [13] in which one of the initial results in this direction was given.

Theorem 1. *Ref. [13] Suppose that a mapping $F : A \cup B \to A \cup B$ is cyclic contraction, where A, B are nonempty, closed, and convex subsets of a metric space (S, d) and $k \in [0, 1)$. If we set $x_{n+1} = Fx_n$ for each $n \in \mathbb{N} \cup \{0\}$, for an arbitrary $x_0 \in A$, then there exists a unique $x \in A$ such that $x_{2n} \to x$ and $d(x, Tx) = d(A, B)$. That is, x is the best proximity point of T.*

In [14], Hitzler and Seda introduced a new notion of metric-like space. In what follows, we recall some notations and definitions:

Definition 2. *For a nonempty set X, distance function $\sigma : X \times X \to \mathbb{R}_0^+$ is named as metric-like (or dislocated) if for any $p, q, r \in X$, if the following conditions are fulfilled*

(1) $\sigma(p, q) = 0 \Rightarrow p = q$;
(2) $\sigma(p, q) = \sigma(q, p)$;
(3) $\sigma(q, r) \leq \sigma(q, p) + \sigma(p, r)$.

Here, we use the couple (X, σ) to describe "a metric-like space".

Let $\{p_n\}$ be a sequence in a metric-like space (X, σ). Then

(1) $\{p_n\}$ converges to $p \in X$ if and only if $\lim_{n\to\infty} \sigma(p_n, p) = \sigma(p, p)$.
(2) if $\lim_{n\to\infty} \sigma(p_n, p_m)$ exists and is finite, then we say that $\{p_n\}$ is fundamental (or, Cauchy)
(3) if each fundamental (Cauchy) sequence is convergent, then we say that (X, σ) is complete.

The characterization of a best proximity point of F in the setting of metric-like space (X, σ) is follows: Let $F : A \to B$ be a mapping where A and B be two nonempty subsets X. Consider the distance of the sets A and B:

$$\sigma(A, B) = inf\{\sigma(a, b) : a \in A,\ b \in B\}.$$

Then, $a \in A$ is called a best proximity point of F if $\sigma(a, Fa) = \sigma(A, B)$.

2. The Best Proximity Point Results of Meir–Keeler–Kannan Type Cyclic Contractions

A mapping $F : A \cup B \to A \cup B$ is called Kannan type cyclic contraction, if there exists $k \in (0, \frac{1}{2})$ such that

$$\sigma(Fp, Fq) \leq k[\sigma(p, Fp) + \sigma(q, Fq)] + (1 - 2k)\sigma(A, B),$$

for all $p \in A$, and $q \in B$ where A, B are nonempty subsets of a metric-like space (X, σ).

In 2016, Aydi and Felhi [15] established the following best proxmity point result for a Kannan type cyclic contraction.

Theorem 2. *Ref. [15] Suppose that a mapping $F : A \cup B \to A \cup B$ is a Kannan type cyclic contraction, where A, B are nonempty, closed subsets of a metric-like space (X, σ) and $k \in [0, \frac{1}{2})$. If we set $p_{n+1} = Fp_n$ for each $n \in \mathbb{N} \cup \{0\}$, for an arbitrary $p_0 \in A$, then*

$$\sigma(p_n, p_{n+1}) \to \sigma(A, B), \quad as \quad n \to \infty.$$

We have the following:

(1) *If $p_0 \in A$ and $\{p_{2n}\}$ has a subsequence $\{p_{2n_k}\}$ which converges to $x^* \in A$ with $\sigma(p^*, p^*) = 0$, then, $\sigma(p^*, Fp^*) = \sigma(A, B)$.*
(2) *If $p_0 \in B$ and $\{p_{2n-1}\}$ has a subsequence $\{p_{2n_k-1}\}$ which converges to $q^* \in B$ with $\sigma(q^*, q^*) = 0$, then $\sigma(q^*, Fq^*) = \sigma(A, B)$.*

A function $\xi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ called Meir–Keeler type (see, [16]), if

$$\forall \theta > 0 \ \ \exists \delta > 0 \ \ \forall t \in \mathbb{R}_0^+ \ \ (\theta \le t < \theta + \delta \Rightarrow \xi(t) < \theta).$$

We shall use $\mathcal{M}$ to denote set of all Meir–Keeler type function ξ. Note that if $\xi \in \mathcal{M}$ then, for all $t \in (0, \infty)$, we have

$$\xi(t) < t.$$

By using the Kannan type cyclic contraction and Meir–Keeler function, we define the new notion of Meir–Keeler–Kannan type cyclic contraction, as follows:

Definition 3. *Let $\phi \in \mathcal{M}$ and $T : A \cup B \to A \cup B$ be a cyclic mapping, where A and B be nonempty subsets of a metric-like space (X, σ). Then, the mapping T is said to be a Meir–Keeler–Kannan type cyclic contraction, if*

$$\sigma(Tx, Ty) - \sigma(A, B) \le \phi(\frac{\sigma(x, Tx) + \sigma(y, Ty)}{2} - \sigma(A, B)),$$

for all $x \in A$ and all $y \in B$.

In this section, we establish the best proximity point results of Meir–Keeler–Kannan type cyclic contraction. Our results generalize and improve Theorem 2.

Lemma 1. *Let $T : A \cup B \to A \cup B$ be a cyclic Meir–Keeler–Kannan type contraction, where A and B be nonempty closed subsets of a metric-like space (X, σ), and $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \in \mathcal{M}$ and it is increasing. For $x_0 \in A \cup B$, define $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$. Then*

$$\sigma(x_n, x_{n+1}) \to \sigma(A, B), \quad as \quad n \to \infty.$$

Proof. Since $T : A \cup B \to A \cup B$ is a Meir–Keeler–Kannan type cyclic contraction, we obtain that for each $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\sigma(x_{n+2}, x_{n+1}) - \sigma(A, B) \\
=&\sigma(Tx_{n+1}, Tx_n) - \sigma(A, B) \\
\le&\phi(\frac{\sigma(x_{n+1}, Tx_{n+1}) + \sigma(x_n, Tx_n)}{2} - \sigma(A, B)) \\
\le&\phi(\frac{\sigma(x_{n+1}, x_{n+2}) + \sigma(x_n, x_{n+1})}{2} - \sigma(A, B)).
\end{aligned}$$

Since $\phi \in \mathcal{M}$, we find that

$$\sigma(x_{n+2}, x_{n+1}) - \sigma(A,B) < \frac{\sigma(x_{n+1}, x_{n+2}) + \sigma(x_n, x_{n+1})}{2} - \sigma(A,B).$$

Attendantly, we deduce, for each $n \in \mathbb{N} \cup \{0\}$, that

$$\sigma(x_{n+2}, x_{n+1}) - \sigma(A,B) \leq \sigma(x_{n+1}, x_n) - \sigma(A,B).$$

In other words, $\{\sigma(x_{n+1}, x_n) - \sigma(A,B)\}$ is bounded below and monotone (non-increasing). Accordingly, there exists $\ell \geq 0$ such that

$$\sigma(x_n, x_{n+1}) - \sigma(A,B) \to \ell, \text{ as } n \to \infty.$$

Notice that

$$\ell = inf\{\sigma(x_n, x_{n+1}) - \sigma(A,B) : \, n \in \mathbb{N} \cup \{0\}\}.$$

In what follows, we assert that $\ell = 0$. Suppose, on the contrary, that $\ell > 0$. Keeping, $\phi \in \mathcal{M}$, in mind, corresponding to ℓ, there exist an η and a positive integer k_0 such that

$$\ell \leq \sigma(x_{k+1}, x_k) - d(A,B) \leq \ell + \eta, \text{ for all } k \geq k_0.$$

Since $T : A \cup B \to A \cup B$ is a Meir–Keeler–Kannan type cyclic contraction, we have

$$\begin{aligned}
&\sigma(x_{k+2}, x_{k+1}) - \sigma(A,B) \\
=&\sigma(Tx_{k+1}, Tx_k) - \sigma(A,B) \\
\leq&\phi(\frac{\sigma(x_{k+1}, Tx_{k+1}) + \sigma(x_k, Tx_k)}{2} - \sigma(A,B) \\
\leq&\phi(\frac{\sigma(x_{k+1}, x_{k+2}) + \sigma(x_k, x_{k+1})}{2} - \sigma(A,B)) \\
\leq&\phi(\sigma(x_{k+1}, x_k) - \sigma(A,B)) < \ell,
\end{aligned}$$

a contradiction. Consequently, we get $\ell = 0$, and we have

$$\sigma(x_n, x_{n+1}) - \sigma(A,B) \to 0, \text{ as } n \to \infty,$$

that is,

$$\sigma(x_n, x_{n+1}) \to \sigma(A,B), \text{ as } n \to \infty.$$

□

By Lemma 1, we shall derive the following result in the framework of best proximity theory.

Theorem 3. *Let $T : A \cup B \to A \cup B$ be a cyclic Meir–Keeler–Kannan type contraction, where A and B are nonempty closed subsets of a complete metric-like space (X, σ).*

If we set $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$, for an arbitrary $x_0 \in A \cup B$, then we have the following:

(1) *If $x_0 \in A$ and $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ which converges to $x^* \in A$ with $\sigma(x^*, x^*) = 0$, then $\sigma(x^*, Tx^*) = \sigma(A,B)$.*
(2) *If $x_0 \in B$ and $\{x_{2n-1}\}$ has a subsequence $\{x_{2n_k-1}\}$ which converges to $x_* \in B$ with $\sigma(x_*, x_*) = 0$, then $\sigma(x_*, Tx_*) = \sigma(A,B)$.*

Proof. Suppose $x_0 \in A$. Due to the fact that T is cyclic, we have $x_{2n} \in A$ and $x_{2n+1} \in B$ for all $n \in \mathbb{N} \cup \{0\}$. Next, if $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ which converges to $x^* \in A$ with $\sigma(x^*, x^*) = 0$, then

$$\lim_{n\to\infty}\sigma(x_{2n},x^*)=\sigma(x^*,x^*)=0.$$

Since $\phi\in\mathcal{M}$ is increasing, we have

$$\begin{aligned}\sigma(x^*,Tx^*)-\sigma(A,B)&\le\sigma(x^*,x_{2n_k})+\sigma(x_{2n_k},Tx^*)-\sigma(A,B)\\&\le\sigma(x^*,x_{2n})+\sigma(Tx_{2n_k-1},Tx^*)-\sigma(A,B)\\&\le\sigma(x^*,x_{2n_k})+\phi(\frac{\sigma(x_{2n_k-1},Tx_{2n_k-1})+\sigma(x^*,Tx^*)}{2}-\sigma(A,B))\\&\le\sigma(x^*,x_{2n_k})+\phi(\frac{\sigma(x_{2n_k-1},x_{2n_k})+\sigma(x^*,Tx^*)}{2}-\sigma(A,B))\\&<\sigma(x^*,x_{2n_k})+\frac{\sigma(x_{2n_k-1},x_{2n_k})+\sigma(x^*,Tx^*)}{2}-\sigma(A,B).\end{aligned}$$

We claim that $\sigma(x^*,Tx^*)-\sigma(A,B)=0$. If not, we assume that

$$\sigma(x^*,Tx^*)-\sigma(A,B)>0.$$

Letting $k\to\infty$, by Lemma 1, we find that

$$\sigma(x^*,Tx^*)-\sigma(A,B)<0+\frac{\sigma(A,B)+\sigma(x^*,Tx^*)}{2}-\sigma(A,B)=\frac{\sigma(x^*,Tx^*)-\sigma(A,B)}{2},$$

which implies a contradiction. Thus, $\sigma(x^*,Tx^*)=\sigma(A,B)$, that is, x^* is a best proximity point of T.

The proof of (2) is a verbatim of (1), thus we omit it. □

By using (Example 2.6, [15]), we give an example to support Theorem 3.

Example 1. *Let $X=\mathbb{R}_0^+\times\mathbb{R}_0^+$ be endowed with the metric-like $\sigma: X\times X\to\mathbb{R}_0^+$ defined by:*

$$\sigma((x_1,y_1),(x_2,y_2))=\begin{cases}|x_1-x_2|+|y_1-y_2|, & \text{if } (x_1,y_1),(x_2,y_2)\in[0,1]\times[0,1];\\ x_1+x_2+y_1+y_2, & \text{if not.}\end{cases}$$

Let $\phi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ be defined by:

$$\phi(t)=\frac{3t}{4}.$$

Clearly, (X,σ) is a complete metric-like space, and ϕ is an increasing Meir–Keeler function.

Take $A=[0,1]\times\{0\}$ and $B=[0,1]\times\{1\}$, and let $T: A\cup B\to A\cup B$ be defined by

$$T((x,0))=(\frac{1}{4}x,1),\ \text{for all}\ \ x\in[0,1],$$

and

$$T((x,1))=(\frac{1}{4}x,0),\ \text{for all}\ \ x\in[0,1].$$

Then we have $\sigma(A,B)=1$ and T is a cyclic mapping.

For $(x_1,0)\in A$ and $(x_2,1)\in B$, we have that $x_1,x_2\in[0,1]$ and

$$\begin{aligned}&\sigma(T((x_1,0)),T((x_2,1)))-\sigma(A,B)\\=&\sigma((\frac{1}{4}x_1,1),(\frac{1}{4}x_2,0))-1\\=&\frac{1}{4}|x_1-x_2|\\\le&\frac{1}{4}(x_1+x_2),\end{aligned}$$

and

$$\begin{aligned}
&\phi(\frac{\sigma((x_1,0),T((x_1,0)))+\sigma((x_2,1),T((x_2,1))}{2}-\sigma(A,B))\\
=&\phi(\frac{\sigma((x_1,0),(x_1,1))+\sigma((x_2,1),(x_2,0))}{2}-1)\\
=&\phi(\frac{\sigma((\frac{1}{4}x_1,0),(x_1,1))+\sigma((\frac{1}{4}x_2,1),(x_2,0))}{2}-1)\\
=&\phi(\frac{3}{8}(x_1+x_2))=\frac{9}{32}(x_1+x_2).
\end{aligned}$$

Then T is a Meir–Keeler–Kannan type cyclic contraction.

Let $\xi_0=(a,0)\in A$. Then for all $n\in\mathbb{N}\cup\{0\}$, we have

$$\xi_{2n+1}=T^{2n+1}((a,0))=(\frac{1}{4^{2n+1}}a,1)\in B,$$

and

$$\xi_{2n+2}=T^{2n+2}((a,0))=(\frac{1}{4^{2n+2}}a,0)\in A.$$

Thus, we get that as $n\to\infty$

$$\sigma(\xi_{2n+1},\xi_{2n+2})=|\frac{1}{4^{2n+2}}-\frac{1}{4^{2n+1}}|a+1\to 1=\sigma(A,B)$$

So, Lemma 1 holds and we also get that $(0,0)\in A$ and $(0,1)\in B$ are the two best proximity points of T.

3. The Best Proximity Point Results of a Generalized Ćirić Type Cyclic φ-Contraction via the $\mathcal{MT}$-Function

A mapping $T:A\cup B\to A\cup B$ is said to be a cyclic Ćirić type contraction if there exists $k\in(0,1)$ such that

$$\sigma(Tx,Ty)\le k\,max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty)\}+(1-k)\sigma(A,B),$$

for all $x\in A$, and $y\in B$, where A and B are nonempty subsets of a metric-like space (X,σ).

In 2016, Aydi and Felhi [15] established the following best proximity point result for the cyclic Ćirić type contraction.

Theorem 4. *Ref. [15] Let a mapping $T:A\cup B\to A\cup B$ be a cyclic Ćirić type contraction, where A and B are nonempty closed subsets of a complete metric-like space (X,σ). If we set $x_{n+1}=Tx_n$ for each $n\in\mathbb{N}\cup\{0\}$ and for an arbitrary $x_0\in A\cup B$, then*

$$\sigma(x_n,x_{n+1})\to\sigma(A,B),\ \ as\ \ n\to\infty.$$

We have the following:

(1) *If $x_0\in A$ and $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ which converges to $x^*\in A$ with $\sigma(x^*,x^*)=0$, then, $\sigma(x^*,Tx^*)=\sigma(A,B)$.*
(2) *If $x_0\in B$ and $\{x_{2n-1}\}$ has a subsequence $\{x_{2n_k-1}\}$ which converges to $x_*\in B$ with $\sigma(x_*,x_*)=0$, then, $\sigma(x_*,Tx_*)=\sigma(A,B)$.*

In what follows, we recall the notion of $\mathcal{MT}$-function (or, called the Reich's function).

Definition 4. *A function $\psi:\mathbb{R}^+\to[0,1)$ is said to be an $\mathcal{MT}$-function, if*

$$\limsup_{s\to t^+}\psi(s)=\inf_{\alpha>0}\sup_{0<s-t<\alpha}\psi(s)<1\ \ for\ all\ \ t\in\mathbb{R}^+.$$

In 2012, Du [17] proved the following theorem and remark.

Theorem 5. *Ref. [17] Let $\psi : \mathbb{R}^+ \to [0,1)$ be a function. Then the following two statements are equivalent.*

(a) *ψ is an $\mathcal{MT}$-function.*
(b) *For any non-increasing sequence $\{\ell_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^+$, we have*

$$0 \le \sup_{n\in\mathbb{N}} \psi(\ell_n) < 1.$$

Remark 1. *Ref. [17] It is obvious that if $\psi : \mathbb{R}^+ \to [0,1)$ is non-increasing or non-decreasing, then ψ is an $\mathcal{MT}$-function.*

In the sequel, assume that a function $\varphi : \mathbb{R}^{+^3} \to \mathbb{R}_0^+$ satisfies the following conditions:

(1) φ is an increasing, continuous function in each coordinate;
(2) for all $t > 0$, $\varphi(t,t,t) \le t$;
(3) $\varphi(t_1, t_2, t_3) = 0$ if and only if $t_1 = t_2 = t_3 = 0$.

By using the above mapping $\varphi : \mathbb{R}^{+^3} \to \mathbb{R}_0^+$ and $\mathcal{MT}$-function, we introduce a new notion of a generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction.

Definition 5. *A mapping $T : A \cup B \to A \cup B$ is said to be a generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction, if*

$$\sigma(Tx, Ty) - \sigma(A,B) \le \psi(\sigma(x,y))[\varphi(\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty)) - \sigma(A,B)],$$

for all $x \in A$ and all $y \in B$, where A and B are nonempty subsets of a metric-like space (X,σ), and ψ is an $\mathcal{MT}$-function.

In this section, we establish the best proximity point results of a generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction. Our results generalize and improve Theorem 4.

Lemma 2. *Let A and B be nonempty closed subsets of a metric-like space (X,σ). Let $T : A \cup B \to A \cup B$ be a generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction. For $x_0 \in A \cup B$, define $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$. Then*

$$\sigma(x_n, x_{n+1}) \to \sigma(A,B), \quad as \quad n \to \infty.$$

Proof. Since $T : A \cup B \to A \cup B$ is a generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction, we obtain that for each $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\sigma(x_{n+2}, x_{n+1}) - \sigma(A,B) \\
=&\sigma(Tx_{n+1}, Tx_n) - \sigma(A,B) \\
\le&\psi(\sigma(x_{n+1}, x_n))[\varphi(\sigma(x_{n+1}, x_n), \sigma(x_{n+1}, Tx_{n+1}), \sigma(x_n, Tx_n)) - \sigma(A,B)] \\
\le&\psi(\sigma(x_{n+1}, x_n))[\varphi(\sigma(x_{n+1}, x_n), \sigma(x_{n+1}, x_{n+2}), \sigma(x_n, x_{n+1})) - \sigma(A,B)].
\end{aligned}$$

If $\sigma(x_{n+2}, x_{n+1}) > \sigma(x_{n+1}, x_n)$ for some n, then by the conditions of the function φ we have that

$$\begin{aligned}
&\sigma(x_{n+2}, x_{n+1}) - \sigma(A,B) \\
=&\sigma(Tx_{n+1}, Tx_n) - \sigma(A,B) \\
\le&\psi(\sigma(x_{n+1}, x_n))[\varphi(\sigma(x_{n+1}, x_{n+2}), \sigma(x_{n+1}, x_{n+2}), \sigma(x_{n+1}, x_{n+2})) - \sigma(A,B)] \\
<&\sigma(x_{n+2}, x_{n+1}) - \sigma(A,B),
\end{aligned}$$

which implies a contraction. So, we conclude that $\sigma(x_{n+2}, x_{n+1}) \leq \sigma(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. On the other hand,

$$\begin{aligned}
&\sigma(x_{n+1}, x_{n+2}) - \sigma(A, B) \\
=&\sigma(Tx_n, Tx_{n+1}) - \sigma(A, B) \\
\leq&\psi(\sigma(x_n, x_{n+1}))[\varphi(\sigma(x_n, x_{n+1}), \sigma(x_n, Tx_n), \sigma(x_{n+1}, Tx_{n+1})) - \sigma(A, B)] \\
\leq&\psi(\sigma(x_n, x_{n+1}))[\varphi(\sigma(x_n, x_{n+1}), \sigma(x_n, x_{n+1}), \sigma(x_{n+1}, x_{n+2})) - \sigma(A, B)].
\end{aligned}$$

If $\sigma(x_{n+1}, x_{n+2}) > \sigma(x_n, x_{n+1})$ for some n, then by the conditions of the function φ we have that

$$\begin{aligned}
&\sigma(x_{n+1}, x_{n+2}) - \sigma(A, B) \\
=&\sigma(Tx_n, Tx_{n+1}) - \sigma(A, B) \\
\leq&\psi(\sigma(x_n, x_{n+1}))[\varphi(\sigma(x_n, x_{n+1}), \sigma(x_n, Tx_n), \sigma(x_{n+1}, Tx_{n+1})) - \sigma(A, B)] \\
<&\sigma(x_{n+1}, x_{n+2}) - \sigma(A, B),
\end{aligned}$$

which implies a contraction. So, we conclude that $\sigma(x_{n+1}, x_{n+2}) \leq \sigma(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

From above argument, the sequence $\{\sigma(x_{n+1}, x_n)\}_{n\in\mathbb{N}}$ is non-increasing and bounded below in $\mathbb{R}_0^+$. Since ψ is an $\mathcal{MT}$-function, by Theorem 5 we conclude that

$$0 \leq \psi(sup_{n\in\mathbb{N}}\sigma(x_n, x_{n+1})) < 1.$$

Let $\lambda = sup_{n\in\mathbb{N}}\psi(\sigma(x_n, x_{n+1})) < 1$. Then

$$0 \leq \psi(\sigma(x_n, x_{n+1})) \leq \lambda, \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $T : A \cup B \to A \cup B$ is a generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction, we obtain that

$$\begin{aligned}
&\sigma(x_n, x_{n+1}) - \sigma(A, B) \\
=&\sigma(Tx_{n-1}, Tx_n) - \sigma(A, B) \\
\leq&\lambda \cdot [\sigma(x_{n-1}, x_n) - \sigma(A, B)] \\
\leq&\lambda^2 \cdot [\sigma(x_{n-2}, x_{n-1}) - \sigma(A, B)] \\
\leq&\cdots \\
\leq&\lambda^n \cdot [\sigma(x_0, x_1) - \sigma(A, B)].
\end{aligned}$$

Since $\lambda < 1$, $\lim_{n\to\infty} \lambda^n = 0$, and we also get that

$$\lim_{n\to\infty} [\sigma(x_n, x_{n+1}) - \sigma(A, B)] = 0,$$

that is,

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = \sigma(A, B).$$

□

By Lemma 2, we obtain the following best proximity point result of a generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction.

Theorem 6. *Let $T : A \cup B \to A \cup B$ be a generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction, where A and B are nonempty closed subsets of a complete metric-like space (X, σ). If we construct a sequence $x_{n+1} = Tx_n$ for each $n \in \mathbb{N} \cup \{0\}$ for an arbitrary $x_0 \in A \cup B$, then we have the following:*

(1) *If* $x_0 \in A$ *and* $\{x_{2n}\}$ *has a subsequence* $\{x_{2n_k}\}$ *which converges to* $x^* \in A$ *with* $\sigma(x^*, x^*) = 0$, *then* $\sigma(x^*, Tx^*) = \sigma(A, B)$.
(2) *If* $x_0 \in B$ *and* $\{x_{2n-1}\}$ *has a subsequence* $\{x_{2n_k-1}\}$ *which converges to* $x_* \in B$ *with* $\sigma(x_*, x_*) = 0$, *then* $\sigma(x_*, Tx_*) = \sigma(A, B)$.

Proof. Suppose that $x_0 \in A$. On account of the fact that T is cyclic, we get $x_{2n} \in A$ and $x_{2n+1} \in B$ for all $n \in \mathbb{N} \cup \{0\}$. Here, if $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ which converges to $x^* \in A$ with $\sigma(x^*, x^*) = 0$, then

$$\lim_{n\to\infty} \sigma(x_{2n}, x^*) = \sigma(x^*, x^*) = 0.$$

Since T is a generalized $\mathcal{MT}$-Ćirić -function type cyclic φ-contraction, we have

$$\begin{aligned}
&\sigma(x^*, Tx^*) - \sigma(A, B) \\
\le\ & \sigma(x^*, x_{2n_k}) + \sigma(x_{2n_k}, Tx^*) - \sigma(A, B) \\
\le\ & \sigma(x^*, x_{2n_k}) + \sigma(Tx_{2n_k-1}, Tx^*) - \sigma(A, B) \\
\le\ & \sigma(x^*, x_{2n_k}) + \psi(\sigma(x_{2n_k-1}, x^*))[\varphi(\sigma(x_{2n_k-1}, x^*), \sigma(x_{2n_k-1}, Tx_{2n_k-1}), \sigma(x^*, Tx^*)) - \sigma(A, B)] \\
\le\ & \sigma(x^*, x_{2n_k}) + \psi(\sigma(x_{2n_k-1}, x^*))[\varphi(\sigma(x_{2n_k-1}, x^*), \sigma(x_{2n_k-1}, x_{2n_k}), \sigma(x^*, Tx^*)) - \sigma(A, B)] \\
\le\ & \sigma(x^*, x_{2n_k}) + \psi(\sigma(x_{2n_k-1}, x^*))[\varphi(\sigma(x_{2n_k-1}, x_{2n_k}) + \sigma(x_{2n_k}, x^*), \sigma(x_{2n_k-1}, x_{2n_k}), \sigma(x^*, Tx^*)) - \sigma(A, B)]
\end{aligned}$$

We claim that $\sigma(x^*, Tx^*) - \sigma(A, B) = 0$. If not, we assume that

$$\sigma(x^*, Tx^*) - \sigma(A, B) > 0.$$

Letting $k \to \infty$, by Lemma 1, we obtain

$$\begin{aligned}
\sigma(x^*, Tx^*) - \sigma(A, B) &< 0 + \varphi(\sigma(A, B) + 0, \sigma(A, B), \sigma(x^*, Tx^*)) - \sigma(A, B) \\
&\le \varphi(\sigma(x^*, Tx^*), \sigma(x^*, Tx^*), \sigma(x^*, Tx^*)) - \sigma(A, B) \\
&\le \sigma(x^*, Tx^*) - \sigma(A, B),
\end{aligned}$$

which implies a contradiction. Thus, $\sigma(x^*, Tx^*) = \sigma(A, B)$, that is, x^* is the best proximity point of T.

The proof of (2) is similar to (1), we omit it. □

Example 2. *Let* $X = \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *be endowed with the metric-like* $\sigma : X \times X \to \mathbb{R}_0^+$ *defined by:*

$$\sigma((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| + |y_1 - y_2|, & \text{if } (x_1, y_1), (x_2, y_2) \in [0, 1] \times [0, 1]; \\ x_1 + x_2 + y_1 + y_2, & \text{if not.} \end{cases}$$

Let $\phi : \mathbb{R}_0^+ \to [0, 1)$ *be defined by*

$$\varphi(t_1, t_2, t_3) = \frac{t_1 + t_2 + t_3}{3},$$

and

$$\psi(t) = \begin{cases} \frac{11}{14} + \frac{1}{14}t, & \text{if } t \in [0, 1] \times [0, 1]; \\ \frac{13}{14}, & \text{if } t > 1. \end{cases}$$

Clearly, (X, σ) *is a complete metric-like space, and* ψ *is an* $\mathcal{MT}$*-function.*

Take $A = [0, 1] \times \{0\}$ *and* $B = [0, 1] \times \{1\}$, *and let* $T : A \cup B \to A \cup B$ *be defined by*

$$T((x, 0)) = (\frac{1}{4}x, 1), \quad \text{for all} \quad x \in [0, 1],$$

and

$$T((x, 1)) = (\frac{1}{4}x, 0), \quad \text{for all} \quad x \in [0, 1].$$

Then we have $\sigma(A,B)=1$ *and* T *is a cyclic mapping.*
For $(x_1,0)\in A$ *and* $(x_2,1)\in B$, *we have that* $x_1,x_2\in[0,1]$ *and*

$$\begin{aligned}&\sigma(T((x_1,0)),T((x_2,1)))-\sigma(A,B)\\ =&\sigma((\frac{1}{4}x_1,1),(\frac{1}{4}x_2,0))-1\\ =&\frac{1}{4}|x_1-x_2|,\end{aligned}$$

and

$$\begin{aligned}&\psi(\sigma((x_1,0),(x_2,0)))[\varphi(\sigma((x_1,0),(x_2,0)),\sigma((x_1,0),T((x_1,0)),\sigma((x_2,1),T((x_2,1)))-\sigma(A,B)]\\ =&(\frac{11}{14}+\frac{1}{14}\sigma((x_1,0),(x_2,0)))[\varphi(\sigma((x_1,0),(x_2,0)),\sigma((x_1,0),(\frac{1}{4}x_1,1)),\sigma((x_2,1),(\frac{1}{4}x_2,0))-1]\\ =&(\frac{11}{14}+\frac{1}{14}(1+|x_1-x_2|))[\frac{1+|x_1-x_2|+1+\frac{3}{4}x_1+1+\frac{3}{4}x_2}{3}-1]\\ =&(\frac{11}{14}+\frac{1}{14}(1+|x_1-x_2|))[\frac{1}{3}|x_1-x_2|+\frac{1}{4}(x_1+x_2)].\end{aligned}$$

Then T *is a generalized* $\mathcal{MT}$*-Ćirić -function type cyclic* φ*-contraction, and we also get that* $(0,0)\in A$ *and* $(0,1)\in B$ *are the two best proximity points of* T.

Example 3. *Let* $A=[0,1]$ *and* $B=[-1,0]$ *be two subsets of* $X=\mathbb{R}$, *and let* $\sigma(x,y)=max\{|x|,|y|\}$. *Define* $T:A\cup B\to A\cup B$ *be defined by* $Tx=\frac{-x}{4}$ *for all* $x\in A\cup B$ *and* $\psi(t)=\frac{1}{3}$. *Let* $\varphi(t_1,t_2,t_3)=\frac{t_1+t_2+t_3}{3}$. *Then* $\sigma(A,B)=0$ *and Theorem 6 holds.*

Example 4. *Let* $A=B=[0,1]$ *be two subsets of* $X=\mathbb{R}$ *and* $\sigma(x,y)=max\{x,y\}$. *Define* $T:A\cup B\to A\cup B$ *be defined by*

$$Tx=\begin{cases}\frac{1}{4}, & if\ x=1;\\ \frac{1}{2}, & if\ x\in[0,1).\end{cases}$$

Let $\psi(t)=\frac{t}{8}$, *and let* $\varphi(t_1,t_2,t_3)=\frac{t_1+t_2+t_3}{3}$. *Then* $\sigma(A,B)=0$ *and Theorem 6 holds.*

Example 5. *Let* $X=\mathbb{R}_0^+\times\mathbb{R}_0^+$ *be endowed with the metric-like* $\sigma:X\times X\to\mathbb{R}_0^+$ *defined by:*

$$\sigma((x_1,y_1),(x_2,y_2))=max\{|x_1|,|x_2|\}+max\{|y_1|,|y_2|\}.$$

For $A=\{a_1=(5,2),a_2=(1,2)\}$, $B=\{b_1=(3,0),b_2=(0,4)$, *define* $T:A\cup B\to A\cup B$ *by* $T(a_1)=b_2$, $T(a_2)=b_1$, $T(b_1)=a_2$, $T(b_2)=a_1$. *Then* $\sigma(A,B)=5$. *All conditions are satisfied and both* a_2 *and* b_1 *are the best proximity points of* T.

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