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On an Exact Relation between $\zeta''(2)$ and the Meijer $\mathcal{G}$-Functions

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Abstract: In this paper we consider some integral representations for the evaluation of the coefficients of the Taylor series for the Riemann zeta function about a point in the complex half-plane $\Re(s) > 1$. Using the standard approach based upon the Euler-MacLaurin summation, we can write these coefficients as $\Gamma(n+1)$ plus a relatively smaller contribution, ξ_n. The dominant part yields the well-known Riemann's zeta pole at $s = 1$. We discuss some recurrence relations that can be proved from this standard approach in order to evaluate $\zeta''(2)$ in terms of the Euler and Glaisher-Kinkelin constants and the Meijer $\mathcal{G}$-functions.

Keywords: Riemann zeta function; Euler-Maclaurin summation; Meijer $\mathcal{G}$-functions

MSC: 11M06; 33E20; 40C10

1. Introduction

The Riemann zeta function, $\zeta(s)$, is holomorphic in the open half-plane [1–4], $\Re(s) > 1$, and, consequently, it can be expanded using the Talyor series about a point s_0, with $\Re(s_0) > 1$, as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s_0}} e^{-(s-s_0)\ln n} = \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{k!} S_k(s_0) \,, \tag{1}$$

where $S_k(s_0)$, $k = 0, 1, 2, \ldots$ are Dirichlet sums given by:

$$S_k(s_0) = \sum_{n=1}^{\infty} \frac{(\ln n)^k}{n^{s_0}} \,, \tag{2}$$

whose convergence for any $k = 0, 1, \ldots$ can be proven using the integral test. For $s_0 = 2$, $k = 0$ and $k = 1$ these sums are known explicitly in terms of classical constants:

$$S_0(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \,, \tag{3}$$

$$S_1(2) = \sum_{n=1}^{\infty} \frac{\ln n}{n^2} = \frac{\pi^2}{6} \left[12 \ln A - \gamma - \ln(2\pi)\right] \,, \tag{4}$$

where A is the Glaisher-Kinkelin constant [5–7] given by the limit:

$$A = \lim_{n\to\infty} \frac{\prod_{k=1}^{n} k^k}{n^{n^2/2+n/2+1/12} e^{-n^2/4}} \,. \tag{5}$$

The first sum in Equation (3) is the solution of the Basel problem by Euler (1741) and the second one corresponds to an identity proved by Glaisher (1894). For $k = 0$ and $s_0 = 2m$, $m = 1, 2, \ldots$ we have also a well-known explicit expression relating these sums to Bernouilli numbers, B_{2n}:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = (-1)^{m+1} \frac{(2\pi)^{2m} B_{2m}}{2(2m)!} , \tag{6}$$

but, apart from these classical results [1,2], it is still interesting to find closed expressions or algorithms for the general case of $S_k(s_0)$, specially for large values of k. Moreover, as these Dirichlet sums provide the coefficients for the Taylor series of $\zeta(s)$ at points s_0 with real part $\Re(s_0) > 1$, they have an intrinsic interest for the theory of the Riemann zeta function. Obviously a brute-force algorithm to estimate $S_k(s_0)$ for large k is not possible because the complex function over the reals:

$$f_k(x) = \frac{(\ln x)^k}{x^{s_0}} , \tag{7}$$

achieves its maximum's modulus for $x = e^{k/\Re(s_0)}$, so the number of terms in these sums, to be taken into account in any reasonable approximation, goes beyond present computational capabilities with the exception of the lowest values of k. For example, for $s_0 = 2$ and $k = 50$ the maximum is achieved at $n = 72{,}004{,}899{,}337$ which makes a direct evaluation, term by term, unattainable.

The standard procedure to evaluate these sums is Euler-MacLaurin's formula, a favourite tool of Ramanujan as pointed out by Berndt [8] and also one of the first to be used in the numerical evaluation of the Riemann zeta function [9]. We will revisit this approach in Section 2. The equivalence with Riemann's integral analytic continuation for $\zeta(s)$ provides another method by which we will find a recurrence relation for $S_k(s_0)$ at any point s_0. This method is developed in Section 3. As a corollary of this recurrence's relation we find an explicit expression for $S_2(s_0 = 2)$ in terms of Glaisher's constant and a convergent series involving Meijer $\mathcal{G}$-functions [10]. The paper is ended with some conclusions in Section 4.

2. Euler-MacLaurin's Summation Approach

Euler-MacLaurin's summation has been used for a long time to obtain approximations of the Riemann zeta function [9]. High-precision computation of the Hurwitz zeta function and its derivatives has also been obtained thanks to these techniques [11]. The objective of this section is to provide only a brief background to the analytical relations of the Riemann zeta derivatives discussed in the next section. For the moment, and without comprimising the generality of our approach, we will focus on the case $s_0 = 2$. For the corresponding sums, that we will denote by S_N, we will show the following:

Theorem 1. *The Dirichlet series S_N, which appear in the Taylor expansion of Riemann's zeta function for $s_0 = 2$, are given by $S_N = \Gamma(N+1) + \xi_N$ where:*

$$\xi_N = \int_1^{\infty} f'_N(x) P_1(x) dx = -\frac{1}{N!} \int_1^{\infty} f_N^{(N)}(x) P_N(x) dx . \tag{8}$$

Here we have assumed that N is an even integer, the prime denote the first derivative and the upper index, (N), denotes the Nth derivative. The functions $P_i(x)$, $i = 1, \ldots, N$ are the Bernoulli periodic functions defined in terms of the Bernoulli polynomials by $P_i(x) = B_i(x - [x])$. For N odd the result is similar but replacing N in the right-hand side of Equation (8), by $N - 1$.

Proof. To prove this we recall the classical Euler-MacLaurin summation formula [12,13]:

$$\begin{aligned}\sum_{i=m+1}^{n} f(i) &= \int_m^n f(x)\,dx + \frac{f(n)-f(m)}{2} \\ &+ \sum_{k=1}^{[p/2]} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right) + R_p\,,\end{aligned} \tag{9}$$

with the remainder term given by:

$$R_p = (-1)^{p+1}\int_m^n f^{(p)}(x)\,\frac{P_p(x)}{p!}\,dx\,, \tag{10}$$

where $P_p(x)$ are Bernoulli periodic functions of order p and B_{2k} denotes the $2k$-th Bernoulli number as usual. The brackets $[\ldots]$ stand for the integer part.

The first identity in Equation (8) is obtained from Equations (9) and (10) by taking $p = 1$ with $m = 1$ and $n \to \infty$. For obtaining the second identity we notice that $\lim_{x\to\infty} f_N^{(j)}(x) = 0$ for any order of the derivative, $j \geq 0$ and that $f_N^{(j)}(x = 1) = 0$ for $0 \leq j \leq N-1$ and, as a consequence, the sum involving the Bernouilli numbers vanishes. Notice also that:

$$\int_1^\infty \frac{(\ln x)^N}{x^2}dx = \int_0^\infty y^N e^{-y}dy = \Gamma\left(N+1\right)\,. \tag{11}$$

□

In the course of the calculation we also find:

Corollary 1. *The difference $S_N - N! = \xi_N$ is given by the series:*

$$\xi_N = \sum_{j=1}^{\infty}\left[\frac{1}{2}\left(\frac{(\ln j)^N}{j^2} + \frac{(\ln(j+1))^N}{(j+1)^2}\right) - \Gamma\left(N+1, \ln j, \ln(j+1)\right)\right]\,, \tag{12}$$

where $\Gamma(n, a, b)$ denotes a generalized incomplete Gamma function:

$$\Gamma(x, a, b) = \int_a^b t^{n-1}\,e^{-t}\,dt\,. \tag{13}$$

Proof. This follows from the definition of $P_1(x) = x - [x] - 1/2$ and the first identity in Equation (8) rewritten as the series:

$$\xi_N = \sum_{n=1}^{\infty}\int_n^{n+1} f_N'(x)\left(x - n - \frac{1}{2}\right)dx\,. \tag{14}$$

Then, integration by parts of Equation (12) and the definition in Equation (13) lead to the statement in the corollary. □

However, we must point out that the series in Equation (14) converges only very slowly for large N. The identity of the two expressions for ξ_N in Equation (8) also implies the following result:

Corollary 2. *For N being even and a function $f_N(x)$ as given by Equation (7) we have:*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{N-1}(-1)^k\left\{f_N^{(N-k)}(n+1) - f_N^{(N-k)}(n)\right\}\frac{B_{N-k+1}}{(N-k+1)!} = 0\,, \tag{15}$$

with B_k denoting the k-th Bernouilli number.

Proof. From Theorem 1 and for N being even we have that:

$$\xi_N = -\frac{1}{N!}\sum_{n=1}^{\infty}\int_n^{n+1} f_N^{(N)}(x) B_N(x-n)\,dx\,. \tag{16}$$

Integration by parts yields:

$$\begin{aligned} N!\,\xi_N &= -\sum_{n=1}^{\infty}\left[f_N^{(N-1)}(n+1) - f_N^{(N-1)}(n)\right] B_N \\ &+ N\sum_{n=1}^{\infty}\int_n^{n+1} f_N^{(N-1)}(x)\,B_{N-1}(x-n)\,dx\,. \end{aligned} \tag{17}$$

Now repeating the procedure of integrating by parts a total of $N-2$ times we arrive at:

$$\begin{aligned} \xi_N &= \sum_{n=1}^{\infty}\sum_{k=1}^{N-1}(-1)^k\left[f_N^{(N-k)}(n+1) - f_N^{(N-k)}(n)\right]\frac{B_{N-k+1}}{(N-k+1)!} \\ &+ \sum_{n=1}^{\infty}(-1)^N\int_n^{n+1} f'(x)\,B_1(x-n)\,dx\,, \end{aligned} \tag{18}$$

but, taking into account that N is even, the second summatory over n in the right-hand side of Equation (18) is precisely ξ_N, according to Theorem 1, and the corollary follows. □

Finally, we should point out that the second identity in Theorem 1 is far more convenient for computing ξ_N numerically than the first one because $f_N^{(N)}(x) = \mathcal{O}((\ln x)^N/x^{N+2})$ as $x \to \infty$.

3. Recurrence Relations and Explicit Evaluations of S_N

In his historic paper on number theory, Riemann deduced a functional equation for $\zeta(s)$ and an integral representation for its analytic continuation over the whole complex plane [2]. A combination of these two results suggests the definition of a function, $H(s)$, holomorphic on $\mathbb{C}$, as follows:

$$H(s) = \frac{1}{2}\,s(s-1)\,\frac{\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}}\,\zeta(s)\,, \tag{19}$$

with $H(s)$ given by:

$$2H(s) = 1 + s(s-1)\int_1^{\infty}\left(x^{s/2-1} + x^{-s/2-1/2}\right)\left[\frac{\vartheta(x)-1}{2}\right]dx\,. \tag{20}$$

Here, $\vartheta(x)$ is Jacobi's ϑ-function defined as:

$$\frac{\vartheta(x)-1}{2} = \sum_{n=1}^{\infty} e^{-n^2\pi x} = J(x)\,. \tag{21}$$

In this section we will use the analytic continuation of $\zeta(s)$ to find a recurrence relation for $S_N(s_0 = 2)$ stated as follows:

Theorem 2. *The Dirichlet series $S_N(s_0)$, with $s_0 = 2$, satisfies a recurrence relation:*

$$\begin{aligned} S_N &= \pi\left[N!\left(1 - \frac{1}{2^{N+1}}\right) + (-1)^N\mathcal{H}_N\right. \\ &\left. - \sum_{j=0}^{N-1}\binom{N}{j}(-1)^{N+j}\,q_{N-j}\,S_j\right]\,, \end{aligned} \tag{22}$$

where:

$$\mathcal{H}_N = \frac{1}{2^N}\int_1^\infty (\ln x)^N \left[1+(-1)^N x^{-3/2}\right] J(x)\, d\, x \tag{23}$$

$$q_N = \frac{1}{2^N}\int_0^\infty (\ln x)^N\, e^{-\pi x}\, dx\,, \tag{24}$$

for any integer $N \geq 0$. The recurrence in Equation (22) would be applied for $N > 0$ starting with $S_0 = \pi^2/6$.

Proof. We start with the definition of the complex analytic function $H(s)$ in Equation (20) and rewrite it in the form:

$$\frac{2H(s)}{s(s-1)} = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{s_0/2-1} e^{\frac{z\ln x}{2}} + x^{-s_0/2-1/2} e^{\frac{-z\ln x}{2}} \right) J(x)\, dx\,, \tag{25}$$

where s_0 is a point with $\Re(s_0) > 1$ and $z = s - s_0$. By Taylor expansion of the exponentials in the integral we have:

$$\frac{2H(s)}{s(s-1)} = \frac{1}{s(s-1)} + \sum_{n=0}^{\infty} \frac{\mathcal{H}_N(s_0)}{n!} z^n\,, \tag{26}$$

with:

$$\mathcal{H}_n(s_0) = \frac{1}{2^n}\int_1^\infty (\ln x)^n \left[x^{s_0/2-1} + (-1)^n x^{-s_0/2-1/2}\right] J(x)\, dx\,. \tag{27}$$

Alternatively, from Equations (26) and (27) we obtain the following Taylor series:

$$\frac{2H(s)}{s(s-1)} = \sum_{n=0}^{\infty} z^n \left\{ \frac{(-1)^n}{(s_0-1)^{n+1}} - \frac{(-1)^n}{s_0^{n+1}} + \frac{\mathcal{H}_n(s_0)}{n!} \right\}\,. \tag{28}$$

We notice also that the function $Q(s) = \Gamma(s/2)/\pi^{s/2}$ can be expanded in a similar way:

$$Q(s) = \int_0^\infty x^{s/2-1}\, e^{-\pi x}\, dx = \sum_{n=0}^{\infty} q_n(s_0)\frac{z^n}{n!}\,, \tag{29}$$

where:

$$q_n(s_0) = \frac{1}{2^n}\int_0^\infty x^{s_0/2-1}\, (\ln x)^n\, e^{-\pi x}\, dx\,. \tag{30}$$

If we consider the Taylor series for $\zeta(s)$ about a point s_0, as given in Equation (1), and the series for $Q(s)$ defined above in Equations (29) and (30), then the series for the product $Q(s)\zeta(s)$ ensues:

$$\begin{aligned} Q(s)\zeta(s) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} q_n(s_0) \sum_{m=0}^{\infty} (-1)^m \frac{z^m}{m!} S_m(s_0) \\ &= \sum_{n=0}^{\infty} z^n \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!}\, q_{n-k}(s_0) S_k(s_0) \right)\,, \end{aligned} \tag{31}$$

in terms of the Cauchy product of the succession of coefficients, $q_n(s_0)$ and $S_n(s_0)$, $n = 0, 1, \ldots$. Now, from Riemann's functional equation in Equation (19) and the convergent series in Equations (28) and (31) we obtain, by identifying the coefficients of the same order, that:

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!}\, q_{n-k}(s_0)\, S_k(s_0) = \frac{(-1)^n}{(s_0-1)^{n+1}} - \frac{(-1)^n}{s_0^{n+1}} + \mathcal{H}_n(s_0)\,. \tag{32}$$

But, noticing that $q_0(s_0) = \Gamma(s_0/2)/\pi^{s_0/2}$, we find that this expression can be rewritten as:

$$\begin{aligned} S_n(s_0) &= \frac{\pi^{s_0/2}}{\Gamma(s_0/2)} \left[n! \left(\frac{1}{(s_0-1)^{n+1}} - \frac{1}{s_0^{n+1}} \right) + (-1)^n \mathcal{H}_n(s_0) \right. \\ &- \left. \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n+k} q_{n-k}(s_0) S_k(s_0) \right] , \end{aligned} \tag{33}$$

which allows for the calculation of $S_n(s_0)$ for any $n > 0$ in terms of $S_0(s_0)$, $S_1(s_0)$, ..., $S_{n-1}(s_0)$. The initial condition for the iteration is $S_0(s_0) = \zeta(s_0)$. The recurrence relation in Equation (22) is a particular case of Equation (33) for $s_0 = 2$. $\square$

The integrals defining the coefficients of the Taylor expansion for $Q(s)$ as given in Equation (30) can be explicitly evaluated for $s_0 = 2$. For example, we have:

$$q_0 = \frac{1}{\pi} , \tag{34}$$

$$q_1 = -\frac{\gamma + \ln \pi}{2\pi} , \tag{35}$$

$$q_2 = \frac{\pi}{24} + \frac{(\gamma + \ln \pi)^2}{4\pi} , \tag{36}$$

and we can apply these results and the recurrence relation in Equation (22) to obtain the following expressions:

Corollary 3. *For $s_0 = 2$ we have that:*

$$H_1(s_0 = 2) = \frac{3}{2} - 4\pi \ln A + \frac{\pi}{6} (\gamma + \ln(4\pi)) , \tag{37}$$

and:

$$\begin{aligned} S_2(s_0 = 2) &= \frac{7\pi}{4} - \frac{\pi^4}{144} - 2\pi^2 (\gamma + \ln \pi) \left(\ln A - \frac{\ln 2}{12} \right) \\ &+ \frac{\pi^2}{8} (\gamma + \ln \pi)^2 + \frac{\pi}{4} \mathcal{H}_2(s_0 = 2) \end{aligned} \tag{38}$$

where A is Glaisher's constant and γ is Euler's constant. The coefficient $\mathcal{H}_2(s_0 = 2)$ can be expressed in terms of Meijer $\mathcal{G}$-functions:

$$\begin{aligned} \mathcal{H}_2(s_0 = 2) &= \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} \mathcal{G}_{2,3}^{3,0} \left(\begin{array}{cc} \{\} & \{1,1\} \\ \{0,0,0\} & \{\} \end{array} \middle| n^2 \pi \right) \\ &+ 2\sqrt{\pi} \sum_{n=1}^{\infty} n \, \mathcal{G}_{3,4}^{4,0} \left(\begin{array}{cc} \{\} & \{1,1,1\} \\ \{-1/2,0,0,0\} & \{\} \end{array} \middle| n^2 \pi \right) . \end{aligned} \tag{39}$$

Proof. The first result is obtained from the recurrence relation in Equation (22), the explicit expression for $S_1(s_0 = 2)$ in Equation (4) and the initial condition $S_0(s_0 = 2) = \pi^2/6$. The second one in Equation (38) follows from the application of the recurrence relation for $n = 2$, Equations (35) and (36) and the representation of $\mathcal{H}_2(s_0 = 2)$ as a Mellin-Barnes integral [14]. $\square$

4. Conclusions

The problem of the analytical properties or, even the evaluation, of the Riemann zeta function has been proven to be very complicated. The explicit evaluation of this function began before the time of Riemann, with the solution of the so-called Basel problem by Euler and the calculation of $\zeta(2)$ [15]. However, the first derivative, $\zeta'(2)$, was only found in connection with the definition of

Glaisher-Kinkelin's constant at the turn of the XIXth century [5,6]. In the XXth century, Vinogradov's method for evaluating Weyl's sums and trigonometric sums were also proved to be useful for obtaining new results in the neighbourhood of the line $\sigma = 1$ [1,16]. Important results on the mean-value of the Riemann zeta function at the critical line were also obtained by Atkinson and they provide some insight on the behaviour of the function in the region in which it shows its richer structure [1]. Emphasis has been in numerical calculations of zeros since the advent of modern powerful computers. These started with Riemann-Siegel's formula in the thirties of the past century, although it was improved by Odlyzko and Schönhage in the late eighties [17]. Notwithstanding these important developments, it seems useful to explore new connections analytically to other special functions as the one shown in this paper. Representations in asymptotic limits could also be helpful in getting a better understanding on the behaviour of $\zeta(s)$ or its extensions. A more general approach along this line will be published elsewhere.

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References

1. Titchmarsh, E.C. *The Theory of the Riemann Zeta-Function*, 2nd ed.; Clarendon Press: Oxford, UK, 1986.
2. Borwein, P.; Choi, S.; Rooney, B.; Weirathmueller, A. *The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike*; Springer: New York, NY, USA, 2007.
3. Chen, W.W.L. Elementary and Analytic Number Theory. 1981. Available online: http://plouffe.fr/simon/math/Elementary%20&%20Analytic%20Number%20Theory.pdf (accessed on 7 December 2016).
4. Cilleruelo, J.; Córdoba, A. *La Teoría de los Números*; Mondadori: Madrid, Spain, 1991.
5. Glaisher, J.W.L. On the constant which occurs in the formula for $1^1 \cdot 2^2 \cdot 3^3 \cdots n^n$. *Mess. Math.* **1894**, *24*, 1–16.
6. Kinkelin, H. Über eine mit der Gammafunktion verwandte Transcendente und deren Anwendung auf die Integralrechnung. *J. Reine Angew. Math.* **1860**, *57*, 122–158. [CrossRef]
7. Guillera, J.; Sondow, J. Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent. *Ramanujan J.* **2008**, *16*, 247–270. [CrossRef]
8. Berndt, B.C. *Ramanujan's Notebooks, Part I*; Springer: New York, NY, USA, 1985.
9. Edwards, H.M. *Riemann's Zeta Function*; Dover Publications Inc.: New York, NY, USA, 2003.
10. Bateman, H.; Erdélyi, A. *Higher Transcendental Functions*; McGraw-Hill: New York, NY, USA, 1953; Volume I.
11. Johansson, F. Rigorous high-precision computation of the Hurwitz zeta function and its derivatives. *Numer. Algorithms* **2015**, *69*, 253–270. [CrossRef]
12. Apostol, T.M. An Elementary View of Euler's Summation Formula. *Am. Math. Month.* **1999**, *106*, 409–418. [CrossRef]
13. Abramowitz, M.; Stegun, I.A. Bernoulli and Euler Polynomials and the Euler-Maclaurin Formula. In *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed.; Dover: New York, NY, USA, 1972; pp. 804–806.
14. Gasper, G.; Rahman, M. Basic hypergeometric series. In *Encyclopedia of Mathematics and Its Applications*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2004.
15. Ayoub, M. Euler and the Zeta function. *Am. Math. Month.* **1974**, *81*, 1067–1086. [CrossRef]
16. Vinogradov, I.M. *Selected Works: Ivan Matveevič Vinogradov*; Springer: Berlin, Germany, 2014.
17. Odlyzko, A.M.; Schönhage, A. Fast Algorithms for Multiple Evaluations of the Riemann Zeta Function. *Trans. Am. Math. Soc.* **1988**, *309*, 797–809. [CrossRef]