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A Study of Third Hankel Determinant Problem for Certain Subfamilies of Analytic Functions Involving Cardioid Domain

Lei Shi [1], Izaz Ali [2], Muhammad Arif [2,*], Nak Eun Cho [3,*], Shehzad Hussain [2] and Hassan Khan [2]

1 School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, China; shimath@163.com
2 Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan; aliizaz719@gmail.com (I.A.); shehzad873822@gmail.com (S.H.); hassanmath@awkum.edu.pk (H.K.)
3 Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea
* Correspondence: marifmaths@awkum.edu.pk (M.A.); necho@pknu.ac.kr (N.E.C.)

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Abstract: In the present article, we consider certain subfamilies of analytic functions connected with the cardioid domain in the region of the unit disk. The purpose of this article is to investigate the estimates of the third Hankel determinant for these families. Further, the same bounds have been investigated for two-fold and three-fold symmetric functions.

Keywords: subordinations; starlike functions; convex functions; close-to-convex functions; cardioid domain; Hankel determinant; m-fold symmetric functions

1. Introduction and Definitions

Let $\mathcal{A}$ be the family of all functions that are holomorphic (or analytic) in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and having the following Taylor–Maclaurin series form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta). \tag{1}$$

Further, let $\mathcal{S}$ represent a subfamily of $\mathcal{A}$, which contains functions that are univalent in Δ. The familiar coefficient conjecture for the function $f \in \mathcal{S}$ of the form (1) was first presented by Bieberbach [1] in 1916 and proven by de-Branges [2] in 1985. In between the years 1916 and 1985, many researchers tried to prove or disprove this conjecture. Consequently, they defined several subfamilies of $\mathcal{S}$ connected with different image domains. Among these, the families $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{K}$ of starlike functions, convex functions, and close-to-convex functions, respectively, are the most fundamental subfamilies of $\mathcal{S}$ and have a nice geometric interpretation. These families are defined as:

$$\begin{aligned}
\mathcal{S}^* &= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \ (z \in \Delta) \right\}, \\
\mathcal{C} &= \left\{ f \in \mathcal{S} : \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, \ (z \in \Delta) \right\}, \\
\mathcal{K} &= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{g(z)} \prec \frac{1+z}{1-z}, \text{ for } g(z) \in \mathcal{S}^*, \ (z \in \Delta) \right\},
\end{aligned}$$

where the symbol "$\prec$" denotes the familiar subordinations between analytic functions and is defined as: the function h_1 is subordinate to a function h_2, symbolically written as $h_1 \prec h_2$ or $h_1(z) \prec h_2(z)$, if we can find a function w, called the Schwarz function, that is holomorphic in Δ with $w(0) = 0$ and $|w(z)| < 1$ such that $h_1(z) = h_2(w(z))$ $(z \in \Delta)$. In the case of the univalency of h_2 in Δ, then the following relation holds:

$$h_1(z) \prec h_2(z) \quad (z \in \Delta) \quad \Longleftrightarrow \quad h_1(0) = h_2(0) \quad \text{and} \quad h_1(\Delta) \subset h_2(\Delta).$$

In [3], Padmanabhan and Parvatham in 1985 defined a unified family of starlike and convex functions using familiar convolution with the function $z/(1-z)^a$, for $a \in \mathbb{R}$. Later on, Shanmugam [4] generalized this idea by introducing the family:

$$\mathcal{S}_h^*(\phi) = \left\{ f \in \mathcal{A} : \frac{z(f*h)'}{(f*h)} \prec \phi(z), \quad (z \in \Delta) \right\},$$

where "$*$" stands for the familiar convolution, ϕ is a convex, and h is a fixed function in $\mathcal{A}$. Furthermore, if we replace h in $\mathcal{S}_h^*(\phi)$ by $z/(1-z)$ and $z/(1-z)^2$, we obtain the families $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ respectively. In 1992, Ma and Minda [5] reduced the restriction to a weaker supposition that ϕ is a function, with $\operatorname{Re}\phi(z) > 0$ in Δ, whose image domain is symmetric about the real axis and starlike with respect to $\phi(0) = 1$ with $\phi'(0) > 0$ and discussed some properties including distortion, growth, and covering theorems. The family $\mathcal{S}^*(\phi)$ generalizes various subfamilies of the family $\mathcal{A}$, for example;

(i). If $\phi(z) = \frac{1+Az}{1+Bz}$ with $-1 \le B < A \le 1$, then $\mathcal{S}^*[A,B] := \mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right)$ is the family of Janowski starlike functions; see [6]. Further, if $A = 1-2\alpha$ and $B = -1$ with $0 \le \alpha < 1$, then we get the family $\mathcal{S}^*(\alpha)$ of starlike functions of order α.

(ii). The family $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$ was introduced by Sokól and Stankiewicz [7], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$.

(iii). For $\phi(z) = 1 + \sin z$, the family $\mathcal{S}^*(\phi)$ leads to the family $\mathcal{S}_{\sin}^*$, introduced in [8].

(iv). When we take $\phi(z) = e^z$, then we have $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$ [9].

(v). The family $\mathcal{S}_R^* := \mathcal{S}^*(\phi(z))$ with $\phi(z) = 1 + \frac{z}{k}\frac{k+z}{k-z}$, $k = \sqrt{2}+1$ was studied in [10].

(vi). By setting $\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, the family $\mathcal{S}^*(\phi)$ reduces to $\mathcal{S}_{car}^*$, introduced by Sharma and his coauthors [11], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the cardioid given by:

$$(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0,$$

and also by the Alexandar-type relation, the authors in [11] defined the family $\mathcal{C}_{car}$ by:

$$\mathcal{C}_{car} = \left\{ f \in \mathcal{A} : zf'(z) \in \mathcal{S}_C^* \quad (z \in \Delta) \right\}; \tag{2}$$

see also [12,13]. For more special cases of the family $\mathcal{S}^*(\phi)$, see [14,15]. We now consider the following family connected with the cardioid domain:

$$\mathcal{R}_{car} = \left\{ f \in \mathcal{A} : f'(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad (z \in \Delta) \right\}. \tag{3}$$

For given parameters $q, n \in \mathbb{N} = \{1, 2, \ldots\}$, the Hankel determinant $H_{q,n}(f)$ was defined by Pommerenke [16,17] for a function $f \in \mathcal{S}$ of the form (1) given by:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{4}$$

The growth of $H_{q,n}(f)$ has been investigated for different subfamilies of univalent functions. Specifically, the absolute sharp bounds of the functional $H_{2,2}(f) = a_2a_4 - a_3^2$ were found in [18,19] for each of the families $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{R}$, where the family $\mathcal{R}$ contains functions of bounded turning. However, the exact estimate of this determinant for the family of close-to-convex functions is still undetermined [20]. Recently, Srivastava and his coauthors [21] found the estimate of the second Hankel determinant for bi-univalent functions involving the symmetric q-derivative operator, while in [22], the authors studied Hankel and Toeplitz determinants for subfamilies of q-starlike functions connected with the conic domain. For more literature, see [23–30].

The Hankel determinant of third order is given as:

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = -a_5a_2^2 + 2a_2a_3a_4 - a_3^3 + a_5a_3 - a_4^2. \tag{5}$$

The estimation of the determinant $|H_{3,1}(f)|$ is very hard as compared to deriving the bound of $|H_{2,2}(f)|$. The very first paper on $H_{3,1}(f)$ was given in 2010 by Babalola [31], in which he obtained the upper bound of $H_{3,1}(f)$ for the families of $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{R}$. Later on, many authors published their work regarding $|H_{3,1}(f)|$ for different subfamilies of univalent functions; see [32–36]. In 2017, Zaprawa [37] improved the results of Babalola as under:

$$|H_{3,1}(f)| \leq \begin{cases} 1, & \text{for} \quad f \in \mathcal{S}^*, \\ \frac{49}{540}, & \text{for} \quad f \in \mathcal{C}, \\ \frac{41}{60}, & \text{for} \quad f \in \mathcal{R}. \end{cases}.$$

and claimed that these bounds are still not the best possible. Further, for the sharpness, he examined the subfamilies of $\mathcal{S}^*$, $\mathcal{C}$, and $\mathcal{R}$ consisting of functions with m-fold symmetry and obtained the sharp bounds. Moreover, in 2018, Kwon et al. [38] improved the bound of Zaprawa for $f \in \mathcal{S}^*$ and proved that $|H_{3,1}(f)| \leq 8/9$, but it is not yet the best possible. The authors in [39–41] contributed in a similar direction by generalizing different families of univalent functions with respect to symmetric points. In 2018, Kowalczyk et al. [42] and Lecko et al. [43] obtained the sharp inequalities:

$$|H_{3,1}(f)| \leq 4/135 \quad \text{and} \quad |H_{3,1}(f)| \leq 1/9,$$

for the recognizable families $\mathcal{K}$ and $\mathcal{S}^*(1/2)$, respectively, where the symbol $\mathcal{S}^*(1/2)$ stands for the family of starlike functions of order $1/2$. Furthermore, we would like to cite the work done by Mahmood et al. [44] in which they studied the third Hankel determinant for a subfamily of starlike functions in the q-analogue. Additionally, Zhang et al. [45] studied this determinant for the family $\mathcal{S}_e^*$ and obtained the bound $|H_{3,1}(f)| \leq 0.565$.

In the present article, our aim is to investigate the estimate of $|H_{3,1}(f)|$ for the subfamilies $\mathcal{S}_{car}^*$, $\mathcal{C}_{car}$, and $\mathcal{R}_{car}$ of analytic functions connected with the cardioid domain. Moreover, we also study this problem for families of m-fold symmetric functions connected with the cardioid domain.

2. A Lemma

Let $\mathcal{P}$ denote the family of all functions p that are analytic in Δ with $\Re\left(p(z)\right) > 0$ and having the following series representation:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \Delta). \tag{6}$$

Lemma 1. *If $p \in \mathcal{P}$ and it has the form (6), then:*

$$\begin{aligned} |c_n| &\leq 2 \text{ for } n \geq 1, &(7)\\ |c_m c_n - c_k c_l| &\leq 4 \text{ for } m+n = k+l, &(8)\\ \left|c_{n+2k} - \mu c_n c_k^2\right| &\leq 2(1+2\mu); \text{ for } \mu \in \mathbb{R}, &(9)\\ \left|c_2 - \frac{c_1^2}{2}\right| &\leq 2 - \frac{|c_1|^2}{2}, &(10)\\ |c_{n+k} - \mu c_n c_k| &\leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2\,|2\mu - 1|, & \text{elsewhere.} \end{cases} &(11) \end{aligned}$$

where the inequalities (7), (10), (11), and (9) are taken from [46].

3. Bound of $|H_{3,1}(f)|$ for the Family $\mathcal{S}^*_{car}$

Theorem 1. *If $f(z)$ of the form (1) belongs to $\mathcal{S}^*_{car}$, then:*

$$|a_2| \leq \tfrac{4}{3}, \quad |a_3| \leq \tfrac{11}{9} \quad \text{and} \quad |a_4| \leq \tfrac{68}{81}.$$

These bounds are the best possible.

Proof. Let $f \in \mathcal{S}^*_{car}$. Then, in the form of the Schwarz function, we have:

$$\frac{zf'(z)}{f(z)} = 1 + \frac{4}{3}w(z) + \frac{2}{3}\left(w(z)\right)^2 \quad (z \in \Delta).$$

Furthermore, we easily get:

$$\begin{aligned} \frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(3a_4 - 3a_2a_3 + a_2^3\right)z^3 \\ + \left(4a_5 - 2a_3^2 - 4a_2a_4 + 4a_2^2a_3 - a_2^4\right)z^4 + \cdots. \end{aligned} \tag{12}$$

and from series expansion of w with simple calculations, we can write:

$$\begin{aligned} 1 + \frac{4}{3}w(z) + \frac{2}{3}\left(w(z)\right)^2 &= 1 + \frac{2}{3}c_1 z + \left(\frac{2}{3}c_2 - \frac{c_1^2}{6}\right)z^2 + \left(\frac{2}{3}c_3 - \frac{1}{3}c_1c_2\right)z^3 \\ &\quad + \left(\frac{2}{3}c_4 + \frac{c_1^4}{24} - \frac{c_2^2}{6} - \frac{c_1c_3}{3}\right)z^4 + \cdots. \end{aligned} \tag{13}$$

By comparing (12) and (13), we get:

$$a_2 = \frac{2}{3}c_1, \tag{14}$$

$$a_3 = \frac{1}{2}\left(\frac{5}{18}c_1^2 + \frac{2}{3}c_2\right), \tag{15}$$

$$a_4 = \frac{1}{3}\left(\frac{c_1c_2}{3} + \frac{2}{3}c_3 - \frac{c_1^3}{54}\right), \tag{16}$$

$$a_5 = \frac{1}{4}\left(\frac{2}{3}c_4 + \frac{c_2^2}{18} + \frac{7}{27}c_1c_3 + \frac{7}{486}c_1^4 - \frac{c_1^2c_2}{9}\right). \tag{17}$$

Applying (7) in (14) and (15), we have:

$$|a_2| \leq \frac{4}{3} \text{ and } |a_3| \leq \frac{11}{9}.$$

Now, reshuffling (16), we get:

$$a_4 = \frac{1}{3}\left\{\frac{2}{3}c_3 + \frac{8}{27}c_1c_2 + \frac{c_1}{27}\left(c_2 - \frac{c_1^2}{2}\right)\right\}.$$

If we insert $|c_1| = x \in [0,2]$, then we have:

$$|a_4| \leq \frac{1}{3}\left\{\frac{4}{3} + \frac{16}{27}x + \frac{x}{27}\left(2 - \frac{x^2}{2}\right)\right\}.$$

The above function has its maximum value at $x = 2$. Therefore:

$$|a_4| \leq \frac{68}{81}.$$

Equalities are obtained if we take:

$$\begin{aligned} f(z) &= \exp\left(\frac{4}{3}z + \ln z + \frac{1}{3}z^2\right) \\ &= z + \frac{4}{3}z^2 + \frac{11}{9}z^3 + \frac{68}{81}z^4 + \frac{235}{486}z^5 + \cdots. \end{aligned} \tag{18}$$

□

Theorem 2. *If $f \in \mathcal{S}^*_{car}$ and it has the series form (1), then:*

$$|H_{3,1}(f)| \leq \frac{874}{729}.$$

Proof. From (5), the third Hankel determinant can be written as:

$$H_3(1) = -a_2^2a_5 + 2a_2a_3a_4 - a_3^3 + a_3a_5 - a_4^2.$$

Inserting (14)–(17), we get:

$$\begin{aligned} H_{3,1}(f) = {} & \frac{7}{729}c_1^4c_2 + \frac{281}{11664}c_1^3c_3 + \frac{c_2c_4}{18} + \frac{23}{324}c_1c_2c_3 - \frac{2083}{419904}c_1^6 - \frac{7}{216}c_2^3 - \frac{11}{216}c_1^2c_4 \\ & - \frac{59}{2592}c_1^2c_2^2 - \frac{4}{81}c_3^2. \end{aligned}$$

Now, rearranging, it yields:

$$\begin{aligned} H_{3,1}(f) &= \frac{2083}{209952}c_1^4\left(c_2-\frac{c_1^2}{2}\right)+\frac{c_4}{18}\left(c_2-\frac{c_1^2}{2}\right)+\frac{281}{23328}c_1^3\left(c_3-\frac{67}{2559}c_1c_2\right)+\frac{5}{216}c_1\left(c_2c_3-c_1c_4\right) \\ &\quad -\frac{c_1c_3}{648}\left(c_2-\frac{c_1^2}{2}\right)+\frac{263}{23328}c_1^2\left(c_1c_3-c_2^2\right)-\frac{4}{81}c_3\left(c_3-c_1c_2\right)-\frac{67}{5832}c_1^2c_2^2-\frac{7}{216}c_2^3. \end{aligned}$$

Applying the triangle inequality:

$$\begin{aligned} \left|H_{3,1}(f)\right| &\le \frac{2083}{209952}|c_1|^4\left|c_2-\frac{c_1^2}{2}\right|+\frac{|c_4|}{18}\left|c_2-\frac{c_1^2}{2}\right|+\frac{281}{23328}|c_1|^3\left|c_3-\frac{67}{2559}c_1c_2\right|+\frac{5}{216}|c_1|\left|c_2c_3-c_1c_4\right| \\ &\quad +\frac{|c_1||c_3|}{648}\left|c_2-\frac{c_1^2}{2}\right|+\frac{263}{23328}|c_1|^2\left|c_1c_3-c_2^2\right|+\frac{4}{81}|c_3|\left|c_3-c_1c_2\right|+\frac{67}{5832}|c_1|^2|c_2|^2+\frac{7}{216}|c_2|^3; \end{aligned}$$

besides, (7), (10), (11) and (8) lead us to:

$$\begin{aligned} \left|H_{3,1}(f)\right| &\le \frac{2083}{209952}|c_1|^4\left(2-\frac{|c_1|^2}{2}\right)+\frac{1}{9}\left(2-\frac{|c_1|^2}{2}\right)+\frac{281}{11664}|c_1|^3+\frac{5}{54}|c_1|+\frac{|c_1|}{324}\left(2-\frac{|c_1|^2}{2}\right) \\ &\quad +\frac{263}{5832}|c_1|^2+\frac{16}{81}+\frac{67}{1458}|c_1|^2+\frac{7}{27}. \end{aligned}$$

If we insert $|c_1| = x \in [0,2]$, then we have:

$$\begin{aligned} \left|H_3(f)\right| &\le \frac{2083}{209952}x^4\left(2-\frac{x^2}{2}\right)+\frac{1}{9}\left(2-\frac{x^2}{2}\right)+\frac{281}{11664}x^3+\frac{5}{54}x+\frac{x}{324}\left(2-\frac{x^2}{2}\right) \\ &\quad +\frac{263}{5832}x^2+\frac{16}{81}+\frac{67}{1458}x^2+\frac{7}{27}=\Phi(x), \text{ say.} \end{aligned}$$

Then, the function $\Phi(x)$ is increasing. Therefore, we get its maximum value by putting $x = 2$,

$$\left|H_{3,1}(f)\right| \le \frac{874}{729}.$$

Thus, the proof follows. □

From the function given by (18), we conclude the following conjecture.

Conjecture 3.1. *Let $f \in \mathcal{S}^*_{car}$ and in the form (1). Then, the sharp bound is:*

$$\left|H_{3,1}(f)\right| \le \frac{827}{13122}.$$

4. Bound of $|H_{3,1}(f)|$ for the Family $\mathcal{C}_{car}$

Theorem 3. *If $f \in \mathcal{C}_{car}$ and has the series form (1), then:*

$$|a_2| \le \tfrac{2}{3}, \quad |a_3| \le \tfrac{11}{27} \quad \text{and} \quad |a_4| \le \tfrac{17}{81}.$$

These bounds are the best possible.

Proof. Let the function $f \in \mathcal{C}_{car}$. Then, by the Alexandar-type relation, we say that $zf' \in \mathcal{S}^*_{car}$, and hence, using the coefficient bounds of the family $\mathcal{S}^*_{car}$, which was proven in the last Theorem, we get the needed bounds. □

Theorem 4. *Let f have the form (1) and belong to $\mathcal{C}_{car}$. Then:*

$$|H_{3,1}(f)| \leq \frac{319}{4374}.$$

Proof. From (5), the third Hankel determinant can be obtained as:

$$H_{3,1}(f) = -a_2^2 a_5 + 2a_2 a_3 a_4 - a_3^3 + a_3 a_5 - a_4^2.$$

Utilizing the definition of the family $\mathcal{C}_{car}$, we easily have:

$$\begin{aligned} H_{3,1}(f) &= \frac{97}{174960}c_1^4 c_2 + \frac{61}{58320}c_1^3 c_3 + \frac{1}{270}c_2 c_4 + \frac{1}{405}c_1 c_2 c_3 - \frac{617}{3149280}c_1^6 - \frac{31}{29160}c_2^3 \\ &\quad -\frac{7}{3240}c_1^2 c_4 - \frac{143}{116640}c_1^2 c_2^2 - \frac{1}{324}c_3^2. \end{aligned}$$

After reordering, it yields:

$$\begin{aligned} H_{3,1}(f) &= \frac{97}{349920}c_1^4(c_2 - \frac{617}{873}c_1^2) - \frac{143}{116640}c_1^2 c_2(c_2 - \frac{97}{429}c_1^2) - \frac{7}{3240}c_1^2(c_4 - \frac{61}{126}c_1 c_3) \\ &\quad + \frac{c_2}{270}(c_4 - \frac{31}{108}c_2^2) - \frac{c_3}{324}(c_3 - \frac{324}{405}c_1 c_2). \end{aligned}$$

Using the triangle inequality, we get:

$$\begin{aligned} |H_{3,1}(f)| &\leq \frac{97}{349920}|c_1|^4 \left|c_2 - \frac{617}{873}c_1^2\right| + \frac{143}{116640}|c_1|^2 |c_2| \left|c_2 - \frac{97}{429}c_1^2\right| + \frac{7}{3240}|c_1|^2 \left|c_4 - \frac{61}{126}c_1 c_3\right| \\ &\quad + \frac{|c_2|}{270}\left|c_4 - \frac{31}{108}c_2^2\right| + \frac{|c_3|}{324}\left|c_3 - \frac{324}{405}c_1 c_2\right|. \end{aligned}$$

The application of (7) and (11) leads us to:

$$\begin{aligned} |H_{3,1}(f)| &\leq \frac{97}{10935} + \frac{143}{7290} + \frac{7}{405} + \frac{4}{270} + \frac{4}{324} \\ &= \frac{319}{4374}. \end{aligned}$$

Thus, the proof is completed. □

5. Bound of $|H_{3,1}(f)|$ for the Family $\mathcal{R}_{car}$

Theorem 5. *Let $f \in \mathcal{R}_{car}$ and be given in the form (1). Then:*

$$|a_2| \leq \tfrac{2}{3}, \quad |a_3| \leq \tfrac{4}{9}, \quad |a_4| \leq \tfrac{1}{3}.$$

These results are the best possible.

Proof. Let $f \in \mathcal{R}_{car}$. Then, we can write (3), in the form of the Schwarz function, as:

$$f'(z) = 1 + \frac{4}{3}w(z) + \frac{2}{3}(w(z))^2, \ (z \in \Delta).$$

Since:

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots, \tag{19}$$

by comparing (19) and (13), we may get:

$$a_2 = \frac{c_1}{3}, \tag{20}$$
$$a_3 = \frac{2}{9}\left(c_2 - \frac{c_1^2}{4}\right), \tag{21}$$
$$a_4 = \frac{1}{6}\left(c_3 - \frac{c_1 c_2}{2}\right), \tag{22}$$
$$a_5 = \frac{1}{15}\left(2c_4 + \frac{c_1^4}{8} - \frac{c_2^2}{2} - c_1 c_3\right). \tag{23}$$

Using (7) in (20), we get:

$$|a_2| \leq \frac{2}{3}.$$

Applying (11) in (21) and (22), we obtain:

$$|a_3| \leq \frac{4}{9} \text{ and } |a_4| \leq \frac{1}{3}.$$

Thus, the proof is completed.

Equalities in each coefficient $|a_2|$, $|a_3|$, and $|a_4|$ are obtained respectively by taking:

$$\begin{aligned} f_1(z) &= z + \frac{2}{3}z^2 + \frac{2}{9}z^3, \\ f_2(z) &= z + \frac{4}{9}z^3 + \frac{2}{15}z^5, \\ f_3(z) &= z + \frac{1}{3}z^4 + \frac{2}{21}z^7. \end{aligned}$$

□

Theorem 6. *Let $f \in \mathcal{R}_{car}$ and be given in the form (1). Then:*

$$|H_{3,1}(f)| \leq \frac{754}{1215}.$$

Proof. From (5), the third Hankel determinant can be written as:

$$H_3(1) = -a_2^2 a_5 + 2a_2 a_3 a_4 - a_3^3 + a_3 a_5 - a_4^2.$$

Utilizing (20)–(23), we have:

$$\begin{aligned} H_{3,1}(f) = {} & \frac{7}{2430}c_1^4 c_2 + \frac{2}{405}c_1^3 c_3 + \frac{4}{135}c_2 c_4 + \frac{61}{1620}c_1 c_2 c_3 - \frac{71}{58320}c_1^6 - \frac{67}{1620}c_2^3 \\ & - \frac{c_1^2 c_4}{45} - \frac{107}{19440}c_1^2 c_2^2 - \frac{c_3^2}{36}. \end{aligned}$$

By rearranging, it yields:

$$\begin{aligned} H_{3,1}(f) = {} & \frac{7}{4860}c_1^4\left(c_2 - \frac{71}{84}c_1^2\right) - \frac{107}{19440}c_1^2 c_2\left(c_2 - \frac{28}{107}c_1^2\right) - \frac{c_1^2}{45}\left(c_4 - \frac{2}{9}c_1 c_3\right) \\ & - \frac{c_3}{36}\left(c_3 - \frac{61}{45}c_1 c_2\right) + \frac{4}{135}c_2\left(c_4 - \frac{67}{108}c_2^2\right). \end{aligned}$$

Implementing the triangle inequality, we have:

$$\begin{aligned} |H_{3,1}(f)| \leq\ & \frac{7}{4860}|c_1|^4\left|c_2-\frac{71}{84}c_1^2\right|+\frac{107}{19440}|c_1|^2|c_2|\left|c_2-\frac{28}{107}c_1^2\right|+\frac{|c_1|^2}{45}\left|c_4-\frac{2}{9}c_1c_3\right| \\ & +\frac{|c_3|}{36}\left|c_3-\frac{61}{45}c_1c_2\right|+\frac{4}{135}|c_2|\left|c_4-\frac{67}{108}c_2^2\right|. \end{aligned}$$

(7) and (11) lead us to:

$$\begin{aligned} |H_{3,1}(f)| & \leq \frac{224}{4860}+\frac{1712}{19440}+\frac{8}{45}+\frac{77}{405}+\frac{16}{135}. \\ & = \frac{754}{1215}. \end{aligned}$$

Thus, the proof of this result is completed. □

6. Bounds of $|H_{3,1}(f)|$ for Two-fold and Three-fold functions

Let $m \in \mathbb{N} = \{1, 2, \ldots\}$. If a rotation $\boldsymbol{\Omega}$ about the origin through an angle $2\pi/m$ carries $\boldsymbol{\Omega}$ on itself, then such a domain $\boldsymbol{\Omega}$ is called m-fold symmetric. An analytic function f is m-fold symmetric in Δ, if:

$$f\left(e^{2\pi i/m}z\right) = e^{2\pi i/m}f(z) \ \ (z \in \Delta).$$

By $\mathcal{S}^{(m)}$, we define the family of m-fold univalent functions having the following Taylor series form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \quad (z \in \Delta). \tag{24}$$

The subfamilies $\mathcal{S}^{*(m)}_{car}$, $\mathcal{C}^{(m)}_{car}$, and $\mathcal{R}^{(m)}_{car}$ of $\mathcal{S}^{(m)}$ are the families of the m-fold symmetric starlike, convex, and bounded turning functions, respectively, associated with the cardioid functions. More intuitively, an analytic function f of the form (24) belongs to the families $\mathcal{S}^{*(m)}_{car}$, $\mathcal{C}^{(m)}_{car}$, and $\mathcal{R}^{(m)}_{car}$ if and only if:

$$\frac{zf'(z)}{f(z)} = 1+\frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right)+\frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2, \ p \in \mathcal{P}^{(m)}, \tag{25}$$

$$1+\frac{zf''(z)}{f'(z)} = 1+\frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right)+\frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2, \ p \in \mathcal{P}^{(m)}, \tag{26}$$

$$f'(z) = 1+\frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right)+\frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2, \ p \in \mathcal{P}^{(m)}, \tag{27}$$

where the family $\mathcal{P}^{(m)}$ is defined by:

$$\mathcal{P}^{(m)} = \left\{p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} c_{mk}z^{mk}, \ (z \in \mathbf{D})\right\}. \tag{28}$$

Now, we prove some theorems concerned with two-fold and three-fold symmetric functions.

Theorem 7. *If $f \in \mathcal{S}^{*(2)}_{car}$ and it has the form given in (24), then:*

$$|H_{3,1}(f)| \leq \frac{2}{9}.$$

Proof. Let $f \in \mathcal{S}^{*(2)}_{car}$. Then, there exists a function $p \in \mathcal{P}^{(2)}$ such that:

$$\frac{zf'(z)}{f(z)} = 1 + \frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2.$$

Using the series form (24) and (28), when $m = 2$ in the above relation, we can get:

$$a_3 = \frac{c_2}{3}, \tag{29}$$

$$a_5 = \frac{1}{4}\left(\frac{c_2^2}{18} + \frac{2}{3}c_4\right). \tag{30}$$

Now:

$$H_3(f) = a_3 a_5 - a_3^3.$$

Utilizing (29) and (30), we get:

$$H_{3,1}(f) = -\frac{7}{216}c_2^3 + \frac{c_2 c_4}{18}.$$

By reordering, it yields:

$$H_{3,1}(f) = \frac{c_2}{18}\left(c_4 - \frac{7}{12}c_2^2\right).$$

Using the triangle inequality long with (11) and (7), we have:

$$|H_{3,1}(f)| \leq \frac{2}{9}.$$

Hence, the proof is done. □

Theorem 8. *If $f \in \mathcal{S}^{*(3)}_{car}$ and it has the form (24), then:*

$$|H_{3,1}(f)| \leq \frac{16}{81}.$$

The result is sharp for the function:

$$f(z) = \exp\left(\ln z + \frac{4}{9}z^3 + \frac{1}{9}z^6\right) = z + \frac{4}{9}z^4 + \frac{17}{81}z^7 + \cdots. \tag{31}$$

Proof. Let $f \in \mathcal{S}^{*(3)}_{car}$. Then, there exists a function $p \in \mathcal{P}^{(3)}$ such that:

$$\frac{zf'(z)}{f(z)} = 1 + \frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2.$$

Utilizing the series form (24) and (28), when $m = 3$ in the above relation, we can obtain:

$$a_4 = \frac{2}{9}c_3.$$

Then,

$$H_{3,1}(f) = -a_4^2 = -\frac{4}{81}c_3^2.$$

Utilizing (7) along with triangle inequality, we have:

$$|H_{3,1}(f)| \leq \frac{16}{81}.$$

Thus, the proof is completed. □

Theorem 9. *Let $f \in \mathcal{C}_{car}^{(2)}$, and it has the form (24), then:*

$$|H_{3,1}(f)| \leq \frac{2}{135}.$$

Proof. Let $f \in \mathcal{C}_{car}^{(2)}$. Then, there exists a function $p \in \mathcal{P}^{(2)}$ such that:

$$1+\frac{zf''(z)}{f'(z)}=1+\frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right)+\frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2.$$

Utilizing the series form (24) and (28), when $m=2$ in the above relation, we can obtain:

$$a_3 = \frac{c_2}{9}, \tag{32}$$

$$a_5 = \frac{1}{20}\left(\frac{c_2^2}{18}+\frac{2}{3}c_4\right). \tag{33}$$

$$H_{3,1}(f)=a_3a_5-a_3^3.$$

Using (32) and (33), we have:

$$H_{3,1}(f)=-\frac{31}{29160}c_2^3+\frac{c_2c_4}{270}.$$

Now, reordering the above equation, we obtain:

$$H_3(f)=\frac{c_2}{270}\left(c_4-\frac{31}{108}c_2^2\right).$$

Application of (7), (11), and the triangle inequality leads us to:

$$|H_{3,1}(f)| \leq \frac{2}{135}.$$

Thus, the required result is completed. □

Theorem 10. *If $f \in \mathcal{C}_{car}^{(3)}$ and it has the form given in (24), then:*

$$|H_{3,1}(f)| \leq \frac{1}{81}.$$

The result is sharp for the function:

$$f(z)=\int_0^z \frac{\exp\left(\ln x+\frac{4}{9}x^3+\frac{1}{9}x^6\right)}{x}dx = z+\frac{1}{9}z^4+\frac{17}{657}z^7+\cdots.$$

Proof. Let $f \in \mathcal{C}_{car}^{(3)}$. Then, there exists a function $p \in \mathcal{P}^{(3)}$ such that:

$$1+\frac{zf''(z)}{f'(z)}=1+\frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right)+\frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2.$$

Utilizing the series form (24) and (28), when $m=3$ in the above relation, we obtain:

$$a_4=\frac{c_3}{18}.$$

Then:

$$H_{3,1}(f) = -a_4^2 = -\frac{c_3^2}{324}.$$

Implementing (7) and the triangle inequality, we have:

$$|H_{3,1}(f)| \le \frac{1}{81}.$$

Hence, the proof is done. □

Theorem 11. *Let $f \in \mathcal{R}_{car}^{(2)}$ be of the form (24). Then:*

$$|H_{3,1}(f)| \le \frac{16}{135}.$$

Proof. Since $f \in \mathcal{R}_{car}^{(2)}$, therefore there exists a function $p \in \mathcal{P}^{(2)}$ such that:

$$f'(z) = 1 + \frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2.$$

For $f \in \mathcal{R}_{car}^{(2)}$, using the series form (24) and (28), when $m = 2$ in the above relation, we can write:

$$a_3 = \frac{2}{6}c_2, \tag{34}$$

$$a_5 = \frac{1}{5}\left(\frac{2}{3}c_4 - \frac{c_2^2}{6}\right). \tag{35}$$

It is clear that for $f \in \mathcal{R}_{car}^{(2)}$,

$$H_{3,1}(f) := a_3 a_5 - a_3^3.$$

Applying (34) and (35), we have:

$$H_{3,1}(f) = \frac{4}{135}c_2 c_4 - \frac{67}{3645}c_2^3.$$

By rearrangement, we have:

$$H_{3,1}(f) = \frac{4}{135}c_2(c_4 - \frac{67}{108}c_2^2).$$

Using Lemma (7), (10), and triangle inequality, we get:

$$|H_{3,1}(f)| \le \frac{16}{135}.$$

Hence, the proof is completed. □

Theorem 12. *If $f \in \mathcal{R}_{car}^{(3)}$ and it is of the form (24), then:*

$$|H_{3,1}(f)| \le \frac{1}{9}.$$

This result is sharp for the function:

$$f(z) = \int_0^z \left(1 + \frac{4}{3}x^3 + \frac{2}{3}x^6\right) dx = z + \frac{1}{3}z^4 + \frac{2}{21}z^7.$$

Proof. Since $f \in \mathcal{R}_{car}^{(3)}$, there exists a function $p \in \mathcal{P}^{(3)}$ such that:

$$f'(z) = 1 + \frac{4}{3}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{2}{3}\left(\frac{p(z)-1}{p(z)+1}\right)^2.$$

For $f \in \mathcal{R}_{car}^{(3)}$, using the series form (24) and (28), when $m = 2$ in the above relation, we can write:

$$a_4 = \frac{c_3}{6}.$$

Then:

$$H_{3,1}(f) := -a_4^2 = -\frac{c_3^2}{36}.$$

Implementing (7), we have:

$$|H_{3,1}(f)| \leq \frac{1}{9}.$$

Hence, the proof is completed. □

7. Conclusions

In this article, we studied the Hankel determinant $H_{3,1}(f)$ for the subfamilies $\mathcal{S}_{car}^*$, $\mathcal{C}_{car}$, and $\mathcal{R}_{car}$ of the analytic function using a very simple technique. Further, these bounds were also discussed for two-fold symmetric and three-fold symmetric functions.

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