Abstract: In this paper, we consider a resolvent operator which depends on the composition of two mappings with ⊕ operation. We prove some of the properties of the resolvent operator, that is, that it is single-valued as well as Lipschitz-type-continuous. An existence and convergence result is proven for a generalized implicit set-valued variational inclusion problem with ⊕ operation. Some special cases of a generalized implicit set-valued variational inclusion problem with ⊕ operation are discussed. An example is constructed to illustrate some of the concepts used in this paper.

Keywords: algorithm; implicit; inclusion; set-valued mapping; ⊕ operation

MSC: 47H09; 49J40
Li and his co-authors [21–23] first used the $\oplus$ operation for solving some classes of variational inclusions and after that, Ahmad and his co-authors [24–26] also solved some generalized variational inclusions with $\oplus$ operation.

In this paper, we consider a resolvent operator with $\oplus$ operation involving composition of two mappings. We proved some properties of the resolvent operator. An iterative algorithm was constructed to solved a generalized implicit set-valued variational inclusion problem with $\oplus$ operation in real ordered positive Hilbert spaces. An existence and convergence result was proven for a generalized implicit set-valued inclusion problem with $\oplus$ operation. Some special cases are discussed and an example is given in support of some of the concepts used in this work.

2. Preliminaries

Let $C$ be a cone with partial ordering “$\leq$”. An ordered Hilbert space with norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$ is called positive if $0 \leq x$ and $0 \leq y$, then $0 \leq \langle x, y \rangle$ holds. Throughout the paper, $H_p$ is assumed to be a real ordered positive Hilbert space. We denote $2^H_p$ (respectively, $C^*(H_p)$) as the family of nonempty (respectively, compact) subsets of $H_p$, and $d$ is the metric induced by the norm and $D(.,.)$ is the Hausdorff metric on $C^*(H_p)$.

Now, we illustrate some known concepts and results which are needed to prove the main result. The following concepts and results can be found in [20–27].

**Definition 1.** A nonempty closed convex subset $C$ of $H_p$ is said to be a cone if:

(i) for any $x \in C$ and any $\lambda > 0$, $\lambda x \in C$;
(ii) if $x \in C$ and $-x \in C$, then $x = 0$.

**Definition 2.** Let $C$ be the cone, then:

(i) $C$ is called a normal cone if there exists a constant $\lambda_N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \lambda_N\|y\|$, for all $x, y \in H_p$;
(ii) for any $x, y \in H_p$, $x \leq y$ if and only if $y - x \in C$;
(iii) $x$ and $y$ are said to be comparative to each other if either $x \leq y$ or $y \leq x$ holds and is denoted by $x \propto y$.

**Definition 3.** For any $x, y \in H_p$, $\text{lub}\{x, y\}$ denotes the least upper bound and $\text{glb}\{x, y\}$ denotes the greatest lower bound of the set $\{x, y\}$. Suppose $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ exist, then some binary operations are given below:

(i) $x \lor y = \text{lub}\{x, y\}$;
(ii) $x \land y = \text{glb}\{x, y\}$;
(iii) $x \oplus y = (x - y) \lor (y - x)$;
(iv) $x \odot y = (x - y) \land (y - x)$.

The operations $\lor, \land, \oplus$, and $\odot$ are called OR, AND, XOR, and XNOR operations, respectively.

**Lemma 1.** If $x \propto y$, then $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ exist such that $x - y \propto y - x$ and $0 \leq (x - y) \lor (y - x)$.

**Lemma 2.** For any natural number $n$, $x \propto y_n$ and $y_n \to y^*$ as $n \to \infty$, then $x \propto y^*$.

**Proposition 1.** Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then the following relations hold for all $x,y,u,v,w \in H_p$ and $\alpha, \beta, \lambda \in \mathbb{R}$:

(i) $x \odot x = 0, x \odot y = y \odot x = -x \oplus y = -(y \oplus x)$;
(ii) if $x \propto 0$, then $-x \oplus 0 \leq x \leq x \oplus 0$;
A set-valued mapping $M$

Definition 8.

A set-valued mapping $T$

Let $F$

Definition 4.

A single-valued mapping $F$

Definition 9.

Let $M$

$a$ constant

$\propto$

Definition 5.

A single-valued mapping $F$

Mathematics

Proposition 2. Let $C$ be a normal cone in $\mathcal{H}_p$ with constant $\lambda_N$, then for each $x, y \in \mathcal{H}_p$, the following relations hold:

(i) $\|0 \oplus 0\| = \|0\| = 0$;
(ii) $\|x \oplus y\| \leq \|x\| \lor \|y\| \leq \|x\| + \|y\|$;
(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_N\|x \oplus y\|$;
(iv) if $x \propto y$, then $\|x \oplus y\| = \|x - y\|$.

Definition 4. Let $F : \mathcal{H}_p \rightarrow \mathcal{H}_p$ be a single-valued mapping, then:

(i) $F$ is said to be comparison mapping, if for each $x, y \in \mathcal{H}_p$, $x \propto y$ then $F(x) \propto F(y)$, $x \propto F(x)$ and $y \propto F(y)$;
(ii) $F$ is said to be strongly comparison mapping, if $F$ is a comparison mapping and $F(x) \propto F(y)$ if and only if $x \propto y$, for all $x, y \in \mathcal{H}_p$.

Definition 5. A single-valued mapping $F : \mathcal{H}_p \rightarrow \mathcal{H}_p$ is said to be $\beta$-ordered compression mapping if $F$ is a comparison mapping and:

$F(x) \oplus F(y) \leq \beta(x \oplus y)$, for $0 < \beta < 1$.

Definition 6. Let $M : \mathcal{H}_p \rightarrow 2^{\mathcal{H}_p}$ be a set-valued mapping. Then:

(i) $M$ is said to be a comparison mapping if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for $v_x \in M(x)$ and $v_y \in M(y)$, $v_x \propto v_y$, for all $x, y \in \mathcal{H}_p$;
(ii) A comparison mapping $M$ is said to be $\alpha$-non-ordinary difference mapping if:

$(v_x \oplus v_y) \oplus \alpha(x \oplus y) = 0$ holds, for all $x, y \in \mathcal{H}_p$, $v_x \in M(x)$ and $v_y \in M(y)$;
(iii) A comparison mapping $M$ is said to be $\theta$-ordered rectangular if there exists a constant $\theta > 0$ such that:

$\langle v_x \oplus v_y, -(x \oplus y) \rangle \geq \theta \|x \oplus y\|^2$ holds, for all $x, y \in \mathcal{H}_p$, there exists $v_x \in M(x)$ and $v_y \in M(y)$.

Definition 7. A set-valued mapping $M : \mathcal{H}_p \rightarrow 2^{\mathcal{H}_p}$ is said to be $\lambda$-XOR-ordered strongly monotone compression mapping if $x \propto y$, then there exists a constant $\lambda > 0$ such that:

$\lambda(v_x \oplus v_y) \geq x \oplus y$, for all $x, y \in \mathcal{H}_p, v_x \in M(x), v_y \in M(y)$.

Definition 8. A set-valued mapping $T : \mathcal{H}_p \rightarrow C^*(\mathcal{H}_p)$ is said to be $D$-Lipschitz continuous if for all $x, y \in \mathcal{H}_p, x \propto y$, there exists a constant $\lambda_T > 0$ such that:

$D(T(x), T(y)) \leq \lambda_T \|x \oplus y\|.$

Definition 9. A single-valued mapping $F : \mathcal{H}_p \rightarrow \mathcal{H}_p$ is said to be Lipschitz-type-continuous if there exists a constant $\delta > 0$ such that:

$\|F(x) \oplus F(y)\| \leq \delta \|x \oplus y\|$, for all $x, y \in \mathcal{H}_p$. 
Let $H, F : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings, we consider the composition of $H$ and $F$ as:

$$(H \circ F)(x) = H(F(x)), \text{ for all } x \in \mathcal{H}_p.$$  

**Definition 10.** Let $H, F : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $H \circ F$ is strongly comparison and $\beta$-ordered compression mapping. Then, a set-valued comparison mapping $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be $(\alpha, \lambda)$-XOR-NODSM if $M$ is an $\alpha$-non-ordinary difference mapping and $\lambda$-XOR-ordered strongly monotone compression mapping and $[(H \circ F) \ominus \lambda M](\mathcal{H}_p) = \mathcal{H}_p$, for $\alpha, \beta, \lambda > 0$.

**Definition 11.** Let $H, F : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mapping such that $H \circ F$ is strongly comparison and $\beta$-ordered compression mapping. Suppose that the set-valued mapping $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is $(\alpha, \lambda)$-XOR-NODSM mapping. We define the resolvent operator $J_{\lambda,M}^{H,F} : \mathcal{H}_p \to \mathcal{H}_p$ by:

$$J_{\lambda,M}^{H,F}(x) = [(H \circ F) \ominus \lambda M]^{-1}(x), \text{ for all } x \in \mathcal{H}_p \text{ and } \alpha, \lambda > 0.$$  

(1)

Now, we present some properties of the resolvent operator defined by (1).

**Proposition 3.** Let $H, F : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings such that $(H \circ F)$ is $\beta$-ordered compression mapping and $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is the set-valued $\theta$-ordered rectangular mapping with $\lambda \theta > \beta$. Then, the resolvent operator $J_{\lambda,M}^{H,F} : \mathcal{H}_p \to \mathcal{H}_p$ is single-valued.

**Proof.** For any given $u \in \mathcal{H}_p$ and $\lambda > 0$, let $x, y \in [(H \circ F) \ominus \lambda M]^{-1}(u)$. Then:

$$v_x = \frac{1}{\lambda}(u \ominus (H \circ F)(x)) = \frac{1}{\lambda}(u \ominus H(F(x))) \in M(x),$$

and:

$$v_y = \frac{1}{\lambda}(u \ominus (H \circ F)(y)) = \frac{1}{\lambda}(u \ominus (H \circ F)(y)) \in M(y).$$

Using (i) and (ii) of Proposition 1, we obtain:

$$v_x \circ v_y = \frac{1}{\lambda}(u \ominus H(F(x))) \ominus \frac{1}{\lambda}(u \ominus H(F(y)))$$

$$= \frac{1}{\lambda}[(u \ominus H(F(x))) \ominus (u \ominus H(F(y)))]$$

$$= -\frac{1}{\lambda}[(u \ominus H(F(x))) \ominus (u \ominus H(F(y)))]$$

$$= -\frac{1}{\lambda}[(u \ominus u) \ominus (H(F(x)) \ominus H(F(y)))]$$

$$= -\frac{1}{\lambda}[(u \ominus H(F(x)) \ominus H(F(y)))]$$

$$\leq -\frac{1}{\lambda}[H(F(x)) \ominus H(F(y))].$$

Thus, we have:

$$v_x \circ v_y \leq -\frac{1}{\lambda}[H(F(x)) \ominus H(F(y))].$$  

(2)
Since $M$ is $\theta$-ordered rectangular mapping, $(H \circ F)$ is $\beta$-ordered compression mapping and using (2), we have:

$$\theta \|x \oplus y\|^2 \leq \langle v_x \circ v_y, - (x \oplus y) \rangle$$

$$\leq \left\langle -\frac{1}{\lambda} [H(F(x)) \oplus H(F(y))], -(x \oplus y) \right\rangle$$

$$\leq \frac{1}{\lambda} ((H \circ F)(x) \oplus (H \circ F)(y), x \oplus y)$$

$$\leq \frac{1}{\lambda} (\beta(x \oplus y), x \oplus y)$$

$$= \frac{\beta}{\lambda} \|x \oplus y\|^2,$$

i.e.,

$$\left(\theta - \frac{\beta}{\lambda}\right) \|x \oplus y\|^2 \leq 0, \text{ for } \lambda \theta > \beta,$$

which shows that:

$$\|x \oplus y\| = 0, \text{ which implies } x \oplus y = 0.$$

Therefore $x = y$, i.e., the resolvent operator $\mathcal{J}^{H,F}_{\lambda,M}$ is single-valued, for $\lambda \theta > \beta$. □

**Proposition 4.** Let $M : \mathcal{H}_p \rightarrow 2^{\mathcal{H}_p}$ be $(\alpha, \lambda)$-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^{H,F}_{\lambda,M}$. Let $H, F : \mathcal{H}_p \rightarrow \mathcal{H}_p$ be the single-valued mappings such that $(H \circ F)$ is strongly comparison mapping with respect to $\mathcal{J}^{H,F}_{\lambda,M}$. Then the resolvent operator $\mathcal{J}^{H,F}_{\lambda,M} : \mathcal{H}_p \rightarrow \mathcal{H}_p$ is a comparison mapping.

**Proof.** Since $M$ is $(\alpha, \lambda)$-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^{H,F}_{\lambda,M}$, i.e., $M$ is $\alpha$-non-ordinary difference as well as $\lambda$-XOR-ordered strongly monotone compression mapping with respect to $\mathcal{J}^{H,F}_{\lambda,M}$. For any $x, y \in \mathcal{H}_p$, let $x \prec y$ and:

$$v^+_x = \frac{1}{\lambda} (x \oplus (H \circ F)(\mathcal{J}^{H,F}_{\lambda,M}(x))) = \frac{1}{\lambda} (x \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))) \in M(\mathcal{J}^{H,F}_{\lambda,M}(x)), \quad (3)$$

and:

$$v^+_y = \frac{1}{\lambda} (y \oplus (H \circ F)(\mathcal{J}^{H,F}_{\lambda,M}(y))) = \frac{1}{\lambda} (y \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))) \in M(\mathcal{J}^{H,F}_{\lambda,M}(y)). \quad (4)$$

Since $M$ is $\lambda$-XOR-ordered strongly monotone compression mapping and using (3) and (4), we have:

$$(x \oplus y) \leq \lambda (v^+_x \oplus v^+_y)$$

$$(x \oplus y) \leq (x \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))) \oplus (y \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y))))$$

$$(x \oplus y) \leq (x \oplus y) \oplus (H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))) \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y))))$$

$$(x \oplus y) \leq H(F(\mathcal{J}^{H,F}_{\lambda,M}(x))) \oplus H(F(\mathcal{J}^{H,F}_{\lambda,M}(y))))$$

$$(x \oplus y) \leq H(F(\mathcal{J}^{H,F}_{\lambda,M}(x)))) - H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))) \lor [H(F(\mathcal{J}^{H,F}_{\lambda,M}(y)))) - H(F(\mathcal{J}^{H,F}_{\lambda,M}(x))))].$$

which implies either:

\[ 0 \leq \left[ H(F(J_{\lambda,M}^{H,F}(x))) - H(F(J_{\lambda,M}^{H,F}(y))) \right] \]  or
\[ 0 \leq \left[ H(F(J_{\lambda,M}^{H,F}(y))) - H(F(J_{\lambda,M}^{H,F}(x))) \right]. \]

Thus, in both cases, we have:

\[ (H \circ F)(J_{\lambda,M}^{H,F}(x)) \preceq (H \circ F)(J_{\lambda,M}^{H,F}(y)). \]

Since \((H \circ F)\) is strongly comparison mapping with respect to \(J_{\lambda,M}^{H,F}\), thus, we have \(J_{\lambda,M}^{H,F}(x) \preceq J_{\lambda,M}^{H,F}(y)\), i.e., the resolvent operator \(J_{\lambda,M}^{H,F}\) is a comparison mapping. \(\square\)

**Proposition 5.** If all the mappings and conditions are the same as those stated in Proposition 3, then the following condition holds:

\[ \| J_{\lambda,M}^{H,F}(x) \oplus J_{\lambda,M}^{H,F}(y) \| \leq \frac{1}{(\lambda \theta - \beta)} \| x \oplus y \|, \text{ for } \lambda \theta > \beta \text{ and } \alpha, \beta, \lambda > 0, \]

i.e., the resolvent operator \(J_{\lambda,M}^{H,F}\) is Lipschitz-type-continuous mapping.

**Proof.** Let \(x, y \in \mathcal{H}_p\), and:

\[ v_x^* = \frac{1}{\lambda} (x \oplus (H \circ F)(J_{\lambda,M}^{H,F}(x))) = \frac{1}{\lambda} (x \oplus H(F(J_{\lambda,M}^{H,F}(x)))) \in M(J_{\lambda,M}^{H,F}(x)), \]

(5)

and:

\[ v_y^* = \frac{1}{\lambda} (y \oplus (H \circ F)(J_{\lambda,M}^{H,F}(y))) = \frac{1}{\lambda} (y \oplus H(F(J_{\lambda,M}^{H,F}(y)))) \in M(J_{\lambda,M}^{H,F}(y)). \]

(6)

Since \((H \circ F)\) is \(\beta\)-ordered compression mapping and using (5) and (6), we have

\[ v_x^* \oplus v_y^* = \frac{1}{\lambda} \left[ (x \oplus H(F(J_{\lambda,M}^{H,F}(x)))) \oplus (y \oplus H(F(J_{\lambda,M}^{H,F}(y)))) \right] \]
\[ = \frac{1}{\lambda} \left[ (x \oplus y) \oplus \left( H(F(J_{\lambda,M}^{H,F}(x))) \oplus H(F(J_{\lambda,M}^{H,F}(y))) \right) \right] \]
\[ \leq \frac{1}{\lambda} \left[ (x \oplus y) \oplus \beta(J_{\lambda,M}^{H,F}(x) \oplus J_{\lambda,M}^{H,F}(y)) \right]. \]

(7)

Since \(M\) is \(\theta\)-ordered rectangular mapping and using (7), for any:

\[ J_{\lambda,M}^{H,F}(x) \in M(J_{\lambda,M}^{H,F}(x)) \text{ and } J_{\lambda,M}^{H,F}(y) \in M(J_{\lambda,M}^{H,F}(y)), \]

we have:
\[ \theta \| \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \|^2 \leq \left\langle v_x^* \odot v_y^*, \left( \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right) \right\rangle \\
\leq \left\langle v_x^* \odot v_y^*, \left( \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right) \right\rangle \\
\leq \frac{1}{\lambda} \left[ \left\| (x \oplus y) \oplus \beta(\mathcal{J}^{H,F}_{\lambda,M}(x)) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right\| \right] \\
\leq \frac{1}{\lambda} \left[ \left\| (x \oplus y) \oplus \beta(\mathcal{J}^{H,F}_{\lambda,M}(x)) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right\| \right] \left\| \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right\| .
\]

Using (iii) of Proposition 2, we have:
\[ \theta \| \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \|^2 \leq \frac{1}{\lambda} \left[ \left\| (x \oplus y) \oplus \beta(\mathcal{J}^{H,F}_{\lambda,M}(x)) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right\| \right] \left\| \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right\| \\
\leq \frac{1}{\lambda} \left[ \left\| (x \oplus y) \right\| - \left( \beta(\mathcal{J}^{H,F}_{\lambda,M}(x)) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right) \left\| \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right\| \right] \\
\leq \frac{1}{\lambda} \left[ \left\| (x \oplus y) \right\| \left\| \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right\| \right] \\
+ \frac{\beta}{\lambda} \left\| \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right\|^2 .
\]

It follows that:
\[ \left\| \mathcal{J}^{H,F}_{\lambda,M}(x) \oplus \mathcal{J}^{H,F}_{\lambda,M}(y) \right\| \leq \frac{1}{(\lambda \theta - \beta)} \| x \oplus y \| , \text{ for } \lambda \theta > \beta .
\]

This completes the proof. \(\square\)

In support of Proposition 3–5, we have the following example.

**Example 1.** Let \( \mathcal{H}_p = [0, \infty) \) with the usual inner product and norm, and let \( C = [0, 1] \) be a normal cone in \([0, \infty)\). Let \( H : \mathcal{H}_p \to \mathcal{H}_p \) and \( F : \mathcal{H}_p \to \mathcal{H}_p \) be the mappings defined by:
\[ H(x) = \frac{x}{3} + 1, \text{ and } F(x) = \frac{x}{2} \forall x \in [0, \infty) .\]
Let $x, y \in \mathcal{H}_p$, $x \propto y$, then we calculate:

\[
(H \circ F)(x) \oplus (H \circ F)(y) = H(F(x)) \oplus H(F(y)) \\
= \left( \frac{F(x)}{3} + 1 \right) \oplus \left( \frac{F(y)}{3} + 1 \right) \\
= \left( \frac{x}{6} + 1 \right) \oplus \left( \frac{y}{6} + 1 \right) \\
= \left( \left( \frac{x}{6} + 1 \right) - \left( \frac{y}{6} + 1 \right) \right) \lor \left( \left( \frac{y}{6} + 1 \right) - \left( \frac{x}{6} + 1 \right) \right) \\
= \frac{1}{6}((x - y) \lor (y - x)) \\
= \frac{1}{6}(x \oplus y) \\
\leq \frac{1}{5}(x \oplus y),
\]

i.e.,

\[
(H \circ F)(x) \oplus (H \circ F)(y) \leq \frac{1}{5}(x \oplus y), \ \forall \ x, y \in [0, \infty).
\]

Hence, $H \circ F$ is $\frac{1}{5}$-ordered compression mapping.

Suppose that $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is a the set-valued mapping defined by:

\[
M(x) = \{x + 1\}, \ \forall \ x \in [0, \infty).
\]

It can be easily verified that $M$ is a comparison mapping, 1-XOR-ordered strongly monotone comparison mapping, and 1-non-ordinary difference mapping.

Let $v_x = x + 1 \in M(x)$ and $v_y = y + 1 \in M(y)$, then we evaluate:

\[
\langle v_x \odot v_y, -(x \oplus y) \rangle = \langle v_x \odot v_y, x \oplus y \rangle \\
= \langle (x + 1) \odot (y + 1), x \oplus y \rangle \\
= \langle x \oplus y, x \oplus y \rangle \\
= \|x \oplus y\|^2 \\
\geq \frac{1}{2}\|x \oplus y\|^2,
\]

i.e.,

\[
\langle v_x \odot v_y, -(x \oplus y) \rangle \geq \frac{1}{2}\|x \oplus y\|^2, \ \forall \ x, y \in [0, \infty).
\]

Thus, $M$ is a $\frac{1}{2}$-ordered rectangular comparison mapping. Further, it is clear that for $\lambda = 1$, $[(H \circ F) \oplus \lambda M][0, \infty) = [0, \infty)$. Hence, $M$ is an $(1, 1)$-XOR-NODSM set-valued mapping.

The resolvent operator defined by (1) is given by:

\[
J_{\lambda, M}^{H, F}(x) = \frac{6x}{5}, \ \forall \ x \in [0, \infty).
\]  

(8)

It is easy to check that the resolvent operator defined above is a comparison and single-valued mapping.
Further:

\[
\| J_{\lambda,M}^{H,F} (x) \oplus J_{\lambda,M}^{H,F} (y) \| = \left\| \frac{6x}{5} \oplus \frac{6y}{5} \right\| = \frac{6}{5} \| x \oplus y \| \leq \frac{10}{3} \| x \oplus y \|,
\]

i.e.,

\[
\| J_{\lambda,M}^{H,F} (x) \oplus J_{\lambda,M}^{H,F} (y) \| \leq \frac{10}{3} \| x \oplus y \|, \forall x, y \in [0, \infty).
\]

That is, the resolvent operator \( J_{\lambda,M}^{H,F} \) is \( \frac{10}{3} \)-Lipschitz-type-continuous.

3. Formulation of The Problem and Existence of Solution

Let \( H_p \) be a real positive Hilbert space. Let \( A, B, C : H_p \to C^*(H_p) \) and \( M : H_p \to 2^{H_p} \) be the set-valued mappings and let \( G : H_p \times H_p \times H_p \to H_p \) be a single-valued mapping. Then, we consider the following problem:

Find \( x \in H_p, w \in A(x), u \in B(x) \) and \( v \in C(x) \) such that:

\[
0 \in G(w, u, v) \oplus M(x). \tag{9}
\]

We call the problem in Equation (9) a generalized implicit set-valued variational inclusion problem with \( \oplus \) operation.

(i) If \( C \equiv 0 \) and \( G(w, u, v) = G(w, v) \), then problem (9) coincides with the problem studied by Ahmad et al. [1].

(ii) If \( B, C \equiv 0 \) and \( A \) is single-valued such that \( G(w, u, v) = A(x) \), then problem (9) reduces to the problem studied by Ahmad et al. [7].

(iii) If \( G \equiv 0 \), then problem (9) becomes the problem studied by Li [22].

It is clear that for suitable choices of operators involved in the formulation of problem (9), one can obtain many related problems.

The following Lemma is a fixed point formulation of the problem in Equation (9).

**Lemma 3.** The generalized implicit set-valued variational inclusion problem involving \( \oplus \) operation (9) has a solution \( x \in H_p, w \in A(x), u \in B(x), v \in C(x) \) if and only if it satisfies the following equation:

\[
x = J_{\lambda,M}^{H,F} \left[ \lambda G(w, u, v) \oplus (H \circ F)(x) \right], \tag{10}
\]

where \( \lambda > 0 \) is a constant.

**Proof.** Using the definition of the resolvent operator \( J_{\lambda,M}^{H,F} \) and Equation (10), we get:

\[
x = J_{\lambda,M}^{H,F} \left[ \lambda G(w, u, v) \oplus (H \circ F)(x) \right] = \left[ (H \circ F) \oplus \lambda M \right]^{-1} \left[ \lambda G(w, u, v) \oplus (H \circ F)(x) \right],
\]

\[
(H \circ F) \oplus \lambda M(x) = \lambda G(w, u, v) \oplus (H \circ F)(x),
\]

which implies that \( 0 \in G(w, u, v) \oplus M(x) \), the required generalized implicit set-valued variational inclusion problem with \( \oplus \) operation (9).
Conversely, suppose that generalized implicit set-valued variational inclusion problem with $\oplus$ operation (9) is satisfied, that is, $x \in \mathcal{H}_p$, $w \in A(x)$, $u \in B(x)$ and $v \in C(x)$ such that:

$$0 \in G(w, u, v) \oplus M(x),$$

which shows that:

$$G(w, u, v) = M(x),$$

$$\lambda G(w, u, v) = \lambda M(x),$$

$$\lambda G(w, u, v) \oplus (H \circ F)(x) = (H \circ F)(x) \oplus \lambda M(x),$$

$$\lambda G(w, u, v) \oplus (H \circ F)(x) = \lambda [(H \circ F) \oplus \lambda M](x),$$

$$x = [(H \circ F) \oplus \lambda M]^{-1}[\lambda G(w, u, v) \oplus (H \circ F)(x)]$$

Thus, Equation (10) is satisfied. \(\square\)

Based on Lemma 3, we establish the following iterative algorithm to obtain the solution of the problem in Equation (9).

**Iterative Algorithm 1.** For any given $x_0 \in \mathcal{H}_p$, choose $w_0 \in A(x_0)$, $u_0 \in B(x_0)$, $v_0 \in C(x_0)$ and using (10), let:

$$x_1 = (1 - \alpha)x_0 + \alpha J_{A,M}^{H, F}[\lambda G(w_0, u_0, v_0) \oplus (H \circ F)(x_0)].$$

Since $w_0 \in A(x_0)$, $u_0 \in B(x_0)$, $v_0 \in C(x_0)$, by the Nadler’s theorem [28], there exists $w_1 \in A(x_1)$, $u_1 \in B(x_1)$, $v_1 \in C(x_1)$, and using Proposition 2, we have:

$$\|w_0 \oplus w_1\| \leq \|w_0 - w_1\| \leq (1 + 1)D(A(x_0), A(x_1)),$$

$$\|u_0 \oplus u_1\| \leq \|u_0 - u_1\| \leq (1 + 1)D(B(x_0), B(x_1)),$$

$$\|v_0 \oplus v_1\| \leq \|v_0 - v_1\| \leq (1 + 1)D(C(x_0), C(x_1)),$$

where $D$ is the Hausdorff metric on $C^*(\mathcal{H}_p)$. Let:

$$x_2 = (1 - \alpha)x_1 + \alpha J_{A,M}^{H, F}[\lambda G(w_1, u_1, v_1) \oplus (H \circ F)(x_1)].$$

Again by Nadler’s theorem [28], there exist $w_2 \in F(x_2)$, $u_2 \in B(x_2)$, $v_2 \in C(x_2)$ such that:

$$\|w_1 \oplus w_2\| \leq \|w_1 - w_2\| \leq (1 + 2^{-1})D(A(x_1), A(x_2)),$$

$$\|u_1 \oplus u_2\| \leq \|u_1 - u_2\| \leq (1 + 2^{-1})D(B(x_1), B(x_2)),$$

$$\|v_1 \oplus v_2\| \leq \|v_1 - v_2\| \leq (1 + 2^{-1})D(C(x_1), C(x_2)).$$

Continuing the above procedure inductively, we have the following scheme:

$$x_{n+1} = (1 - \alpha)x_n + \alpha J_{A,M}^{H, F}[\lambda G(w_n, u_n, v_n) \oplus (H \circ F)(x_n)].$$
Since \( w_{n+1} \in A(x_{n+1}), u_{n+1} \in B(x_{n+1}), v_{n+1} \in C(x_{n+1}) \), such that:

\[
\left\| w_n \oplus w_{n+1} \right\| \leq \left\| w_n - w_{n+1} \right\| \leq (1 + (n + 1)^{-1})D(A(x_n), A(x_{n+1})),
\]

\[
\left\| u_n \oplus u_{n+1} \right\| \leq \left\| u_n - u_{n+1} \right\| \leq (1 + (n + 1)^{-1})D(B(x_n), B(x_{n+1})),
\]

\[
\left\| v_n \oplus v_{n+1} \right\| \leq \left\| v_n - v_{n+1} \right\| \leq (1 + (n + 1)^{-1})D(C(x_n), C(x_{n+1})).
\]

where \( \alpha \in [0, 1], n = 0, 1, 2, \ldots \).

**Theorem 1.** Let \( C \subseteq \mathcal{H}_p \) be a normal cone with constant \( \lambda N, H, F : \mathcal{H}_p \to \mathcal{H}_p \) and \( G : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p \) be the single-valued mappings such that \( (H \circ F) \) be strongly comparison, \( \beta \)-ordered compression mapping, \( G \) is \( \beta_1 \)-ordered compression mapping in the first argument, \( \beta_2 \)-ordered compression mapping in the second argument, and \( \beta_3 \)-ordered compression mapping in the third argument. Let \( A, B, C : \mathcal{H}_p \to \mathcal{C}(\mathcal{H}_p) \) be the set-valued mappings such that \( A \) is \( \lambda A \)-Lipschitz-continuous, \( B \) is \( \lambda B \)-D-Lipschitz-continuous, and \( C \) is \( \lambda C \)-D-Lipschitz-continuous. Suppose that \( M : \mathcal{H}_p \to 2^{\mathcal{H}_p} \) is \((\alpha, \lambda)\)-XOR-NODSM set-valued mapping with respect to \( J_{\lambda M}^{H,F} \) and \( \theta \)-ordered rectangular mapping with \( \lambda \theta > \beta \). If \( x_{n+1} \alpha x_n, n = 0, 1, 2, \ldots \) and the following condition is satisfied:

\[
| \lambda | [ \beta_1 \lambda A + \beta_2 \lambda B + \beta_3 \lambda C ] + \beta < 1 - \frac{\lambda N (1 - \alpha)}{\lambda N \theta'}, \tag{11}
\]

where \( \theta' = \frac{1}{\alpha \theta - \beta} \) and \( \lambda \theta > \beta; \beta_1, \beta_2, \beta_3, \lambda A, \lambda B, \lambda C, \lambda N, \alpha, \theta, \beta \) all are positive constants. Then, the generalized implicit set-valued variational inclusion problem with \( \oplus \) operation (9) has a solution \( x \in \mathcal{H}_p, w \in A(x), u \in B(x), \) and \( v \in C(x) \). Moreover, the iterative sequences \( \{x_n\}, \{w_n\}\{u_n\}, \) and \( \{v_n\} \) generated by Algorithm 1 converge strongly to \( x, w, u, \) and \( v \), the solution of generalized implicit set-valued variational inclusion problem with \( \oplus \) operation (9).

**Proof.** By Algorithm 1 and Proposition 1, we have:

\[
0 \leq x_{n+1} \oplus x_n
\]

\[
= \left[ (1 - \alpha)x_n + \alpha \left( J_{\lambda M}^{H,F} [\lambda G(w_n, u_n, v_n) \oplus (H \circ F)(x_n)] \right) \right]
\]

\[
\oplus \left[ (1 - \alpha)x_n - 1 + \alpha \left( J_{\lambda M}^{H,F} [\lambda (G(w_{n-1}, u_{n-1}, v_{n-1}) \oplus (H \circ F)(x_{n-1})) \right) \right]
\]

\[
= (1 - \alpha)(x_n \oplus x_{n-1}) + \alpha \left( J_{\lambda M}^{H,F} [\lambda G(w_n, u_n, v_n) \oplus (H \circ F)(x_n)] \right)
\]

\[
\oplus J_{\lambda M}^{H,F} [\lambda G(w_{n-1}, u_{n-1}, v_{n-1}) \oplus (H \circ F)(x_{n-1})].
\]
where $\theta' = \frac{1}{\lambda A - \beta}$.

Since $G$ is $\beta_1$-compression mapping in the first argument, $\beta_2$-compression mapping in the second argument, and $\beta_3$-compression mapping in third argument, $A$ is $\lambda_A \cdot \mathcal{D}$-Lipschitz-continuous, $B$ is $\lambda_B \cdot \mathcal{D}$-Lipschitz-continuous, and $C$ is $\lambda_C \cdot \mathcal{D}$-Lipschitz-continuous, using Algorithm 1, we have:

\[
\|G(w_n, u_n, v_n) \oplus G(w_{n-1}, u_{n-1}, v_{n-1})\| = \|G(w_n, u_n, v_n) \oplus G(w_{n-1}, u_{n-1}, v_{n-1})\|
\]

\[
\leq \beta_1 \|w_n + w_{n-1}\| + \beta_2 \|u_n + u_{n-1}\| + \beta_3 \|v_n + v_{n-1}\|
\]

\[
\leq \beta_1 \|w_n - w_{n-1}\| + \beta_2 \|u_n - u_{n-1}\| + \beta_3 \|v_n - v_{n-1}\|
\]

\[
\leq \beta_1 (1 + n^{-1}) \mathcal{D}(A(x_n), A(x_{n-1})) + \beta_2 (1 + n^{-1}) \mathcal{D}(B(x_n), B(x_{n-1}))
\]

\[
+ \beta_3 (1 + n^{-1}) \mathcal{D}(C(x_n), C(x_{n-1}))
\]

\[
\leq \beta_1 (1 + n^{-1}) \lambda_A \|x_n - x_{n-1}\| + \beta_2 (1 + n^{-1}) \lambda_B \|x_n - x_{n-1}\|
\]

\[
+ \beta_3 (1 + n^{-1}) \lambda_C \|x_n - x_{n-1}\|
\]

\[
= (\beta_1 \lambda_A + \beta_2 \lambda_B + \beta_3 \lambda_C)(1 + n^{-1}) \|x_n - x_{n-1}\|. \tag{14}
\]

As $(H \circ F)$ is $\beta$-ordered compression mapping, we have:

\[
\|(H \circ F)(x_n) + (H \circ F)(x_{n-1})\| \leq \beta \|x_n + x_{n-1}\|. \tag{15}
\]

Using Equations (14) and (15), (13) becomes:

\[
\|x_{n+1} \oplus x_n\| \leq \lambda_N (1 - \alpha) \|x_n \oplus x_{n-1}\| + \lambda_N \alpha \theta' |\lambda|[(\beta_1 \lambda_A + \beta_2 \lambda_B + \beta_3 \lambda_C)(1 + 1/n)] \|x_n - x_{n-1}\|
\]

\[
+ \lambda_N \alpha \theta' \|\lambda\|[(\beta_1 \lambda_A + \beta_2 \lambda_B + \beta_3 \lambda_C)(1 + 1/n)] \|x_n - x_{n-1}\|
\]

\[
= \lambda_N (1 - \alpha) \|x_n - x_{n-1}\| + \lambda_N \alpha \theta' |\lambda|[(\beta_1 \lambda_A + \beta_2 \lambda_B + \beta_3 \lambda_C)(1 + 1/n)] \|x_n - x_{n-1}\|
\]

\[
+ \lambda_N \alpha \theta' \|\lambda\|[(\beta_1 \lambda_A + \beta_2 \lambda_B + \beta_3 \lambda_C)(1 + 1/n)] \|x_n - x_{n-1}\|
\]

\[
= \partial(P_n) \|x_n - x_{n-1}\|. \tag{16}
\]

As $x_{n+1} \sim x_n$, $n = 0, 1, 2, \ldots$, we have:

\[
\|x_{n+1} - x_n\| \leq \partial(P_n) \|x_n - x_{n-1}\|
\]

where $\partial(P_n) = \lambda_N (1 - \alpha) + \lambda_N \alpha \theta' |\lambda|[(\beta_1 \lambda_A + \beta_2 \lambda_B + \beta_3 \lambda_C)(1 + 1/n)] + \lambda_N \alpha \theta' \beta$. Let $\partial(P) = \lambda_N (1 - \alpha) + \lambda_N \alpha \theta' |\lambda|[(\beta_1 \lambda_A + \beta_2 \lambda_B + \beta_3 \lambda_C) + \lambda_N \alpha \theta' \beta$.

We know that $\partial(P_n) \rightarrow \partial(P)$ as $n \rightarrow \infty$. It follows from condition (11) that $0 < \partial(p) < 1$, and consequently, $\{x_n\}$ is a cauchy sequence in $\mathcal{H}_p$ and since $\mathcal{H}_p$ is complete, there exists an $x \in \mathcal{H}_p$ such that $x_n \rightarrow x$, as $n \rightarrow \infty$. From Algorithm 1, we have:

\[
\|w_n \oplus w_{n+1}\| \leq \|w_n - w_{n+1}\|
\]

\[
\leq (1 + (n + 1)^{-1}) \mathcal{D}(A(x_n), A(x_{n+1}))
\]

\[
\leq (1 + (n + 1)^{-1}) \lambda_F \|x_n - x_{n-1}\|. \tag{17}
\]

\[
\|u_n \oplus u_{n+1}\| \leq \|u_n - u_{n+1}\|
\]

\[
\leq (1 + (n + 1)^{-1}) \mathcal{D}(B(x_n), B(x_{n+1}))
\]

\[
\leq (1 + (n + 1)^{-1}) \lambda_B \|x_n - x_{n-1}\|. \tag{18}
\]

and $\|v_n \oplus v_{n+1}\| \leq \|v_n - v_{n+1}\|

\[
\leq (1 + (n + 1)^{-1}) \mathcal{D}(C(x_n), C(x_{n+1}))
\]

\[
\leq (1 + (n + 1)^{-1}) \lambda_C \|x_n - x_{n-1}\|. \tag{19}
\]
It is clear from Equations (17)–(19) that \{w_n\}, \{u_n\}, and \{v_n\} are also cauchy sequences in \( \mathcal{H}_p \). Let \( w_n \to w, u_n \to u \) and \( v_n \to v \), as \( n \to \infty \). In view of Lemma 3, we conclude that \((x, w, u, v)\), such that \( x \in \mathcal{H}_p, w \in A(x), u \in B(x) \) and \( v \in C(x) \) is a solution of a generalized implicit set-valued variational inclusion problem with \( \oplus \) operation (9). Now, we show that with \( w \in A(x) \), we have:

\[
d(w, A(x)) \leq \|w \oplus w_n\| + d(w_n, A(x)) \\
\leq \|w - w_n\| + \|w_n \oplus A(x)\| \\
\leq \|w - w_n\| + D(A(x_n), A(x)) \\
\leq \|w - w_n\| + \lambda \|x_n - x\| \to 0, \text{ as } n \to \infty,
\]

which implies that \( d(w, A(x)) = 0 \), and since \( A(x) \in C^*(\mathcal{H}_p) \), it follows that \( w \in A(x) \). Similarly, we can show that \( u \in B(x) \) and \( v \in C(x) \), respectively. This complete the proof. \( \square \)

4. Conclusions

In this paper, we considered a generalized implicit set-valued variational inclusion problem with \( \oplus \) operation, which includes many previously studied problems in ordered spaces as special cases. A resolvent operator which involves composition of two mappings was considered, and we proved some properties of it. An existence and convergence result was proven for our problem in real ordered positive Hilbert spaces.

We remark that our results may be generalized further in higher dimensional spaces.

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