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The Gegenbauer Wavelets-Based Computational Methods for the Coupled System of Burgers' Equations with Time-Fractional Derivative

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Abstract: In this study, Gegenbauer wavelets are used to present two numerical methods for solving the coupled system of Burgers' equations with a time-fractional derivative. In the presented methods, we combined the operational matrix of fractional integration with the Galerkin method and the collocation method to obtain a numerical solution of the coupled system of Burgers' equations with a time-fractional derivative. The properties of Gegenbauer wavelets were used to transform this system to a system of nonlinear algebraic equations in the unknown expansion coefficients. The Galerkin method and collocation method were used to find these coefficients. The main aim of this study was to indicate that the Gegenbauer wavelets-based methods is suitable and efficient for the coupled system of Burgers' equations with time-fractional derivative. The obtained results show the applicability and efficiency of the presented Gegenbaur wavelets-based methods.

Keywords: Gegenbauer wavelets; coupled Burgers' equations; operational matrix of fractional integration; Galerkin method; collocation method

1. Introduction

The aim of this study is to present the numerical solutions by aid of the Gegenbauer wavelet collocation method with an operational matrix of fractional integration and the Gegenbauer wavelet Galerkin method for the following coupled system of Burgers' equations with time-fractional derivative [1]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2u(x,t)\frac{\partial u(x,t)}{\partial x} - \alpha_1 \frac{\partial (u(x,t)v(x,t))}{\partial x} + q_1(x,t), \quad x \in [0,1],\ t \in [0,1],\ \ 0 < \alpha \le 1 \tag{1}$$

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = \frac{\partial^2 v(x,t)}{\partial x^2} + 2v(x,t)\frac{\partial v(x,t)}{\partial x} - \alpha_2 \frac{\partial (u(x,t)v(x,t))}{\partial x} + q_2(x,t), \quad x \in [0,1],\ t \in [0,1],\ \ 0 < \alpha \le 1 \tag{2}$$

with initial and boundary conditions

$$u(x,0) = f_1(x), \quad v(x,0) = f_2(x) \qquad x \in [0,1] \tag{3}$$

and

$$\begin{array}{lll} u(0,t) = g_1(t), & u(1,t) = g_2(t), & t \in [0,1] \\ v(0,t) = h_1(t), & v(1,t) = h_2(t), & t \in [0,1] \end{array} \tag{4}$$

in which α parameter depicts the order of time fractional derivatives. α_1 and α_2 are arbitrary constants hinging on the system such as the Peclet number, Stokes velocity of particles due to gravity, and Brownian diffusivity [2]. $u(x,t)$ and $v(x,t)$ are the velocity components, $u(x,t)\frac{\partial u(x,t)}{\partial x}$ is the nonlinear convection term, $\frac{\partial^2 u(x,t)}{\partial x^2}$ is the diffusion term. The coupled system of Burgers' equations is known as the coupled viscous Burgers' equation. Esipov derived the coupled viscous Burgers' equation to examine the model of polydispersive sedimentation [3]. This system of coupled viscous Burgers' equation is a model of sedimentation and evolution of scaled volume concentrations of two sorts of particles in fluid suspensions. Moreover, this system can be taken as colloids under the effect of gravity. The Burgers' equation can be linearized by Hopf-Cole transformation [4]. Mathematical models of requisite flow equations depicting unsteady transport issues comprising of systems of nonlinear hyperbolic and parabolic partial differential equations. The coupled Burgers' equations constitute a considerable type of such partial differential equations. These equations happen in a huge number of physical problems such as the phenomena of turbulence flow through a shock wave traveling in a viscous fluid [5].

To solve the coupled system of Burgers' equations with time-fractional derivative numerically, there are various approaches which have been studied by many authors. Some of these approaches are the Chebyshev collocation method and the hybrid spectral exponential Chebyshev method presented by Albuohimad and Adibi in references [1] and [6], respectively, new coupled fractional reduced differential transform method proposed by Ray [7], the generalized differential transform method (GDTM), and the homotopy perturbation method (HPM) given by Khan et al. [8], the fractional variational iteration method established by Prakash et al. [9], the homotopy algorithm introduced by Singh et al. [10], the conformable double Laplace decomposition method studied by Eltayeb et al. [11], the new iterative method developed by Al-luhaibi [12], the Adomian decomposition method studied by Chen and An [13], and the modified extended tanh-function method applied by Zayed et al. [14]. Liu and Hou [15] used the generalized two-dimensional differential transform method (DTM) to solve this system, Kaplan [16] applied the modified simple equation method for solving the space-time fractional coupled Burgers' equations. Zhao et al. [17] solved the space-time fractional coupled Burgers' equations by using the extended fractional sub-equation method. In reference [18], the numerical/analytical solutions of the Burgers and coupled Burgers equations were applied to the differential transformation method by Abazari and Borhanifar. Srivastava et al. solved the one-dimensional coupled Burgers' equation by an implicit logarithmic finite-difference method [19]. D. Kaya used the decomposition method to find the solution of the homogenous and inhomogeneous coupled viscous Burgers equations [20]. Khater et al. used the Chebyshev spectral collocation method to get approximate solutions of the coupled Burgers equations [21]. Jima et al. applied the differential quadrature method based on the Fourier expansion basis to the coupled viscous burgers' equation [22].

Islam and Akbar [23] applied the generalized (G′/G)-expansion method to obtain exact wave solutions of the space-time fractional-coupled Burgers equations. In reference [24], the projected differential transform method (PDTM) was used to obtain solution of nonlinear coupled Burgers' equations with time and space fractional derivative by Elzaki.

Wavelet methods, improved mostly over the last 30 years, have been used to solve differential equations. Heretofore, a huge number of studies dedicated to this topic. Some methods used in these studies are the Legendre wavelet operational matrix method presented by Secer and Altun [25], the new spectral method using Legendre wavelets given by Yin et al. [26], the Chebyshev Wavelet Method, the Haar wavelet method, the Haar wavelet-finite difference hybrid method used by Oruc et al. [27–29], Hermite wavelet method applied by Saeed et al. [30], Harmonic wavelet method proposed by Cattani and Kudreyko [31], Wavelets Galerkin method, the Legendre wavelets method, the Chebyshev wavelets method studied by Heydari et al. [32–34], and the wavelet collocation method shown by Singh et al. [35]. In these studies, wavelets coefficients were calculated using the collocation and Galerkin method. However, too few articles deal with the application of Gegenbauer wavelets in handling fractional-order partial differential equations. Therefore, we focus on the numerical analysis of the coupled system of Burgers' equations with time-fractional derivative using the Gegenbauer wavelet collocation method with the operational matrix of fractional integration and the Gegenbauer wavelet Galerkin method in this paper. The most important advantage of the presented methods is that these methods present an understandable procedure to reduce the coupled system of Burgers' equations with time-fractional derivative and this system to a system of algebraic equations, which can be solved easily.

Firstly, we begin by presenting some basic definitions and fundamental relations of fractional calculus in Section 2. In Section 3, the properties of Gegenbauer wavelets are described. The approximation of a function by using Gegenbauer wavelets are briefly presented in Section 4. The operational matrix of fractional integration is defined in Section 5. In Section 6, to find the approximation solution for the coupled system of Burgers' equations with time-fractional derivative, the presented methods are presented. Finally, the last section includes the conclusions.

2. Mathematical Preliminaries of Fractional Calculus

We present some basic definitions and properties of the fractional calculus theory used in this paper.

Definition (Riemann-Liouville Integral): *The Riemann- Liouville fractional integration operator $I^{\alpha}(\alpha > 0)$ of a function $u(t)$, is defined as [36,37]*

$$I^{\alpha}u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-\zeta)^{\alpha-1}u(\zeta)d\zeta, & \alpha > 0, \ \alpha \in \Re^+ \\ u(t), & \alpha = 0 \end{cases}$$

in which $\Re^+$ is the set of positive real numbers. Some properties of the Riemann-Liouville fractional integral are as follows:

$$I^{\alpha}I^{\beta}u(t) = I^{\alpha+\beta}u(t), \quad (\alpha > 0, \beta > 0)$$
$$I^{\alpha}I^{\beta}u(t) = I^{\beta}I^{\alpha}u(t)$$
$$I^{\alpha}t^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\alpha+\delta+1)}t^{\alpha+\delta}, \ (\delta > -1)$$

Definition (Caputo Fractional Derivative): *The fractional derivative of $u(t)$ in the Caputo sense is defined as [36,37]:*

$$D_t^{\alpha}u(t) = I^{n-\alpha}D^n u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int\limits_0^t \frac{1}{(t-\zeta)^{(\alpha-n+1)}}\frac{d^n u(\zeta)}{d\zeta^n}d\zeta, & n-1 < \alpha < n, \ n \in \mathbb{N} \\ \frac{d^n u(\zeta)}{d\zeta^n}, & \alpha = n, \ n \in \mathbb{N} \end{cases}$$

The Caputo fractional derivative has the following well-established properties:

(i) $I^{\alpha}D^{\alpha}u(t) = u(t) - \sum\limits_{m=0}^{n-1} u^{(m)}(0^+)\frac{t^m}{m!}, \ n-1 < \alpha \le n, \ n \in \mathbb{N}$

(ii) $D^{\alpha}I^{\alpha}u(t) = u(t)$

(iii) $D^{\alpha}t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, & \beta > \alpha - 1 \\ 0, & \beta \le \alpha - 1 \end{cases}$

3. Gegenbauer Polynomials and Gegenbauer Wavelets

For Gegenbauer polynomials [38,39], or ultraspherical harmonics polynomials, $C_m^{\beta}(x)$ is of order m which satisfy the following singular Sturm- Liouville equation in $[-1,1]$:

$$\frac{d}{dn}\left[(1-x^2)^{\beta+\frac{1}{2}}\frac{d}{dx}C_m^{\beta}(x)\right] + m(m+2\beta)(1-x^2)^{\beta-\frac{1}{2}}C_m^{\beta}(x) = 0, \ \beta > -\frac{1}{2}, m \in \mathbb{Z}^+$$

and defined on the interval $[-1,1]$. Gegenbauer polynomials' recurrence formulae are given by:

$$C_0^\beta(x) = 1,\ C_1^\beta(x) = 2\beta x,$$
$$C_{m+1}^\beta(x) = \frac{1}{m+1}\Big(2(m+\beta)xC_m^\beta(x) - (m+2\beta-1)C_{m-1}^\beta(x)\Big),\quad m = 1,2,3,\ldots.$$

Gegenbauer polynomials are defined by the generating function as [40],

$$\frac{1}{(1-2xt+t^2)^\beta} = \sum_{m=0}^{\infty} C_m^\beta(x)t^m.$$

Gegenbauer polynomials have the following relations as given [40].

$$\frac{d}{dx}\Big(C_m^\beta(x)\Big) = 2\beta C_{m-1}^{\beta+1}(x),\quad \frac{d^k}{dx^k}\Big(C_m^\beta(x)\Big) = 2^k\beta^k C_{m-k}^{\beta+k}(x),\quad m \geq 1$$
$$(m+\beta)C_m^\beta(x) = \beta\Big(C_m^{\beta+1}(x) - C_{m-2}^{\beta+1}(x)\Big),\ m \geq 2$$
$$\frac{d}{dx}\Big(C_{m+1}^\beta(x) - C_{m-1}^\beta(x)\Big) = 2\beta\Big(C_m^{\beta+1}(x) - C_{m-2}^{\beta+1}(x)\Big) = 2(m+\beta)C_m^\beta(x).$$

The following integral formula can be obtained from the Rodrigues formula [40].

$$\int \left(1-x^2\right)^{\beta-1/2} C_m^\beta(x)dx = -\frac{2\beta\left(1-x^2\right)^{\beta+1/2}}{m(m+2\beta)} C_{m-1}^{\beta+1}(x),\quad m \geq 1.$$

According to the weight function $w(x) = \left(1-x^2\right)^{\beta-\frac{1}{2}}$, Gegenbauer polynomials are orthogonal on $[-1,1]$. That is,

$$\int_{-1}^{1} \left(1-x^2\right)^{\beta-\frac{1}{2}} C_m^\beta(x)C_n^\beta(x)dx = L_m^\beta \delta_{mn}, \beta > -\frac{1}{2}$$

in which $L_m^\beta = \frac{\pi 2^{1-2\beta}\Gamma(m+2\beta)}{m!(m+\beta)(\Gamma(\beta))^2}$ is called the normalizing factor, and δ is the Kronecker delta function [39].

From the Gegenbauer polynomials, for $\beta = 0, \beta = 1$ and $\beta = \frac{1}{2}$ we get the first- kind Chebyshev polynomials as [38]:

$$T_m(x) = \frac{m}{2}\lim_{\beta\to 0}\frac{C_m^\beta(x)}{\beta} \qquad (m \geq 1),$$

second kind Chebyshev polynomials as [38]:

$$U_m(x) = C_m^1(x)$$

and Legendre polynomial as [38]:

$$L_m(x) = C_m^{\frac{1}{2}}(x)$$

respectively.

Gegenbauer wavelets are written as

$$\psi_{m,n}(x) = \psi(k,n,m,x)$$

in which $k = 1,2,3,\ldots,$ is the level of resolution, $n = 1,2,3,\ldots,2^{k-1}, \hat{n} = 2n-1$, is the translation parameter, and $m = 0,1,2,\ldots,M-1$ is the order of the Gegenbauer polynomials, $M > 0$.

Gegenbauer wavelets are defined on the interval $[0,1]$ by

$$\psi^{\beta}_{n,m}(x)=\begin{cases}\frac{1}{\sqrt{L^{\beta}_{m}}}2^{\frac{k}{2}}C^{\beta}_{m}\left(2^{k}x-\hat{n}\right), & \frac{\hat{n}-1}{2^{k}}\leq x\leq\frac{\hat{n}+1}{2^{k}}\\ 0, & elsewhere\end{cases},$$

in which $C^{\beta}_{m}\left(2^{k}x-\hat{n}\right)$ are Gegenbauer polynomials of degree m and β is the known ultraspherical parameter. Corresponding to each $\beta>-\frac{1}{2}$, we get a different family of wavelets, i.e., when $\beta=\frac{1}{2}$, we have Legendre wavelets. For $\beta=0$ and $\beta=1$, we get the first kind Chebyshev wavelet and the second kind Chebyshev wavelet, respectively.

Gegenbauer wavelets are orthogonal on $[0,1]$ with respect to the weight function as follows:

$$w_{n}(x)=\begin{cases}w\left(2^{k}x-2n+1\right)=\left(1-\left(2^{k}x-2n+1\right)^{2}\right)^{\beta-\frac{1}{2}}, & x\in\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right]\\ 0, & otherwise\end{cases}.$$

4. Function Approximation by Gegenbauer Wavelets

A square integrable function $u(x)$ on the interval $[0,1]$ can be expanded by Gegenbauer wavelets as:

$$u(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{nm}(x)$$

in which c_{nm} values are wavelet coefficients, and these coefficients can be calculated with the inner product $c_{nm}=\langle u(x),\psi_{nm}(x)\rangle_{w_n}$. If the infinite series expansion in Equation (5) is truncated, then Equation (5) can be written as:

$$u(x)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(x)=C^{T}\Psi(x) \tag{5}$$

where T points to transposition, and C and $\Psi(x)$ are vectors given by:

$$\begin{aligned}C^{T}&=\left[\begin{array}{ccc}c_{10},c_{11},\ldots,c_{1M-1} & c_{20},c_{21},\ldots,c_{2M-1} & \cdots & c_{2^{k-1}0},c_{2^{k-1}1},\ldots,c_{2^{k-1}M-1}\end{array}\right]\\ \Psi(x)&=\left[\begin{array}{ccc}\psi_{10},\psi_{11},\ldots,\psi_{1M-1} & \psi_{20},\psi_{21},\ldots,\psi_{2M-1} & \cdots & \psi_{2^{k-1}0},\psi_{2^{k-1}1},\ldots,\psi_{2^{k-1}M-1}\end{array}\right]^{T}.\end{aligned} \tag{6}$$

For a more compact notation, Equation (5) can be written as:

$$u(x)\simeq\sum_{i=1}^{\hat{m}}c_{i}\psi_{i}(x) \tag{7}$$

where $\hat{m}=\left(2^{k-1}M\right)$, $C\triangleq[c_{1},c_{2},\ldots,c_{\hat{m}}]^{T}$,

$$\Psi(x)\triangleq[\psi_{1}(x),\ldots,\psi_{\hat{m}}(x)]^{T} \tag{8}$$

and the relation $i=M(n-1)+m+1$ is used for finding the index i.

In the same manner, a square integrable function $u(x,t)$ on the domain $[0,1]\times[0,1]$ may be represented in terms of a Gegenbauer wavelet as:

$$u(x,t)\simeq\sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}}u_{ij}\psi_{i}(x)\psi_{j}(t)=\Psi^{T}(x)U\Psi(t) \tag{9}$$

in which u_{ij} wavelets coefficients can be calculated with the inner product

$$u_{ij} = \left\langle \psi_i(x), \left\langle u(x,t), \psi_j(t) \right\rangle_{w_n} \right\rangle_{w_n} \tag{10}$$

By taking the collocation points as:

$$x_i = \frac{2i-1}{2\hat{m}}, i = 1, 2, \ldots, \hat{m}$$

and by substituting the collocation points into Equation (8), we can define the Gegenbauer wavelet matrix $\Phi_{\hat{m}\times\hat{m}}$ as:

$$\Phi_{\hat{m}\times\hat{m}} = \left[\Psi\left(\frac{1}{2\hat{m}}\right), \Psi\left(\frac{3}{2\hat{m}}\right), \ldots, \Psi\left(\frac{2\hat{m}-1}{2\hat{m}}\right) \right]. \tag{11}$$

5. Operational Matrix of Fractional Integration

The fractional integration of order α of the vector $\Psi(x)$, which is defined in Equation (8), can be defined as:

$$I^\alpha \Psi(x) \simeq P^\alpha \Psi(x)$$

in which the $\hat{m} \times \hat{m}$ matrix P^α is the operational matrix of fractional integration of order α for Gegenbauer wavelets. As shown in reference [41], the matrix P^α can be approximated as:

$$P^\alpha \simeq \Phi_{\hat{m}\times\hat{m}} P_B^\alpha \Phi_{\hat{m}\times\hat{m}}^{-1}$$

in which the $\hat{m} \times \hat{m}$ matrix P_B^α is called the operational matrix of integration for block pulse functions and is taken in reference [41] as:

$$P_B^\alpha = \frac{1}{\hat{m}^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \gamma_1 & \gamma_2 & \cdots & \gamma_{\hat{m}-1} \\ 0 & 1 & \gamma_1 & \cdots & \gamma_{\hat{m}-2} \\ 0 & 0 & 1 & \cdots & \gamma_{\hat{m}-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

in which $\gamma_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}$ [41].

6. Description of the Presented Methods

6.1. Gegenbauer Wavelets Collocation Method (GWCM)

We consider the coupled system of Burgers' equations with time-fractional derivative given by Equations (1) and (2) with initial conditions given by Equation (3) and boundary conditions given by Equation (4).

For solving this system, we assume:

$$\frac{\partial^{\alpha+2} u(x,t)}{\partial t^\alpha \partial x^2} = \Psi(x)^T U \Psi(t) \tag{12}$$

$$\frac{\partial^{\alpha+2} v(x,t)}{\partial t^\alpha \partial x^2} = \Psi(x)^T V \Psi(t) \tag{13}$$

in which $U = \left[u_{ij}\right]_{\hat{m}\times\hat{m}}$ and $V = \left[v_{ij}\right]_{\hat{m}\times\hat{m}}$ are unknown matrices which should be determined. By integrating of order α of Equation (12) with respect to t and considering the initial condition, we get:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \Psi(x)^T U P^\alpha \Psi(t) + f_1''(x). \tag{14}$$

When we integrate Equation (12) two times with respect to x, we obtain:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \Psi(x)^T \left(P^2\right)^T U\Psi(t) + \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}|_{x=0} + x\frac{\partial}{\partial x}\left(\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right)|_{x=0}. \tag{15}$$

By putting $x = 1$ into Equation (15) and considering boundary conditions, we get:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \Psi(x)^T \left(P^2\right)^T U\Psi(t) - x\Psi(1)^T \left(P^2\right)^T U\Psi(t) + \frac{\partial^\alpha g_1(t)}{\partial t^\alpha} + x\left(\frac{\partial^\alpha g_2(t)}{\partial t^\alpha} - \frac{\partial^\alpha g_1(t)}{\partial t^\alpha}\right). \tag{16}$$

Now we integrate of order α of Equation (16) with respect to t, we obtain

$$u(x,t) = \Psi(x)^T \left(P^2\right)^T UP^\alpha\Psi(t) - x\Psi(1)^T \left(P^2\right)^T UP^\alpha\Psi(t) + G_1(x,t) \tag{17}$$

in which

$$\begin{gathered} G_1(x,t) = f_1(x) + g_1(t) - g_1(0) + x(g_2(t) - g_2(0) - g_1(t) + g_1(0)). \\ \frac{\partial u(x,t)}{\partial x} = \Psi(x)^T (P)^T UP^\alpha\Psi(t) - \Psi(1)^T \left(P^2\right)^T UP^\alpha\Psi(t) + \frac{\partial G_1(x,t)}{\partial x}. \end{gathered} \tag{18}$$

Similarly, we get:

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \Psi(x)^T VP^\alpha\Psi(t) + f_2''(x). \tag{19}$$

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = \Psi(x)^T \left(P^2\right)^T V\Psi(t) + \frac{\partial^\alpha v(x,t)}{\partial t^\alpha}|_{x=0} + x\frac{\partial}{\partial x}\left(\frac{\partial^\alpha v(x,t)}{\partial t^\alpha}\right)|_{x=0}. \tag{20}$$

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = \Psi(x)^T \left(P^2\right)^T V\Psi(t) - x\Psi(1)^T \left(P^2\right)^T V\Psi(t) + \frac{\partial^\alpha h_1(t)}{\partial t^\alpha} + x\left(\frac{\partial^\alpha h_2(t)}{\partial t^\alpha} - \frac{\partial^\alpha h_1(t)}{\partial t^\alpha}\right). \tag{21}$$

$$v(x,t) = \Psi(x)^T \left(P^2\right)^T VP^\alpha\Psi(t) - x\Psi(1)^T \left(P^2\right)^T VP^\alpha\Psi(t) + G_2(x,t) \tag{22}$$

$$\begin{gathered} G_2(x,t) = f_2(x) + h_1(t) - h_1(0) + x(h_2(t) - h_2(0) - h_1(t) + h_1(0)). \\ \frac{\partial v(x,t)}{\partial} = \Psi(x)^T (P)^T VP^\alpha\Psi(t) - \Psi(1)^T \left(P^2\right)^T VP^\alpha\Psi(t) + \frac{\partial G_2(x,t)}{\partial x} \end{gathered} \tag{23}$$

for $v(x,t)$.

When we substitute Equations (14), (16), (18), (19), and (21)–(23) into Equations (1) and (2) and we take the collocation points for both t and x, we get a nonlinear system of algebraic equations. From this system, the wavelet coefficients u_{ij} and v_{ij} can be successively calculated.

6.2. Gegenbauer Wavelets Galerkin Method (GWGM)

The Gegenbauer wavelet expansion, together with the operational matrix of integration, is utilized to solve the coupled system of Burgers' equations with time-fractional derivative, given by:

$$\begin{gathered} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2u(x,t)\frac{\partial u(x,t)}{\partial x} - \alpha_1\frac{\partial(u(x,t)v(x,t))}{\partial x} + q_1(x,t), \quad x \in [0,1],\ t \in [0,1],\ \ 0 < \alpha \le 1 \\ \frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = \frac{\partial^2 v(x,t)}{\partial x^2} + 2v(x,t)\frac{\partial v(x,t)}{\partial x} - \alpha_2\frac{\partial(u(x,t)v(x,t))}{\partial x} + q_2(x,t), \quad x \in [0,1],\ t \in [0,1],\ \ 0 < \alpha \le 1 \end{gathered}$$

with initial and boundary conditions

$$u(x,0) = f_1(x), \quad v(x,0) = f_2(x) \qquad x \in [0,1]$$

and

$$\begin{gathered} u(0,t) = g_1(t), \quad u(1,t) = g_2(t), \quad t \in [0,1] \\ v(0,t) = h_1(t), \quad v(1,t) = h_2(t), \quad t \in [0,1]. \end{gathered}$$

For solving this system, by integrating of order α each equation of this system with respect to t and considering the initial conditions, we find the integral form of the coupled system of Burgers' equations with time-fractional derivative as follows:

$$u(x,t) = f_1(x) + \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2} d\tau + 2\int_0^t u(x,\tau)\frac{\partial u(x,\tau)}{\partial x} d\tau - \alpha_1 \int_0^t \frac{\partial(u(x,\tau)v(x,\tau))}{\partial x} d\tau - \int_0^t q_1(x,t) d\tau \tag{24}$$

$$v(x,t) = f_2(x) + \int_0^t \frac{\partial^2 v(x,\tau)}{\partial x^2} d\tau + 2\int_0^t v(x,\tau)\frac{\partial v(x,\tau)}{\partial x} d\tau - \alpha_1 \int_0^t \frac{\partial(u(x,\tau)v(x,\tau))}{\partial x} d\tau - \int_0^t q_2(x,t) d\tau. \tag{25}$$

Now, we approximate $\frac{\partial^2 u(x,t)}{\partial x^2}$ by the Gegenbauer wavelets as follows:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \Psi(x)^T U \Psi(x) \tag{26}$$

in which $U = \left[u_{ij}\right]_{\hat{m}\times\hat{m}}$ is an unknown matrix which should be determined. When we integrate Equation (26) two times with respect to x, we get:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial x}|_{x=0} + \Psi(x)^T P^T U \Psi(t) \tag{27}$$

and

$$u(x,t) = u(0,t) + x\left(\frac{\partial u(x,t)}{\partial x}|_{x=0}\right) + \Psi(x)^T \left(P^2\right)^T U \Psi(t), \tag{28}$$

And we put $x = 1$ in Equation (28) and we consider the boundary conditions, we have:

$$\frac{\partial u(x,t)}{\partial x}|_{x=0} = g_2(t) - g_1(t) - \Psi(1)^T \left(P^2\right)^T U \Psi(t). \tag{29}$$

$g_1(t)$ and $g_2(t)$ can be expressed by a terminated Gegenbauer wavelet series at the value $\hat{m}$ as follows:

$$\begin{aligned} g_1(t) &= G_1^T \Psi(t) \\ g_2(t) &= G_2^T \Psi(t) \end{aligned} \tag{30}$$

in which G_1 and G_2 are the Gegenbauer wavelet coefficients vectors. If we substitute Equation (30) into Equation (29), we have:

$$\frac{\partial u(x,t)}{\partial x}|_{x=0} = \left(G_2^T - G_1^T - \Psi(1)^T \left(P^2\right)^T U\right)\Psi(t) = \widehat{U}^T \Psi(t). \tag{31}$$

By substituting Equation (31) into Equations (27) and (28), we obtain:

$$\frac{\partial u(x,t)}{\partial x} = \Psi(x)^T \left(E\widehat{U} + P^T U\right)\Psi(t) = \Psi(x)^T A_1 \Psi(t) \tag{32}$$

$$u(x,t) = \Psi(x)^T \left(E G_1^T + X\widehat{U} + \left(P^2\right)^T U\right)\Psi(t) = \Psi(x)^T A_2 \Psi(t) \tag{33}$$

in which $x = \Psi(x)^T X$ and $1 = \Psi(x)^T E$. Furthermore, we can be expressed by a terminated Gegenbauer wavelet series at the value $\hat{m}$ as follows:

$$f_1(x) = \Psi(x)^T F_1, \quad q_1(x,t) = \Psi(x)^T Q_1 \Psi(t) \tag{34}$$

where F_1 and Q_1 are the Gegenbauer wavelet coefficients matrices.

Similarly, we approximate $\frac{\partial^2 v(x,t)}{\partial x^2}$ by the Gegenbauer wavelets as follows:

$$\frac{\partial^2 v(x,t)}{\partial x^2} = \Psi(x)^T V \Psi(x) \tag{35}$$

in which $V = \left[v_{ij}\right]_{\hat{m}\times\hat{m}}$ is an unknown matrix which should be determined. When we integrate Equation (35) two times with respect to x, we get:

$$\frac{\partial v(x,t)}{\partial x} = \frac{\partial v(x,t)}{\partial x}|_{x=0} + \Psi(x)^T P^T V \Psi(t) \tag{36}$$

and

$$v(x,t) = v(0,t) + x\left(\frac{\partial v(x,t)}{\partial x}|_{x=0}\right) + \Psi(x)^T \left(P^2\right)^T V \Psi(t), \tag{37}$$

And we put $x = 1$ in Equation (37) and we consider the boundary conditions, we have:

$$\frac{\partial v(x,t)}{\partial x}|_{x=0} = h_2(t) - h_1(t) - \Psi(1)^T \left(P^2\right)^T V \Psi(t). \tag{38}$$

$g_1(t)$ and $g_2(t)$ can be expressed by a terminated Gegenbauer wavelet series at the value $\hat{m}$ as follows:

$$\begin{aligned} h_1(t) &= H_1^T \Psi(t) \\ h_2(t) &= H_2^T \Psi(t) \end{aligned} \tag{39}$$

in which H_1 and H_2 are the Gegenbauer wavelet coefficients vectors. If we substitute Equation (39) into Equation (38), we have:

$$\frac{\partial v(x,t)}{\partial x}|_{x=0} = \left(H_2^T - H_1^T - \Psi(1)^T \left(P^2\right)^T V\right)\Psi(t) = \widehat{V}^T \Psi(t). \tag{40}$$

By substituting Equation (40) into Equations (36) and (37), we obtain:

$$\frac{\partial v(x,t)}{\partial x} = \Psi(x)^T \left(E\widehat{V} + P^T V\right)\Psi(t) = \Psi(x)^T A_3 \Psi(t) \tag{41}$$

$$v(x,t) = \Psi(x)^T \left(EH_1^T + X\widehat{V} + \left(P^2\right)^T V\right)\Psi(t) = \Psi(x)^T A_4 \Psi(t) \tag{42}$$

in which $x = \Psi(x)^T X$ and $1 = \Psi(x)^T E$. Furthermore, it can be expressed by a terminated Gegenbauer wavelet series at the value $\hat{m}$ as follows:

$$f_2(x) = \Psi(x)^T F_2, \quad q_2(x,t) = \Psi(x)^T Q_2 \Psi(t) \tag{43}$$

where F_2 is the Gegenbauer wavelet coefficients vector.

Now by substituting Equations (26), (32)–(34), (41) and (42) into Equations (24) and (32), (33), (35), and (41)–(43) into Equation (25), respectively, then using operational matrices of integration, we get the residuals functions $R_1(x,t)$ and $R_2(x,t)$ for this system as follows:

$$R_1(x,t) = \Psi(x)^T \left[A_2 - F_1 E^T - UP - 2\mathrm{K}_1 P + \alpha_1 \mathrm{K}_3 P + \alpha_1 \mathrm{K}_4 P - Q_1 P\right]\Psi(t) \tag{44}$$

$$R_2(x,t) = \Psi(x)^T \left[A_4 - F_2 E^T - VP - 2\mathrm{K}_2 P + \alpha_2 \mathrm{K}_3 P + \alpha_2 \mathrm{K}_4 P - Q_2 P\right]\Psi(t) \tag{45}$$

in which

$$\begin{aligned}
\left[\Psi(x)^T A_1 \Psi(t)\right]\left[\Psi(x)^T A_2 \Psi(t)\right] &= \Psi(x)^T K_1 \Psi(t) \\
\left[\Psi(x)^T A_3 \Psi(t)\right]\left[\Psi(x)^T A_4 \Psi(t)\right] &= \Psi(x)^T K_2 \Psi(t) \\
\left[\Psi(x)^T A_1 \Psi(t)\right]\left[\Psi(x)^T A_4 \Psi(t)\right] &= \Psi(x)^T K_3 \Psi(t) \\
\left[\Psi(x)^T A_2 \Psi(t)\right]\left[\Psi(x)^T A_3 \Psi(t)\right] &= \Psi(x)^T K_4 \Psi(t).
\end{aligned}$$

As in Galerkin method [42], for u_{ij} and $v_{ij},$, $i = 1, 2, \ldots, \hat{m}$ we get $2\hat{m}^2$ non-linear algebraic equations as follows:

$$\begin{aligned}
\int_0^1\int_0^1 R_1(x,t)\psi_i(x)\psi_j(t)\omega_n(x)\omega_n(t)dxdt = 0, \quad i,j = 1,2,\ldots,\hat{m} \\
\int_0^1\int_0^1 R_2(x,t)\psi_i(x)\psi_j(t)\omega_n(x)\omega_n(t)dxdt = 0, \quad i,j = 1,2,\ldots,\hat{m}.
\end{aligned} \tag{46}$$

Eventually, by solving this system for the unknown matrices U and V, we obtain approximate solutions for the coupled system of Burgers' equations with time- fractional derivative using Equations (33) and (42).

7. Test Problem

In this section, we give test problem to show the performance of the presented methods by measuring the absolute error and maximum error L_∞ at points $(x_i, t_i) \in [0,1] \times [0,1]$. The absolute error and maximum error L_∞ are defined as

$$\begin{aligned}
E(x_i,t_i) &= \left|u_{exactsol}(x_i,t_i) - u(x_i,t_i)\right| \\
L_\infty &= \max_{1\le i\le \hat{m}}\left|u_{exactsol}(x,t_i) - u(x,t_i)\right|.
\end{aligned} \tag{47}$$

The obtained errors are showed in tables. Here, our test problem is solved by the Gegenbauer wavelet collocation method for $k = 2, M = 3$. The Gegenbauer wavelet Galerkin method is applied to this problem for $k = 1, M = 3$.

Problem. We consider the coupled system of Burgers' equations with time-fractional derivative with $\alpha_1 = \alpha_2 = \frac{5}{2}, q_1(x,t) = q_2(x,t) = 0$ [18]. And we have

$$\begin{aligned}
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \frac{\partial^2 u(x,t)}{\partial x^2} + 2u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{5}{2}\frac{\partial(u(x,t)v(x,t))}{\partial x}, \qquad 0 < \alpha \le 1 \\
\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} &= \frac{\partial^2 v(x,t)}{\partial x^2} + 2v(x,t)\frac{\partial v(x,t)}{\partial x} - \frac{5}{2}\frac{\partial(u(x,t)v(x,t))}{\partial x}, \qquad 0 < \alpha \le 1.
\end{aligned}$$

The exact solution of the coupled system of Burgers' equations for $\alpha = 1$ is

$$u(x,t) = v(x,t) = \lambda\left[1 - \tanh\left(\frac{3}{2}\lambda(x - 3\lambda t)\right)\right]$$

Boundary conditions and initial conditions are obtained from exact solution and λ is an arbitrary constant.

Tables 1 and 2 show the maximum errors in the collocation points for different values of β, $\alpha = 0.75$ and $\alpha = 0.90$, respectively. We can see that as the value of α approaches 1, approximate results converge to the exact solution. Tables 3 and 4 present the absolute errors obtained by the Gegenbauer wavelet Galerkin method and the Gegenbauer wavelet Collocation method for $\beta = 1/2$, $\alpha = 0.75$, $\alpha = 0.90$ and $\alpha = 1$. As numerical results in Tables 3 and 4 reveal, the numerical results obtained using the Gegenbauer wavelet collocation method are better than the numerical results obtained using the Gegenbauer wavelet Galerkin method.

Table 1. For $\lambda = 0.005$ and $\alpha = 0.75$, the maximum error of example with the Gegenbauer wavelet collocation method for various values of β.

x	β=−0.49	β=0.5	β=1.5	β=2.5
0.08333333333	$5.56091888323387 \times 10^{-9}$	$5.56091887990606 \times 10^{-9}$	$5.56091888573703 \times 10^{-9}$	$5.56091888078351 \times 10^{-9}$
0.2500000000	$1.38776963054663 \times 10^{-8}$	$1.38776962940260 \times 10^{-8}$	$1.38776963109158 \times 10^{-8}$	$1.38776962990505 \times 10^{-8}$
0.4166666667	$1.81658498433006 \times 10^{-8}$	$1.81658498265648 \times 10^{-8}$	$1.81658498502864 \times 10^{-8}$	$1.81658498408983 \times 10^{-8}$
0.5833333333	$1.82082635178889 \times 10^{-8}$	$1.82082634993186 \times 10^{-8}$	$1.82082635154496 \times 10^{-8}$	$1.82082634759442 \times 10^{-8}$
0.7500000000	$1.40062564340391 \times 10^{-8}$	$1.40062564027202 \times 10^{-8}$	$1.40062564202905 \times 10^{-8}$	$1.40062564107182 \times 10^{-8}$
0.9166666667	$5.69678491962975 \times 10^{-9}$	$5.69678489380017 \times 10^{-9}$	$5.69678490879393 \times 10^{-8}$	$5.69678493132917 \times 10^{-9}$

Table 2. For $\lambda = 0.005$ and $\alpha = 0.90$, Maximum error (L_∞) of example with the Gegenbauer wavelet Collocation method for various values of β.

x	β=−0.49	β=0.5	β=1.5	β=2.5
0.08333333333	$2.48736492703210 \times 10^{-9}$	$2.48736492339469 \times 10^{-9}$	$2.48736492649088 \times 10^{-9}$	$2.48736492450080 \times 10^{-9}$
0.2500000000	$6.33715105072939 \times 10^{-9}$	$6.33715104907696 \times 10^{-9}$	$6.33715105527478 \times 10^{-9}$	$6.33715105248488 \times 10^{-9}$
0.4166666667	$8.31540044959192 \times 10^{-9}$	$8.31540045263156 \times 10^{-9}$	$8.31540046084903 \times 10^{-9}$	$8.31540046005840 \times 10^{-9}$
0.5833333333	$8.33164878981551 \times 10^{-9}$	$8.33164879335693 \times 10^{-9}$	$8.33164879823062 \times 10^{-9}$	$8.33164878389264 \times 10^{-9}$
0.7500000000	$6.39224777036601 \times 10^{-9}$	$6.39224776493989 \times 10^{-9}$	$6.39224777102112 \times 10^{-9}$	$6.39224776947591 \times 10^{-9}$
0.9166666667	$2.59610208919879 \times 10^{-9}$	$2.59610208092303 \times 10^{-9}$	$2.59610208695513 \times 10^{-9}$	$2.59610209831542 \times 10^{-9}$

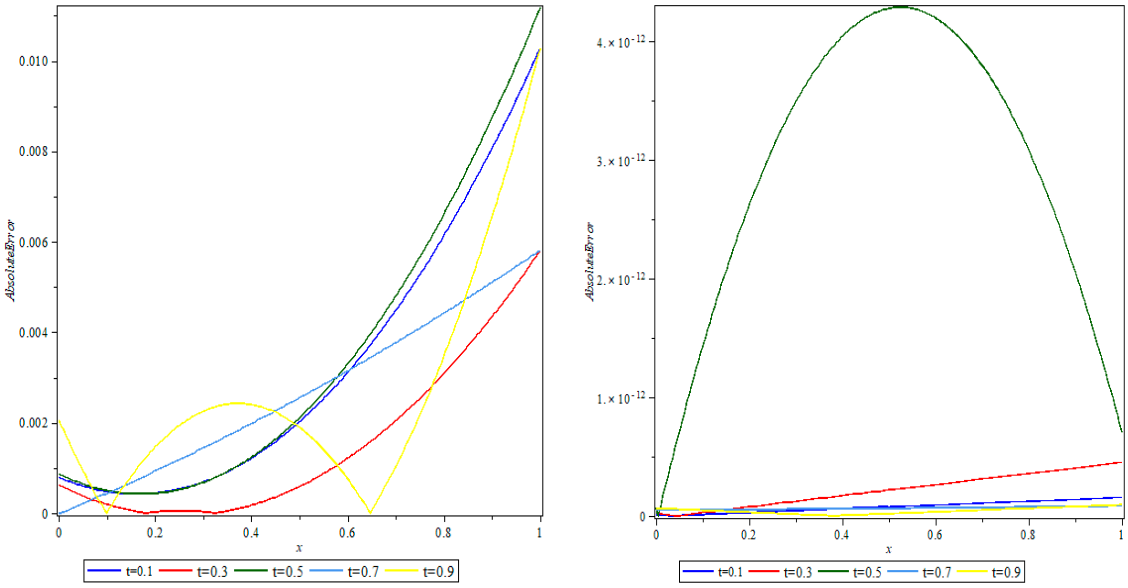
Table 3. Absolute errors of the approximate solutions obtained using the Gegenbauer wavelet collocation method and the Gegenbauer wavelet Galerkin Method at various points of x and t for $\beta = 0.5$.

	α=0.75, β=1/2		α=0.90, β=1/2	
t	$\lvert u_{exactsol} - u_{GWGM} \rvert$	$\lvert u_{exactsol} - u_{GWCM} \rvert$	$\lvert u_{exactsol} - u_{GWGM} \rvert$	$\lvert u_{exactsol} - u_{GWCM} \rvert$
$(0.1, 0.1)$	$6.13025693527975 \times 10^{-4}$	$6.66661856928545 \times 10^{-9}$	$5.29045965455882 \times 10^{-4}$	$2.62846240061929 \times 10^{-9}$
$(0.2, 0.2)$	$1.96106407883844 \times 10^{-4}$	$1.23238756518206 \times 10^{-8}$	$2.20293920176328 \times 10^{-4}$	$5.47257522501428 \times 10^{-9}$
$(0.3, 0.3)$	$5.76008412287644 \times 10^{-4}$	$4.1995860831132 \times 10^{-8}$	$2.27882222894827 \times 10^{-4}$	$6.55618936267295 \times 10^{-9}$
$(0.4, 0.4)$	$1.80785975016607 \times 10^{-3}$	$1.05634171814097 \times 10^{-8}$	$1.20101728292158 \times 10^{-3}$	$4.61118750103048 \times 10^{-9}$
$(0.5, 0.5)$	$3.23042755391540 \times 10^{-3}$	$7.21275298414753 \times 10^{-10}$	$2.59246452767595 \times 10^{-3}$	$1.32748682069936 \times 10^{-10}$
$(0.6, 0.6)$	$4.20113073168401 \times 10^{-3}$	$4.01824477114943 \times 10^{-9}$	$3.80339566853798 \times 10^{-3}$	$1.25973539069898 \times 10^{-9}$
$(0.7, 0.7)$	$3.70382715260461 \times 10^{-3}$	$1.19582281105710 \times 10^{-9}$	$3.74280086149575 \times 10^{-3}$	$7.03441479770895 \times 10^{-11}$
$(0.8, 0.8)$	$3.48813645794342 \times 10^{-4}$	$6.88844784126224 \times 10^{-10}$	$8.27488706145341 \times 10^{-4}$	$6.44803361647952 \times 10^{-10}$
$(0.9, 0.9)$	$7.62717399664530 \times 10^{-3}$	$1.18956826178091 \times 10^{-9}$	$7.01791375130911 \times 10^{-3}$	$6.98040561718981 \times 10^{-10}$

Table 4. Absolute errors of example using the Gegenbauer wavelet collocation method and the Gegenbauer wavelet Galerkin Method at various points of x and t.

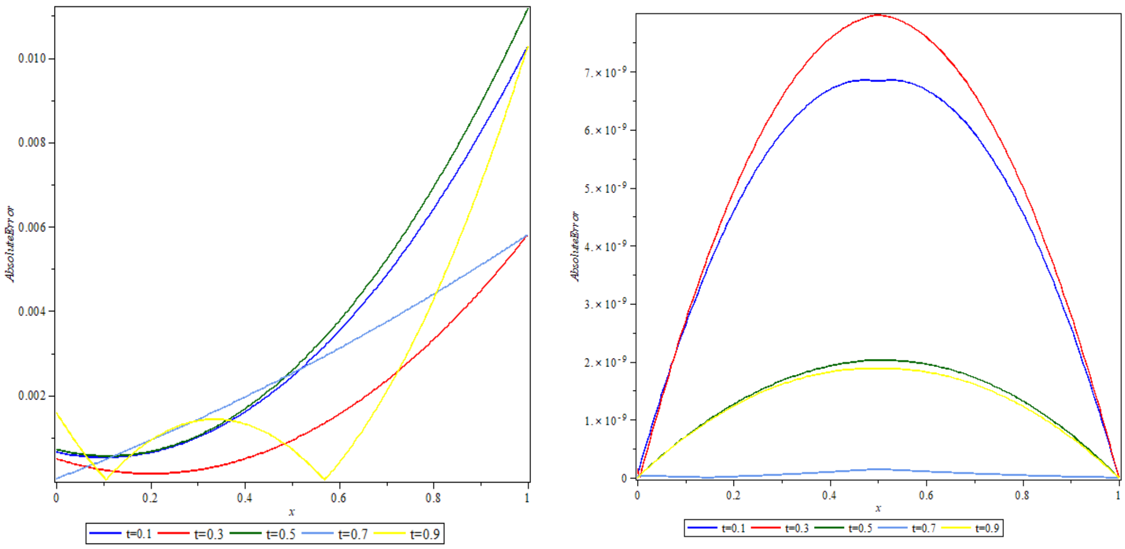
	α=1, β=1/2	
t	$\lvert u_{exactsol} - u_{GWGM} \rvert$	$\lvert u_{exactsol} - u_{GWCM} \rvert$
$(0.1, 0.1)$	$4.41008546402439 \times 10^{-4}$	$2.79364047818072 \times 10^{-13}$
$(0.2, 0.2)$	$2.36885253966561 \times 10^{-4}$	$3.67973403970119 \times 10^{-15}$
$(0.3, 0.3)$	$2.97862564205458 \times 10^{-5}$	$1.78829812543299 \times 10^{-14}$
$(0.4, 0.4)$	$7.46813081452347 \times 10^{-4}$	$2.42305348966039 \times 10^{-14}$
$(0.5, 0.5)$	$2.11220965345779 \times 10^{-3}$	$4.11579218934484 \times 10^{-12}$
$(0.6, 0.6)$	$3.50273601972239 \times 10^{-3}$	$5.42193099334388 \times 10^{-13}$
$(0.7, 0.7)$	$3.77176040769366 \times 10^{-3}$	$6.38553105209211 \times 10^{-13}$
$(0.8, 0.8)$	$1.18968671114004 \times 10^{-3}$	$4.01833606504698 \times 10^{-14}$
$(0.9, 0.9)$	$6.55604550684911 \times 10^{-3}$	$4.42734126124196 \times 10^{-13}$

For $\alpha = 1$, $\alpha = 0.90$ and $\alpha = 0.75$, the physical behaviors of the absolute errors obtained using the Gegenbauer wavelet Galerkin method and the Gegenbauer wavelet collocation method at different times are depicted in Figures 1–3, respectively. Figure 4 is drawn to show that the Maple code written for the Gegenbauer wavelet collocation method is faster than the Maple code written for the Gegenbauer wavelet Galerkin method for $k = 1$, $M = 3$ and $\beta = 1/2$. All of the above computations were computed using the computer code written in Maple 18.



(**a**) (**b**)

Figure 1. (**a**) The absolute errors $|u_{exactsol} - u_{GWGM}|$ at different times, when $\alpha = 1$; (**b**) the absolute errors $|u_{exactsol} - u_{GWCM}|$ at different times, when $\alpha = 1$.



(**a**) (**b**)

Figure 2. (**a**) The absolute errors $|u_{exactsol} - u_{GWGM}|$ at different times, when $\alpha = 0.90$; (**b**) the absolute errors $|u_{exactsol} - u_{GWCM}|$ at different times, when $\alpha = 0.90$.

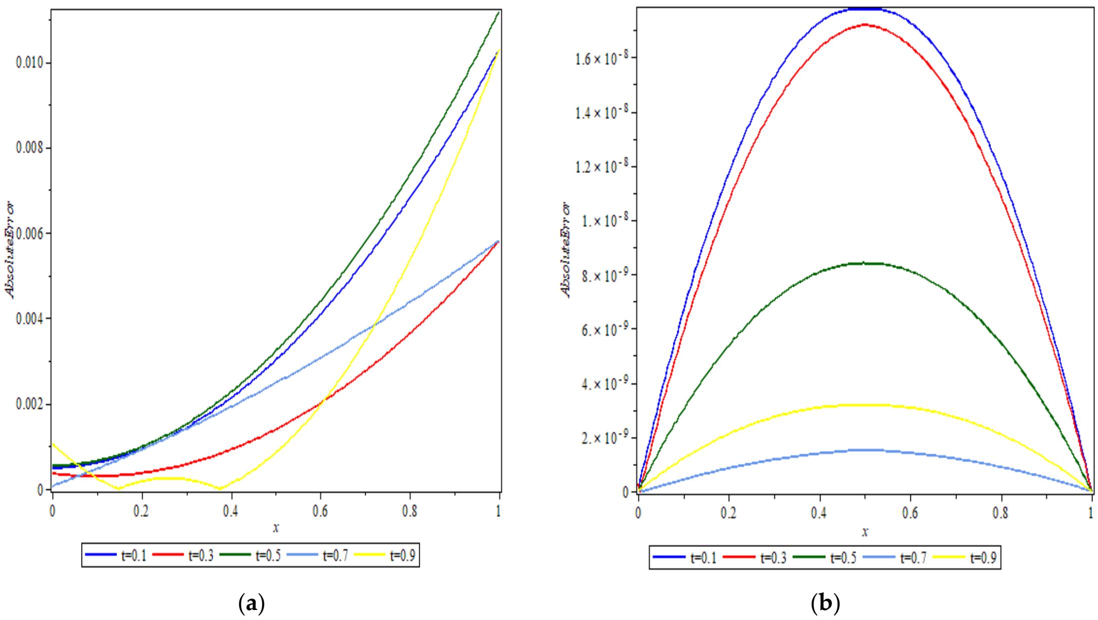


Figure 3. (**a**) The absolute errors $|u_{exactsol} - u_{GWGM}|$ at different times, when $\alpha = 0.75$; (**b**) the absolute errors $|u_{exactsol} - u_{GWCM}|$ at different times, when $\alpha = 0.75$.

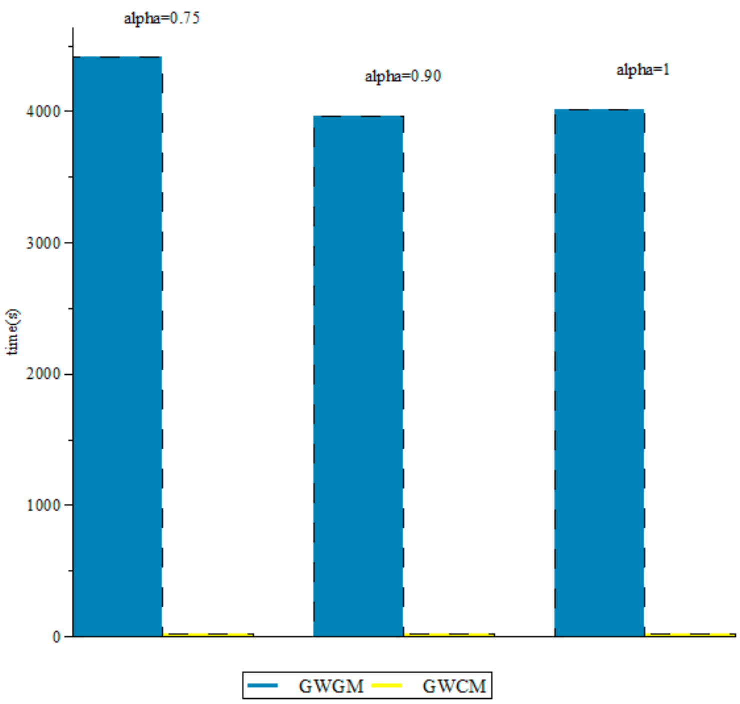


Figure 4. Computation times of Maple codes written for the Gegenbauer wavelets Galerkin method (GWGM) and Gegenbauer wavelets collocation method (GWCM).

8. Conclusions

The main goal of this paper is to build up for obtaining numerical solutions of the coupled system of Burgers' equations with time-fractional derivative using the Gegenbauer wavelet collocation method and the Gegenbauer wavelet Galerkin method at different values of x, t, and α. The obtained numerical results are compared with the exact solution. Consequently, it is manifestly seen that the Gegenbauer wavelet collocation method is more effective method than the Gegenbauer wavelet Galerkin method and the Gegenbauer wavelet collocation method construct the acceptable results for the numerical solution of the coupled system of Burgers' equations with time-fractional derivative. Another profit of these methods are that the proposed schemes, with some modifications, appear to be easily extended to find numerical solutions of partial differential equations and the systems of partial differential equations from different branches of science and engineering.

Authors Contributions

Conceptualization, A.S. and N.O.; methodology, A.S.; software, A.S.; validation, N.O., A.S and M.B.; investigation, N.O.; resources, A.S and N.O.; writing—original draft preparation, N.O. and M.B.; writing—review and editing, M.B.

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