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Dynamic Pricing in a Multi-Period Newsvendor Under Stochastic Price-Dependent Demand

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Abstract: The faster growth of technology stipulates the rapid development of new products; with the spread of new technologies old ones are outdated and their market demand declines sharply. The combined impact of demand uncertainty and short life-cycles complicate ordering and pricing decision of retailers that leads to a decrease in the profit. This study deals with the joint inventory and dynamic pricing policy for such products considering stochastic price-dependent demand. The aim is to develop a discount policy that enables the retailer to order more at the start of the selling season thus increase the profit and market share of the retailer. A multi-period newsvendor model is developed under the distribution-free approach and the optimal stocking quantities, unit selling price, and the discount percentage are obtained. The results show that the proposed discount policy increases the expected profit of the system. Additionally, the stocking quantity and the unit selling price also increases in the proposed discount policy. The robustness of the proposed model is illustrated with numerical examples and sensitivity analysis. Managerial insights are given to extract significant insights for the newsvendor model with discount policy.

Keywords: price discounts; stochastic-price dependent demand; newsvendor; pricing policy

1. Introduction

In today's competitive economic environment, fluctuation in market demand is causing problems for businesses, especially those who are dealing with products having short life cycles. The example includes apparel products, fashion goods, seasonal clothing, personal computers, and other electronic goods. Demand uncertainty either leads to stock outs or overstocking; the short life cycles, further, aggravates the severity of the problem. In such situations, an important question arises: How to deal with these uncertain items? The newsvendor model offers a solution to such problems, in which the decision maker faces stochastic and exogenous demand. The newsvendor has to decide the order quantity ahead of the selling season by using the available information. The aim is to find the trade-off between overstocking and under-stocking conditions. In the classical newsvendor problem, the common practice is to dispose of the product at a salvage price considering no discount during the selling season. However, the past 30 years has reshaped the landscape of the retail industry; retailers prefer to avoid the stock outs by ordering larger lot sizes, and a discount is offered on the remaining stock at the end of the selling season. This increases the market share of the retailer because a strategic consumer waits patiently and buys the product when the price drops. This paper deals with such a business problem, in which the consumer can switch to alternative options with discounted prices; the retailer decides the lot size, finished products price, and the discount percentage on the price of the finished product at the end of the selling season.

As demand is selling price dependent, reducing the price, at the end of the selling season, increases the net demand. Moreover, in these strategic situations, the firms make decisions considering pricing

and stocking. The retailer who deals in short-life-cycle/seasonal items faces stochastic demand and information on demand distribution is limited; however, the newsvendor has only an educated guess of the mean and variance of the demand. To find the optimal inventory level normal distribution of demand is employed; although, this does not provide the best performance compared to other distributions with the same mean and variance. Scarf [1] first addressed the distribution-free newsboy problem where mean and variance of demand were given, whereas the demand distribution assumption was not available. Considering the distribution-free approach, Scarf [1] developed a model for the optimal ordering quantity to get the maximum expected profit with the worst possible distribution of the demand. The min–max distribution-free approach developed by Scarf is good but is complex and difficult to understand. This problem was solved by Gallego and Moon [2], where they simplified Scarf's ordering rule for the newsboy problem.

This paper extends the classical newsvendor distribution-free model; the seller offers a progressive discount to generate more revenue. This model considers a retailer who is selling seasonal items and facing a stochastic demand. The retailer has only one opportunity to place the order before the start of the selling season. Reorders are not possible during the season; because the lead time is longer than the selling season. The newsboy takes the ordering decision based on historical data, which may possess high variance to the demand. In the classical newsvendor model, the selling price is assumed constant throughout the selling season, which prevents the retailer from generating revenue by price adjustment. This model is more realistic because it provides a price adjustment policy, which helps in generating extra revenue. This model specifically answers the following questions. What selling price should the retailer choose? How much to order considering the discounted policy later? And how much discount percent must the retailer offer at the end of the season?

The remainder of this paper is assembled as follows. In Section 2, this paper reviews the relevant literature. Section 3 provides notation and develops the mathematical model. Section 4 tests the developed model with numerical examples and provides a sensitivity analysis. In Section 5, the paper is concluded.

2. Literature Review

In this section we provide a summary of the relevant literature. This paper directly contributes to three streams of research, namely newsvendor models, price dependent stochastic demand, and price discounts in newsvendor. Therefore, relevant papers were reviewed in this section.

The classical newsboy problem is designed to find the optimal quantity of the items in a single period, probabilistic framework. The aim of calculating order quantity is to minimize the expected cost of overestimating and underestimating the probabilistic demand in the selling season. The newsboy problem has gained considerable attention since the pioneering paper of Arrow et al. [2]. Scarf [1] developed the min–max distribution-free procedure to calculate the optimal order quantity with only the mean and variance. Scarf proved the worst distribution of demand would be positive at two points. Gallego and Moon [3] simplified the proof of Scarf [1], the Scarf rule consists of a lengthy mathematical argument and is difficult to understand for researchers and managers. Anvari [4] maximizes the market value of the firm using the capital asset pricing model. An extensive literature review on the newsvendor problem can be found in the literature, for example in Khouja [5]. Bitran and Mondschein [6] dealt with seasonal items, they also developed a pricing policy based on time and inventory. As most of the newsvendor models deal with profit maximization, however, a few models deal with the probability of exceeding a specified minimum profit, as in Lau [7] and Parlar and Kevin Weng [8]. Sarkar [9] considered a service level constraint and variable lead time with the min–max distribution free approach, Sarkar [10] considered distribution-free approach with buyback contracts.

There exists an enormous amount of literature concerning the extension of the classical newsvendor model [11,12]. The comparison of applying the normal distribution and the scenario of the distribution-free approach (only mean and variance of the demand is available) with discounts is given by Moon et al. [13]; however, they did not consider price dependent demand, furthermore, their model is of a single period

while this paper considered a multi period newsvendor problem. The consignment policy is considered in Sarkar et al. [14] in a single-period newsvendor problem with the distribution-free approach and retailer cost is reduced by the sustainable consignment contract. The classical inventory problem is extended by Kogan and Lou [15] to the multi-stage newsboy problem; further, they divided product flow into sequential stages for the reduction in underage or overage costs at the end-of-season. Matsuyama [16] considers a multi-period inventory problem; in their model, the ordering quantity is divided into many cycles, and if the inventory of the present cycle did not match that will affect the next cycle. The effect of coordination on stocking and pricing for two consecutive periods was studied by Lee [17], he analyzed a normal sales period and a leftover markdown sales period. Chen et al. [18] considers a multi-period supply chain model where the supplier is selling products to a multi-period newsvendor and calculated the optimal pricing for the supplier.

Petruzzi and Dada [19] argue that demand is price dependent in the newsvendor model, Khouja [20] developed the price-dependent demand policy and proved the concavity of the newsvendor problem. Arcelus et al. [21] considered the ordering policies in a newsvendor framework with a stochastic price-dependent demand. The classical single-period newsvendor model is extended in Chung et al. [22] with a price adjustment for the retailer in-season after receiving the demand; however, they assumed demand follows a normal distribution while this paper did not assume any distribution for the demand. Banerjee and Meitei [23] studied the effect of the selling price reduction and analyzed the optimal ordering policy for a single period inventory model. The demand is a stochastic function of the selling price in Abad [24] and he used the service level constraint to avoid the economic consequences of a stock-out situation. The relationships of the purchasing cost, price-dependent stochastic demand, and salvage value are shown by Ma et al. [25], they studied impacts of discounted schemes on the expected profit in a single-period newsvendor framework. Arcelus et al. [26] considered risk tolerance in the newsvendor problem with a price-dependent demand, and Hu and Su [27] optimized newsvendor expected profit with joint procurement planning and a pricing procedure.

The additive and multiplicative demand cases under the behavior of the strategic consumer are studied in Ye and Sun [28] in the newsvendor problem. Arcelus et al. [29] evaluated the pricing and ordering policies of a newsvendor facing risk-averse, risk-neutral, and risk-seeking situations with price-dependent stochastic demand. In their paper, they offer three types of sales policy: Pricing, rebates, and advertising for additive and multiplicative demand structure. He et al. [30] investigated the channel coordination issue in the newsvendor framework with the stochastic price and sales effort dependent demand. A coordination contract is studied by Chen et al. [31] in a stochastic demand case for fashionable products concerning the supplier–retailer channel. In their model, the supplier allows a fixed capacity of production, and retailers are not allowed to change the demand after the demand is realized. A decentralized supply chain is investigated in Chen and Bell [32], in this study the retailer determines the price and order quantity while facing the stochastic price-dependent demand and returns from the customer. Jadidi et al. [33] considered a single period model with quantity discounts, a transportation capacity problem, and price-dependent stochastic demand. Modak and Kelle [34] examined the dual-channel supply chain considering a stochastic demand dependent on retailer price and delivery-time.

The uncertain demand in the newsboy problem creates the need for price setting, which can lead to revenue maximization that is accomplished by increasing total sales. The newsboy deals with the overstocking situation by introducing discounts at the end of the season, he must clear the maximum leftover inventory by offering discounts on sales. Khouja [20] introduces the concept of multiple discounts in the newsvendor model, he further elaborated that discounts are applicable on the products until these items remain on the shelf. Khouja [35] extended Khouja [20] by offering quantity discounts with multiple discounts in the single period inventory problem. Cachon and Kök [36] develop the newsvendor model with a clearance price, they also showed how much discount should be offered if the inventory is remaining at the end of the season. The clearance sales theory is expressed by Nocke and Peitz [37], in their model the selling price was originally high; however, at the end-of selling

period, they reduced the price to clear the remaining inventory. The effect of discounts on gift cards is analyzed by Khouja et al. [38] in the newsvendor framework. A discrete-time model is developed by Gupta et al. [39] for deciding clearance prices in the single period inventory problem by setting a bound on prices and the expected revenue. Cachon and Kök [36] considered a newsvendor problem with clearance pricing: Before the start of the selling season demand is stochastic and exogenous, during the season demand is endogenous and deterministic in their model. The optimal values for order quantity are calculated in Ullah and Sarkar [40]. Jammernegg and Kischka [41] developed a procedure to calculate the optimal values of the order quantity, expected profit, and selling price in the newsvendor framework. Mandal et al. [42] elaborates on the consumers switch to the alternative option as soon as they realize the lower price; likewise, firms make their pricing and stocking decisions. Hu and Su [27] eliminated the assumption on salvage and considered a clearance price as a decision variable. The revision of the interval environment is considered in Ruidas et al. [43].

3. Mathematical Model

The newsboy deals in short lifecycle items, which become obsolete after some time, the demand reduces after the selling season and other products with better performance replace them. This paper develops a joint pricing and inventory model for a multi-period newsboy with price-dependent stochastic demand. The newsboy also offers a discount at the end of the period to increase its revenue and market share. No assumption on demand is considered except that demand belongs to a class φ of the probability distribution functions with a known mean and variance; where a distribution-free approach is applied to solve the model.

3.1. Proposed Model

A multi-period newsvendor model was developed based on joint stocking and pricing decisions under demand uncertainty. The decision maker decides the order quantity at the start of the selling season. The newsvendor offers a discount on leftover inventory at the end of the season to enhance the sales; inventory still left after the discounted sales will be salvage in the next selling season. The combine multi-period and discount policy provide multiple selling opportunities, which reduce the risk considering the overstocking and understocking situations in the newsvendor model. The newsvendor is facing a price-dependent stochastic demand of the form $d_i(p, X) = a(p) + X_i$. The demand comprises of two components; the price dependent deterministic demand ($a(p)$) and random error (X_i), which is independent of the price; such type of demand considerations are common in the literature, for example see [19,41,44,45]. The deterministic price dependent demand is $a(p) = y - z * p$ that is linearly decreasing the function of the price p and it follows increasing the price elasticity (see [12,45,46]). Here y represents the market share and the z is the price sensitivity. The price dependent deterministic demand $a(p)$ is assumed to be positive. The stochastic price independent demand factor X_i, a random variable, is additive in nature because of the additive modeling framework. Further, the assumptions are considered that X_i follows a probability distribution function $f(X_i)$ and a cumulative probability distribution function $F(X_i)$ with mean μ_i and standard deviation σ_i.

The objective was to determine the stocking quantity (Q_i), selling price (p), and discount percentage (β) to maximize the expected profit of the newsvendor. The profit of the proposed system can be determined by calculating the revenue minus the total cost. In the proposed system, the revenue is obtained from three sources; selling finished products with price p, selling with discounted price $p(1-\beta)$, and selling the leftover inventory from the previous period $((1-\alpha(\beta))E(Q_{i-1} - d_{i-1})^+)$ with salvage value s. If the demand in the selling period does not exceed Q_i units, then the revenue is $pd_i(p, X)$, and the remaining units will be sold with a discount percentage β of the selling price (p). Otherwise the revenue is pQ_i, therefore, the total expected revenue, from selling with price p can be expressed as $pE(\min(Q_i, d_i))$. The cost consists of a purchasing cost cQ_i for Q_i units. The leftover after the discounted sales are held at a cost h per unit after the selling season that are salvaged in the next period with a salvage value s. Similarly, if the demand exceeds Q_i units, and the revenue

is pQ_i, the shortage may arise and a penalty cost (*b* per unit) will be considered for the observed shortage. The profit for the multi-period newsvendor is the difference between the revenue generated $pE(\min(Q_i, d_i))$ and the cost incurred per item. The multi-period newsvendor expected profit function can be expressed as:

$$
\begin{aligned}
\pi \;=\; & pE(\min(Q_i, d_i)) - cQ_i - bE(d_i - Q_i)^+ + \alpha(\beta)p(1-\beta)E(Q_i - d_i)^+ - (1-\\
& \alpha(\beta))hE(Q_i - d_i)^+ + s(1-\alpha(\beta))E(Q_{i-1} - d_{i-1})^+.
\end{aligned}
\tag{1}
$$

In Equation (1) the first term $pE(\min(Q_i, d_i))$ is the expected profit generated from selling products with the full price. The second term is the purchase cost, the third term is the shortage cost, when the demand exceeds the order quantity, and the fourth term shows the expected revenue from discounted sales. As $E(Q_i - d_i)^+$ is the expected leftover after the end of season when the full price sales are completed, $\alpha(\beta)$ is the percentage of expected sales when the discount is offered. Where, $\alpha(\beta)$ is an increasing function of the discount percentage (β). $p(1-\beta)$ shows the discounted price, thus, the total revenue from the discounted sales can be formulated as $(\alpha(\beta)p(1-\beta)E(Q_i - d_i)^+)$. From the initial leftover $(E(Q_i - d_i)^+)$, $\alpha(\beta)$ percent is sold with a discount then $(1 - \alpha(\beta))$ percent remains after the discounted sales, for which the expected holding cost is $(1 - \alpha(\beta))hE(Q_i - d_i)^+$, and revenue from salvaging the expected leftover from the previous period is $s(1 - \alpha(\beta))E(Q_{i-1} - d_{i-1})^+$.

Where the discounted sales percentage $\alpha(\beta)$ is an increasing function of the discount percentage β (price), such that $\alpha(\beta) = \left(1 - e^{-\frac{\zeta * \beta}{\varrho}}\right)$. Where ζ and ϱ are the parameters; furthermore, $\alpha(\beta)$ shows a declining increment to increase in β and its behaviour is illustrated in Figure 1, when $\zeta = 0.2$ and $\varrho = 0.05$. Where, $\alpha(\beta)$ is a monotonically increasing function over β, as $\frac{\partial \alpha(\beta)}{\partial \beta} > 0$ and $\frac{\partial^2 \alpha(\beta)}{\partial \beta^2} < 0$, $\forall \beta$.
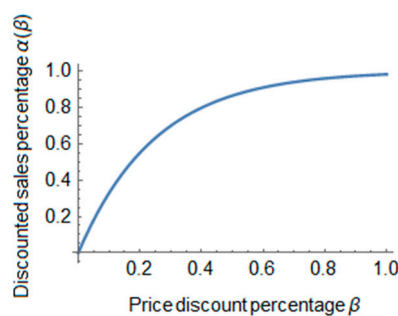

Figure 1. Discounted sales percentage plotted against the price discount percentage.

Observing that $\min(Q_i, d_i) = d_i - (d_i - Q_i)^+$, the expected profit can be written as,

$$
\begin{aligned}
\pi \;=\; & pE(d_i) - cQ_i - (b+p)E(d_i - Q_i)^+ + (\alpha(\beta)p(1-\beta) - (1-\alpha(\beta))h)E(Q_i - d_i)^+ \\
& + (1 - \alpha(\beta))sE(Q_{i-1} - d_{i-1})^+.
\end{aligned}
\tag{2}
$$

As the newsvendor has partial (no distribution knowledge) information on demand, considering this situation, the expected leftover stock is $(Q_i - d_i)^+$ can be formulated as:

$$
(Q_i - d_i)^+ = \frac{E|Q_i - d_i| + E(Q_i - d_i)}{2}.
\tag{3}
$$

By Cauchy Schwartz inequality:

$$
\begin{aligned}
E|Q_i - d_i| &\leq \sqrt{E(Q_i - d_i)^2} \\
&= \sqrt{E(Q_i^2 - 2Q_id_i + d_i^2)};
\end{aligned}
$$

$$
E|Q_i - d_i| \leq \sqrt{E(Q_i^2) - E(2Q_id_i) + E(d_i^2)}.
\tag{4}
$$

The price dependent stochastic demand for period i is:

$$d_i = d_i(p, X).$$

The expected value of the price dependent stochastic price-dependent demand is equal to the expected value of random error plus the deterministic price dependent demand, which is greater than zero as:

$$E(d_i) = \mu_i + (a(p))^+.$$

The price dependent deterministic demand during the season is the maximum perceived cumulative deterministic (riskless) demand, i.e., market share plus the price sensitivity for the cumulative deterministic demand multiplied by price, such that:

$$a(p) = y - z * p.$$

Notice that:

$$E\left(Q_i^2\right) = (Q_i)^2,$$

and

$$E\left(d_i^2\right) = (\mu_i + a)^2 + \sigma_i^2.$$

Putting the values in inequality (4):

$$E|Q_i - d_i| \leq \sqrt{(Q_i)^2 - 2(Q_i)(\mu_i + a) + (\mu_i)^2 + \sigma_i^2}$$

$$= \sqrt{\sigma_i^2 + (Q_i)^2 - 2(Q_i)(\mu_i + a) + (\mu_i + a)^2}.$$

After simplification it can be written as:

$$E|Q_i - d_i| \leq \sqrt{\sigma_i^2 + \sigma_r^2 + (Q_i - \mu_i - a)^2}.$$

Putting the $E|Q_i - d_i|$ in Equation (3),

$$E(Q_i - d_i)^+ \leq \frac{1}{2}\left(\sqrt{\sigma_i^2 + (Q_i - \mu_i - a)^2} + (Q_i - \mu_i - a)\right).$$

Similarly, one can easily prove that,

$$E(d_i - Q_i)^+ \leq \frac{1}{2}\left(\sqrt{\sigma_i^2 + (\mu_i + a - Q_i)^2} - (Q_i - \mu_i - a)\right),$$

and

$$E(Q_{i-1} - d_{i-1})^+ \leq \frac{1}{2}\left(\sqrt{\sigma_{i-1}^2 + (Q_{i-1} - \mu_{i-1} - a)^2} + (Q_{i-1} - \mu_{i-1} - a)\right).$$

Utilizing these inequalities, the expected profit can be written as:

$$\pi = \sum_{i=1}^{n}\left[\begin{array}{l} p(\mu_i + a) - cQ_i - \frac{(b+p)}{2}\left(\sqrt{\sigma_i^2 + (\mu_i + a - Q_i)^2} - (Q_i - \mu_i - a)\right) \\ + \left(\left(1 - e^{-\frac{\zeta\beta}{\varrho}}\right)p(1-\beta) - h\left(e^{-\frac{\zeta\beta}{\varrho}}\right)\right)\frac{1}{2}\left(\sqrt{\sigma_i^2 + (Q_i - \mu_i - a)^2} + (Q_i - \mu_i - a)\right) \\ + \left(e^{-\frac{\zeta\beta}{\varrho}}\right)\frac{s}{2}\left(\sqrt{\sigma_{i-1}^2 + (Q_{i-1} - \mu_{i-1} - a)^2} + (Q_{i-1} - \mu_{i-1} - a)\right) \end{array}\right]. \quad (5)$$

3.2. Optimal Policies

Taking the first derivative of (5) with respect to Q_i, p, and β, and equating to zero the optimal values of Q_i^*, p^*, and β^* can be obtained, such that,

$$\frac{\partial \pi}{\partial Q_i} = \sum_{i=1}^{n} \left(-c + \frac{1}{2} \left(-e^{-\frac{\beta\zeta}{\varrho}} h + \left(1 - e^{-\frac{\beta\zeta}{\varrho}} \right) p(1-\beta) \right) \left(1 + \frac{-y+pz+Q_i-\mu_i}{\sqrt{(-y+pz+Q_i-\mu_i)^2+\sigma_i^2}} \right) - \right.$$
$$\left. \frac{1}{2}(b+p) \left(-1 - \frac{y-pz-Q_i+\mu_i}{\sqrt{(y-pz-Q_i+\mu_i)^2+\sigma_i^2}} \right) \right) = 0;$$

$$\frac{\partial \pi}{\partial p} = \sum_{i=1}^{n} \left(y - 2pz + \mu_i + \frac{1}{2}e^{-\frac{\beta\zeta}{\varrho}} s(z + \frac{z(-y+pz+Q_{-1+i}-\mu_{-1+i})}{\sqrt{(-y+pz+Q_{-1+i}-\mu_{-1+i})^2+\sigma_{-1+i}^2}}) + \frac{1}{2}(-e^{-\frac{\beta\zeta}{\varrho}} h + \right.$$
$$(1 - e^{-\frac{\beta\zeta}{\varrho}})p(1-\beta))(z + \frac{z(-y+pz+Q_i-\mu_i)}{\sqrt{(-y+pz+Q_i-\mu_i)^2+\sigma_i^2}}) + \frac{1}{2}(1 - e^{-\frac{\beta\zeta}{\varrho}})(1-\beta)(-y+pz+Q_i-$$
$$\mu_i + \sqrt{(-y+pz+Q_i-\mu_i)^2+\sigma_i^2}) - \frac{1}{2}(b+p)(-z - \frac{z(y-pz-Q_i+\mu_i)}{\sqrt{(y-pz-Q_i+\mu_i)^2+\sigma_i^2}}) + \frac{1}{2}(-y+$$
$$pz + Q_i - \mu_i - \sqrt{(y-pz-Q_i+\mu_i)^2+\sigma_i^2}) \right) = 0;$$

$$\frac{\partial \pi}{\partial \beta} = \sum_{i=1}^{n} \left(-\frac{e^{-\frac{\beta\zeta}{\varrho}} s\zeta \left(-y+pz+Q_{-1+i}-\mu_{-1+i}+\sqrt{(-y+pz+Q_{-1+i}-\mu_{-1+i})^2+\sigma_{-1+i}^2} \right)}{2\varrho} + \frac{1}{2}(-(1-$$
$$e^{-\frac{\beta\zeta}{\varrho}})p + \frac{e^{-\frac{\beta\zeta}{\varrho}} h\zeta}{\varrho} + \frac{e^{-\frac{\beta\zeta}{\varrho}} p(1-\beta)\zeta}{\varrho})(-y+pz+Q_i-\mu_i+$$
$$\sqrt{(-y+pz+Q_i-\mu_i)^2+\sigma_i^2}) \right) = 0.$$

Q_i^*, p^*, and β^* obtained from the above first order conditions are the global optimal if the hessian matrix of π is negative semidefinite.

See Appendix A for proof.

4. Numerical Example

The developed model is tested with a numerical experiment and sensitivity analysis of the input parameters. The numerical experiment considers a newsvendor problem with two periods. The data for the given example was taken from Alfares and Elmorra [47]. Here $c = 35.1$ \$/unit, $b = 14$ \$/unit, $h = 14$ \$/unit/period, $s = 10$ \$/unit, $\mu_1 = 100$, $\sigma_1 = 15$, $\mu_2 = 100$, $\sigma_2 = 15$, $y = 500$ units/period, $z = 5$, $\zeta = 0.05$, and $\rho = 0.08$.

Case 1. Case 1 is the proposed model with the discount policy. The newsvendor decides on the stocking quantity and price based on the available information. Initially, the product is offered to the consumer with the full price, and at the end of the season, the newsvendor offers a discount on the leftover inventory. The inventory left after the discounted sales are salvaged in the next season.

Case 2. This case is considered to study the impacts of a discount on the expected profit and optimal policies of the newsvendor. This case examines the traditional newsvendor that does not offer a discount. In this case, the newsvendor decides on the stocking quantity and price based on the available information. No discount is offered and, thus, the discount percentage is zero. All the leftover inventory after the selling season is salvaged. Results of both cases are summarised in Table 1.

Table 1. Optimal results of Case 1 and Case 2.

Decision Variables	Case 1	Case 2
Q_1 (Units)	219.77	218.25
Q_2 (Units)	217.95	216.54
P ($/product)	77.12	76.88
β (%)	51	0
Expected Profit π ($)	16,763.5	16,530

From the results, it is clear that the expected profit of the discount policy was higher compared to the non-discount policy. Furthermore, both the stocking quantities and the optimal price were higher in the discount policy compared to the non-discount policy. The higher stocking quantity decreases the risk of stock outs, and the overstocking risk was neutralized by the discount policy. Thus, the discount policy was more flexible compared to the traditional non-discount policy.

4.1. Sensitivity Analysis

The sensitivity analysis was performed for all the key parameters and results were compiled in Tables 2–4. The percentage of variation for all parameters is in the range of −50% to +50%. The sensitivity analysis results from Table 2 revealed the following insights on model parameters:

- Considering the profit of the system, the most effective parameter was purchasing cost; decreasing it by 50% increases the expected profit by 50.99%. However, on the positive side, the effect was a little lesser compared to the negative side. Increasing purchasing cost by 50% decreases the profit by 40.85%. Increasing purchasing cost decreases order quantity, increases the optimal price, and the discount percentage is almost unaffected. This shows that the discount percentage applies to both expensive and inexpensive products.
- The impacts of shortage cost, on the profit of the system, were almost symmetrical towards both negative and positive changes. Order quantity increased with an increase in the shortage cost, whereas, the effect on other variables was negligible.
- The holding cost directly influenced the discount percentage, increasing the holding cost increases the discount percentage. This provides interesting results for newsvendors with higher holding costs. They can increase their profits by discounted sales policy. We can see that the profit was more sensitive towards negative changes compared to the positive changes in the holding cost. The order quantity decreased whereas the price was unaffected.

Compared to the holding cost that had a direct relation with the discount percentage; the salvage value had an indirect relation with the discount percentage. Increasing the salvage value decreases the discount percentage; the expected profit behaves in an almost symmetric way, with little high changes on the positive side. The sensitivity of the holding cost and salvage value provides instructing results for the decision maker. For retailers where holding was higher, a high discount policy was better; and for retailers with high salvage value, a low discount policy was the optimal one.

Table 2. Sensitivity analysis for the key operational parameter.

Parameter	Percent Change in Value	Decision Variables				Percent Change in Expected Profit
		Q_1	Q_2	p	β	
c	−50	271.7	266.8	68.6	0.51	50.99
	−25	244.6	241.9	72.8	0.51	24.18
	+25	195.7	194.3	81.3	0.50	−21.66
	+50	172.1	170.9	85.6	0.50	−40.85
b	−50	218.9	217.1	77.0	0.51	0.483
	−25	219.3	217.5	77.1	0.51	0.237
	+25	220.1	218.3	77.1	0.51	−0.230
	+50	220.5	218.6	77.1	0.51	−0. 453
h	−50	221.1	218.8	77.1	0.47	0.647
	−25	220.4	218.3	77.1	0.49	0.314
	+25	219.2	217.5	77.0	0.53	−0.297
	+50	218.2	217.2	77.0	0.54	−0.581
s	−50	218.9	218.0	77.1	0.52	−0.222
	−25	219.3	218.0	77.1	0.51	−0.113
	+25	220.2	217.8	77.1	0.50	0.117
	+50	220.8	217.8	77.1	0.49	0.240

This paper assumed a stochastic price-dependent demand that composed of a deterministic price dependent part and a random error. Table 3 provides a sensitivity analysis of the demand parameters; the following insights are obtained from the results:

- Although changing the mean of the random errors in the demand affects the profit, the results were symmetric in both the direction. Increasing μ_1 or μ_2 by 50% increased the profit by 12.36%, because, the expected value of demand $d_i(p, X_i)$ increased with increasing μ_1 and μ_2. Price of the finished product increased with increases in demand; however, the discount percentage remained the same. This means the discount policy applied to both low and higher demands newsvendors.

- Compared to the random error μ_1 and μ_2 the standard deviations had much less impact on the profit of the system. However, the changes, in profit, were symmetric to both positive and negative changes in standard deviations of the demand. The stocking quantity increased with an increase in standard deviation. The result was clear because increasing standard deviation increased the uncertainty; therefore, the stocking quantity was increased. Furthermore, the price remained unaffected; however, the discount percentage decreased with increasing σ_1. This means, for a less uncertain demand, the newsvendor should increase the discount percentage to increase its profit and market share. However, as the uncertainty increased the discount rate was reduced to avoid extra loses from high-expected salvage quantity.

- The deterministic price-dependent demand had two parameters, which were y and z. Where y is the potential market size for the deterministic demand and z is the price sensitivity of the consumer. Increasing or decreasing y directly increased or decreased the expected demand; therefore, the profit of the system was affected accordingly. However, the results were asymmetric and the effect grew as y increased.

- On the other hand, increasing z reduced the profit; however, the impacts were asymmetric and the decrease in profit declined as z increased. For a higher value of z, the consumer was more sensitive to price; therefore, the optimal price decreased with increasing z. The discount percentage, on the other hand, continuously increased with an increase in z. This means a higher discount percentage applied to a consumer having high price sensitivity and vice versa. Stocking quantity decreased with an increase in z because the realized deterministic demand decreased with increasing z.

Table 3. Sensitivity analysis for the demand parameter.

Parameter	Percent Change in Value	Decision Variables				Percent Change Expected Profit
		Q_1	Q_2	p	β	
μ_1	−50	181.9	230.1	74.6	0.51	−12.15
	−25	200.8	224.0	75.86	0.51	−6.17
	+25	238.9	211.8	78.3	0.51	6.361
	+50	527.6	205.7	79.6	0.50	12.91
μ_2	−50	231.9	180.1	74.6	0.51	−12.15
	−25	225.8	199.0	75.86	0.51	−6.17
	+25	213.6	236.8	78.3	0.51	6.361
	+50	207.6	255.7	79.64	0.50	12.91
σ_1	−50	216.5	217.4	77.2	0.52	1.763
	−25	218.1	217.7	77.1	0.51	0.881
	+25	221.3	218.1	77.0	0.50	−0.880
	+50	222.9	218.41	77.0	0.50	−1.761
σ_2	−50	219.2	215.5	77.2	0.50	1.98
	−25	219.4	216.7	77.1	0.50	0.99
	+25	220.0	219.1	77.0	0.51	−0.989
	+50	220.3	220.2	77.0	0.51	−1.979
z	−50	217.8	268.6	137.3	0.48	203.7
	−25	244.4	242.2	97.15	0.50	66.14
	+25	196.0	194.3	65.1	0.51	−37.67
	+50	172.7	171.1	57.1	0.52	−61.17
y	−50	81.4	79.8	53.7	0.52	−87.0
	−25	155.3	153.7	64.5	0.52	−53.2
	+25	284.0	281.9	89.7	0.50	72.0
	+50	348.1	345.9	102.2	0.49	162.8

The discounted portion of sales ($\alpha(\beta)$) is a function of the discount percentage with parameters ζ and ϱ. Increasing the discount percentage reduces the price and discounted sales increases. A sensitivity analysis of the parameters of the discounted portion is given in Table 4; the results showed that:

- With the increase in the parameter (ζ) value, the expected profit of the system showed an asymmetric increase. For a 25% decrease, the expected profit decreased by −0.32%; however, with further decreases the profit remains unaffected. On the positive side, the profit increase by 0.31% for a 25% increase, and 0.628% for a 50% increase in the value of ζ. The discount percent decreased with an increase in ζ, price remains unaffected, and the stocking quantity increased with increasing ζ.
- By increasing the value of the parameter (ϱ), the asymmetric negative change occurred in the profit, the order quantity decreased, the price almost remained the same, and the discount percentage increased.

Table 4. Sensitivity analysis for the key discounted sales parameter.

Parameter	Percent Change in Value	Decision Variables				Percent Change in Expected Profit
		Q_1	Q_2	p	β	
ζ	−50	218.2	217.2	77.0	0.53	−0.329
	−25	219.3	217.0	77.0	0.52	−0.329
	+25	220.1	218.3	77.1	0.50	0.318
	+50	220.6	218.7	77.2	0.49	0.628
ϱ	−50	221.6	219.5	77.3	0.47	1.226
	−25	220.3	218.4	77.2	0.49	0.423
	+25	219.4	217.6	77.0	0.51	−0.262
	+50	219.2	217.4	77.0	0.52	−0.441

4.2. Managerial Insights

Based on the obtained results, the following recommendations were suggested to managers:

- The proposed discount policy increases both the price and order quantities, thus, managers can order high quantity compared to non-discount policy. Higher ordering quantities decrease the risk of shortage cost, whereas, the discount percentage decreases the risk of overstocking and leftovers. Therefore, in the proposed policy, both the risks are minimized.

- Another important insight is that the discount percentage increases with a decrease in market uncertainty, this means, a retailer having low variable demand can order more with a higher discount percentage. However, retailers with a highly variable demand, orders low quantity with a low discount percentage.

- The optimal discount percentage increases as the consumer price sensitivity increases, therefore, managers that are dealing with markets having higher consumer sensitivity are advised to offer higher discount percentage. This increases the discounted percentage and higher profit and market shares can be achieved.

- For the single period problem, the optimal ordering quantity and price decreases, where the discount percentage increases; the discounts on the successive periods leads to an increase in order quantity compared to the single period newsvendor problem. The increase in the ordered quantity and price is the result of discount percentage that reduces salvage quantity for the subsequent period; however, the selling price of the items is still more than the salvaged value of the product.

- The associated risk with salvaging leftover and shortage is high in the stochastic environment and it can be reduced by implying the distribution free approach with discount offering—discounts are helpful for mangers in increasing the profit generation. The major objective of the manager is to maximize the expected profit, which can be achieved by the proposed discounted policy for leftover items in successive periods.

5. Conclusions

This paper studied joint pricing and inventory policies for the newsvendor model with discounted sales. The classical multi-period distribution-free newsvendor model was extended with a discount policy to increase the sales and profit of the retailer. Stochastic-price dependent demand was considered, and a distribution-free approach was applied to solve the model. No specific assumptions on the distribution of the random error in the demand were considered, except that it had a known mean and variance. Two numerical examples were considered, and the results showed that the proposed discount policy increased the sales and profit of the system. Furthermore, the discount policy provided more flexibility to the newsvendor in deciding the optimal price. With this policy, the newsvendor, initially, decided a higher price compared to the one without the discount policy, and later on, the leftover is discounted with a lower price. Thus, the newsvendor can catch both the strategic and non-strategic consumer at the same time. The sensitivity results showed that the discount policy was applicable

to both expensive and inexpensive products; retailers with a higher holding cost can use this policy to increase their profit. This study considered the inventory and pricing policy for only one player, however, in practice, every business directly deals with upstream and downstream linkages. Therefore, this study can be extended by considering more than one player, such as the models developed by Sarkar [9,10]; in this case, two different discount policies can be considered, discount for the final consumer and discount for the newsvendor. Another limitation of this study was that we considered only the single discount per period, considering multiple discounts in one period is a more practical extension of this study. A third possible extension is to consider deteriorating products, such as the study done by Ullah et al. [48]; in this case, the salvage value of the deteriorated product must be zero.

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Notation

The following notation are used to establish the mathematical model:

Decision variables

p	Price of finished product per unit ($/unit)
β	Discount percentage (percent of finished product price)
Q_i	Ordering quantity ith period (units)

Parameters

i	Index for selling period, where $i = 1 \ldots \ldots n$
b	Shortage cost per unit ($/unit)
c	Purchasing cost per unit ($/unit)
h	Holding cost ($/unit/period)
s	Salvage value ($/unit)
$d_i(p, X_i)$	Price dependent stochastic demand of period i
X_i	Random error in demand
μ_i	The expected value of random error
σ_i	The standard deviation of demand
$a(p) = y - z * p$	Deterministic price dependent demand in-season
y	Maximum perceived cumulative deterministic (riskless) demand i.e., market share (units/unit time)
z	Price sensitivity for cumulative deterministic demand
$E(d_i) = \mu_i + a(p)$	The expected value of price dependent stochastic demand
X^+	Max [X,0]
π	Expected profit ($)

Appendix A

If,

$$a_{1,1} = \frac{\partial}{\partial Q_1} \frac{\partial \pi}{\partial Q_1}; a_{1,2} = \frac{\partial}{\partial p} \frac{\partial \pi}{\partial Q_1}; a_{1,3} = \frac{\partial}{\partial \beta} \frac{\partial \pi}{\partial Q_1}; a_{1,4} = \frac{\partial}{\partial Q_2} \frac{\partial \pi}{\partial Q_1}; a_{2,1} = \frac{\partial}{\partial Q_1} \frac{\partial \pi}{\partial p}; a_{2,2} =$$

$$\frac{\partial}{\partial p} \frac{\partial \pi}{\partial p}; a_{2,3} = \frac{\partial}{\partial \beta} \frac{\partial \pi}{\partial p}; a_{2,4} = \frac{\partial}{\partial Q_2} \frac{\partial \pi}{\partial p} a_{3,1} = \frac{\partial}{\partial Q_1} \frac{\partial \pi}{\partial \beta}; a_{3,2} = \frac{\partial}{\partial p} \frac{\partial \pi}{\partial \beta}; a_{3,3} = \frac{\partial}{\partial \beta} \frac{\partial \pi}{\partial \beta}; a_{3,4} = \frac{\partial}{\partial Q_2} \frac{\partial \pi}{\partial \beta};$$

$$a_{4,1} = \frac{\partial}{\partial Q_1} \frac{\partial \pi}{\partial Q_2}; a_{4,2} = \frac{\partial}{\partial p} \frac{\partial \pi}{\partial Q_2}; a_{4,3} = \frac{\partial}{\partial \beta} \frac{\partial \pi}{\partial Q_2}; a_{4,4} = \frac{\partial}{\partial Q_2} \frac{\partial \pi}{\partial Q_2},$$

then the Hessian matrix of (4) can be expressed as:

$$H = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}.$$

At the optimal solutions $Q_1{}^* = 219.76$, $Q_2{}^* = 217.94$, $p^* = 77.12$, and $\beta^* = 0.51$ the Hessian matrix can be written as,

$$H = \begin{pmatrix} -2.32 & -11.20 & -1.43 & 0.00 \\ -11.20 & -139.07 & -0.79 & 0.00 \\ -1.43 & -0.79 & -1677.99 & 0.00 \\ 0.00 & -13.49 & 1.47 & -2.79 \end{pmatrix}.$$

The first four principle minors are -2.32, $+197.81$, $-331,668.04$, and $+926,141.46$. Hence total profit is strictly concave at $Q_1{}^* = 219.76$, $Q_2{}^* = 217.94$, $p^* = 77.12$, and $\beta^* = 0.51$.

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