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The Langevin Equation in Terms of Generalized Liouville–Caputo Derivatives with Nonlocal Boundary Conditions Involving a Generalized Fractional Integral

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Abstract: In this paper, we establish sufficient conditions for the existence of solutions for a nonlinear Langevin equation based on Liouville-Caputo-type generalized fractional differential operators of different orders, supplemented with nonlocal boundary conditions involving a generalized integral operator. The modern techniques of functional analysis are employed to obtain the desired results. The paper concludes with illustrative examples.

Keywords: Langevin equation; generalized fractional integral; generalized Liouville–Caputo derivative; nonlocal boundary conditions; existence; fixed point

1. Introduction

The topic of fractional calculus has emerged as an interesting area of investigation in view of its widespread applications in social sciences, engineering and technical sciences. Mathematical models based on fractional order differential and integral operators are considered to be more realistic and practical than their integer-order counterparts as such models can reveal the history of the ongoing phenomena in systems and processes. This branch of mathematical analysis is now very developed and covers a wide range of interesting results, for instance [1–7].

The Langevin equation is an effective tool of mathematical physics, which can describe processes like anomalous diffusion in a descent manner. Examples of such processes include price index fluctuations [8], harmonic oscillators [9], etc. A generic Langevin equation for noise sources with correlations also plays a central role in the theory of critical dynamics [10]. The nature of the quantum noise can be understood better by means of a generalized Langevin equation [11]. The role of the Langevin equation in fractional systems, such as fractional reaction-diffusion systems [12,13], is very rich and beautiful. The fractional analogue (also known as the stochastic differential equation) of the usual Langevin equation is suggested for systems in which the separation between microscopic and macroscopic time scales is not observed; for example, see [8]. In [14], the author investigated moments, variances, position and velocity correlation for a Riemann-Liouville-type fractional Langevin equation

in time and compared the results obtained with the ones derived for the same generalized Langevin equation involving the Liouville-Caputo fractional derivative. Some recent results on the Langevin equation with different boundary conditions can be found in the papers [15–20] and the references cited therein.

Motivated by the aforementioned work on the Langevin equation and its variants, in this paper, we introduce and study a new form of Langevin equation involving generalized Liouville-Caputo derivatives of different orders and solve it with nonlocal generalized fractional integral boundary conditions. In precise terms, we investigate the problem:

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{a+}({}^{\rho}_{c}D^{\beta}_{a+}+\lambda)x(t)=f(t,x(t)), \;\; t\in J:=[a,T], \;\; \lambda\in\mathbb{R}, \\ x(a)=0, \;\; x(\eta)=0, \;\; x(T)=\mu\,{}^{\rho}I^{\gamma}_{a+}x(\xi), \quad a<\eta<\xi<T, \end{cases} \tag{1}$$

where ${}^{\rho}_{c}D^{\alpha}_{a+}$, ${}^{\rho}_{c}D^{\beta}_{a+}$ denote the Liouville–Caputo-type generalized fractional differential operators of order $1<\alpha\leq 2$, $0<\beta<1$, $\rho>0$, respectively, ${}^{\rho}I^{\gamma}_{a+}$ is the generalized fractional integral operator of order $\gamma>0$ and $\rho>0$, and $f:[a,T]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

Here, we emphasize that the present work may have useful applications in fractional quantum mechanics and fractional statistical mechanics, in relation to further generalization of the Feynman and Weiner path integrals [21].

We compose the rest of the article as follows. Section 2 contains the basic concepts of generalized fractional calculus and an auxiliary lemma dealing with the linear variant of the given problem. In Section 3, we present the main results and illustrative examples.

2. Preliminaries

Definition 1 ([22]). *The generalized left-sided fractional integral of order $\beta>0$ and $\rho>0$ of $g\in X^{p}_{c}(a,b)$ for $-\infty<a<t<b<\infty$, is defined by:*

$$({}^{\rho}I^{\beta}_{a+}g)(t)=\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_{a}^{t}\frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\beta}}g(s)ds, \tag{2}$$

where $X^{p}_{c}(a,b)$ denotes the space of all complex-valued Lebesgue measurable functions ϕ on (a,b) equipped with the norm:

$$\|\phi\|_{X^{p}_{c}}=\Big(\int_{a}^{b}|x^{c}\phi(x)|^{p}\frac{dx}{x}\Big)^{1/p}<\infty, \;\; c\in\mathbb{R}, 1\leq p\leq\infty.$$

Similarly, the right-sided fractional integral ${}^{\rho}I^{\beta}_{b-}g$ is defined by:

$$({}^{\rho}I^{\beta}_{b-}g)(t)=\frac{\rho^{1-\alpha}}{\Gamma(\beta)}\int_{t}^{b}\frac{s^{\rho-1}}{(s^{\rho}-t^{\rho})^{1-\beta}}g(s)ds. \tag{3}$$

Definition 2 ([23]). *For $\beta>0$, $n=[\beta]+1$, $\rho>0$ and $0\leq a<x<b<\infty$, we define the generalized fractional derivatives in terms of the generalized fractional integrals (2) and (3) as:*

$$\begin{aligned} ({}^{\rho}D^{\beta}_{a+}g)(t) &= \Big(t^{1-\rho}\frac{d}{dt}\Big)^{n}({}^{\rho}I^{n-\beta}_{a+}g)(t) \\ &= \frac{\rho^{\beta-n+1}}{\Gamma(n-\beta)}\Big(t^{1-\rho}\frac{d}{dt}\Big)^{n}\int_{a}^{t}\frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{\beta-n+1}}g(s)ds, \end{aligned} \tag{4}$$

and:

$$({}^{\rho}D^{\beta}_{b-}g)(t) = \Big(-t^{1-\rho}\frac{d}{dt}\Big)^{n}({}^{\rho}I^{n-\beta}_{b-}g)(t)$$

$$= \frac{\rho^{\beta-n+1}}{\Gamma(n-\beta)}\Big(-t^{1-\rho}\frac{d}{dt}\Big)^n \int_t^b \frac{s^{\rho-1}}{(s^\rho - t^\rho)^{\beta-n+1}} g(s)ds, \tag{5}$$

if the integrals in the above expressions exist.

Definition 3 ([24]). *For $\beta > 0, n = [\beta]+1$ and $g \in AC_\delta^n[a,b]$, the Liouville–Caputo-type generalized fractional derivatives ${}_c^\rho D_{a+}^\beta g$ and ${}_c^\rho D_{b-}^\beta g$ are respectively defined via (4) and (5) as follows:*

$${}_c^\rho D_{a+}^\beta g(x) = {}^\rho D_{a+}^\beta \Big[g(t) - \sum_{k=0}^{n-1} \frac{\delta^k g(a)}{k!}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^k\Big](x), \ \ \delta = x^{1-\rho}\frac{d}{dx}, \tag{6}$$

$${}_c^\rho D_{b-}^\beta g(x) = {}^\rho D_{b-}^\beta \Big[g(t) - \sum_{k=0}^{n-1} \frac{(-1)^k\delta^k g(b)}{k!}\Big(\frac{b^\rho - t^\rho}{\rho}\Big)^k\Big](x), \ \ \delta = x^{1-\rho}\frac{d}{dx}, \tag{7}$$

where $AC_\delta^n[a,b]$ denotes the class of all absolutely-continuous functions g possessing δ^{n-1}-derivative ($\delta^{n-1}g \in AC([a,b],\mathbb{R})$), equipped with the norm $\|g\|_{AC_\delta^n} = \sum_{k=0}^{n-1}\|\delta^k g\|_C$.

Remark 1 ([24]). *For $\alpha \geq 0$ and $g \in AC_\delta^n[a,b]$, the left and right generalized Liouville–Caputo derivatives of g are respectively defined by the expressions:*

$${}_c^\rho D_{a+}^\beta g(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t \Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{n-\beta-1}\frac{(\delta^n g)(s)ds}{s^{1-\rho}}, \tag{8}$$

$${}_c^\rho D_{b-}^\beta g(t) = \frac{1}{\Gamma(n-\beta)}\int_t^b \Big(\frac{s^\rho - t^\rho}{\rho}\Big)^{n-\alpha-1}\frac{(-1)^n(\delta^n g)(s)ds}{s^{1-\rho}}. \tag{9}$$

Lemma 1 ([24]). *Let $g \in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$ and $\beta \in \mathbb{R}$. Then:*

$${}^\rho I_{a+}^\beta {}_c^\rho D_{a+}^\beta g(x) = g(x) - \sum_{k=0}^{n-1}\frac{(\delta^k g)(a)}{k!}\Big(\frac{x^\rho - a^\rho}{\rho}\Big)^k,$$

$${}^\rho I_{b-}^\beta {}_c^\rho D_{b-}^\beta g(x) = g(x) - \sum_{k=0}^{n-1}\frac{(-1)^k(\delta^k g)(a)}{k!}\Big(\frac{b^\rho - x^\rho}{\rho}\Big)^k.$$

In particular, for $0 < \beta \leq 1$, we have:

$${}^\rho I_{a+}^\beta {}_c^\rho D_{a+}^\beta g(x) = g(x) - g(a), \ {}^\rho I_{b-}^\beta {}_c^\rho D_{b-}^\beta g(x) = g(x) - g(b).$$

Definition 4. *A function $x \in C([a,T],\mathbb{R})$ is called a solution of (1) if x satisfies the equation ${}_c^\rho D_{a+}^\alpha({}_c^\rho D_{a+}^\beta + \lambda)x(t) = f(t,x(t))$ on $[a,T]$, and the conditions $x(a) = 0, x(\eta) = 0, x(T) = \mu^\rho I_{a+}^\gamma x(\xi)$.*

In the next lemma, we solve the linear variant of Problem (1).

Lemma 2. *Let $h \in C([a,T],\mathbb{R})$, $x \in AC_\delta^3(J)$ and:*

$$\Omega = \Big[\frac{(T^\rho - a^\rho)^\beta(T^\rho - \eta^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)} - \frac{\mu(\xi^\rho - a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho - \eta^\rho) - \gamma(\eta^\rho - a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big] \neq 0. \tag{10}$$

Then, the unique solution of linear problem:

$$\begin{cases} {}_c^\rho D_{a+}^\alpha({}_c^\rho D_{a+}^\beta + \lambda)x(t) = h(t), \ \ t \in J := [a,T], \\ x(a) = 0, \ \ x(\eta) = 0, \ \ x(T) = \mu^\rho I_{a+}^\gamma x(\xi), \ \ a < \eta < \xi < T, \end{cases} \tag{11}$$

is given by:

$$\begin{aligned} x(t) &= {}^{\rho}I_{a+}^{\alpha+\beta}h(t) - \lambda^{\rho}I_{a+}^{\beta}x(t) + \frac{(t^{\rho}-a^{\rho})^{\beta}(\eta^{\rho}-t^{\rho})}{\rho^{\beta+1}\Gamma(\beta+2)\Omega}\Big\{{}^{\rho}I_{a+}^{\alpha+\beta}h(T) - \lambda^{\rho}I_{a+}^{\beta}x(T) \\ &\quad -\mu^{\rho}I_{a+}^{\alpha+\beta+\gamma}h(\xi) + \mu\lambda^{\rho}I_{a+}^{\beta+\gamma}x(\xi)\Big\} - \frac{(t^{\rho}-a^{\rho})^{\beta}}{\Omega(\eta^{\rho}-a^{\rho})^{\beta}}\Big(\frac{(T^{\rho}-a^{\rho})^{\beta}(T^{\rho}-t^{\rho})}{\rho^{\beta+1}\Gamma(\beta+2)} \\ &\quad -\frac{\mu(\xi^{\rho}-a^{\rho})^{\beta+\gamma}[(\beta+1)(\xi^{\rho}-t^{\rho})-\gamma(t^{\rho}-a^{\rho})]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big)\Big\{{}^{\rho}I_{a+}^{\alpha+\beta}h(\eta) - \lambda^{\rho}I_{a+}^{\beta}x(\eta)\Big\}. \end{aligned} \tag{12}$$

Proof. Applying ${}^{\rho}I_{a+}^{\alpha}$ on the fractional differential equation in (11) and using Lemma 1 yield:

$$({}_{c}^{\rho}D_{a+}^{\beta} + \lambda)x(t) = {}^{\rho}I_{a+}^{\alpha}h(t) + c_1 + c_2\frac{(t^{\rho}-a^{\rho})}{\rho}, \tag{13}$$

for some $c_1, c_2 \in \mathbb{R}$.

Applying ${}^{\rho}I_{a+}^{\beta}$ to both sides of Equation (13), the general solution of the Langevin equation in (11) is found to be:

$$x(t) = {}^{\rho}I_{a+}^{\alpha+\beta}h(t) - \lambda^{\rho}I_{a+}^{\beta}x(t) + c_1\frac{(t^{\rho}-a^{\rho})^{\beta}}{\rho^{\beta}\Gamma(\beta+1)} + c_2\frac{(t^{\rho}-a^{\rho})^{\beta+1}}{\rho^{\beta+1}\Gamma(\beta+2)} + c_3, \tag{14}$$

where $c_3 \in \mathbb{R}$.

Using the condition $x(a) = 0$ in (14), we find that $c_3 = 0$. Inserting the value of c_3 in (14) and then applying the operator ${}^{\rho}I_{a+}^{\gamma}$ on the resulting equation, we get:

$${}^{\rho}I_{a}^{\gamma}x(t) = {}^{\rho}I_{a+}^{\alpha+\beta+\gamma}h(t) - \lambda^{\rho}I_{a+}^{\beta+\gamma}x(t) + c_1\frac{(t^{\rho}-a^{\rho})^{\beta+\gamma}}{\rho^{\beta+\gamma}\Gamma(\beta+\gamma+1)} + c_2\frac{(t^{\rho}-a^{\rho})^{\beta+\gamma+1}}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)}. \tag{15}$$

Using the boundary conditions $x(\eta) = 0$ and $x(T) = \mu^{\rho}I_{a+}^{\gamma}x(\xi)$ together with (14) and (15) leads to a system of algebraic equations in c_1 and c_2, which, upon solving, yields:

$$\begin{aligned} c_1 &= \frac{\rho^{\beta}\Gamma(\beta+1)}{\Omega(\eta^{\rho}-a^{\rho})^{\beta}}\Big\{\frac{(\eta^{\rho}-a^{\rho})^{\beta+1}}{\rho^{\beta+1}\Gamma(\beta+2)}\Big({}^{\rho}I_{a+}^{\alpha+\beta}h(T) - \lambda^{\rho}I_{a+}^{\beta}x(T) - \mu^{\rho}I^{\alpha+\beta+\gamma}h(\xi) + \mu\lambda^{\rho}I^{\beta+\gamma}x(\xi)\Big) \\ &\quad -\Big(\frac{(T^{\rho}-a^{\rho})^{\beta+1}}{\rho^{\beta+1}\Gamma(\beta+2)} - \frac{\mu(\xi^{\rho}-a^{\rho})^{\beta+\gamma+1}}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)}\Big)\Big({}^{\rho}I_{a+}^{\alpha+\beta}h(\eta) - \lambda^{\rho}I_{a+}^{\beta}x(\eta)\Big)\Big\}, \\ c_2 &= -\frac{\rho^{\beta}\Gamma(\beta+1)}{\Omega(\eta^{\rho}-a^{\rho})^{\beta}}\Big\{\frac{(\eta^{\rho}-a^{\rho})^{\beta}}{\rho^{\beta}\Gamma(\beta+1)}\Big({}^{\rho}I_{a+}^{\alpha+\beta}h(T) - \lambda^{\rho}I_{a+}^{\beta}x(T) - \mu^{\rho}I^{\alpha+\beta+\gamma}h(\xi) + \mu\lambda^{\rho}I^{\beta+\gamma}x(\xi)\Big) \\ &\quad -\Big(\frac{(T^{\rho}-a^{\rho})^{\beta}}{\rho^{\beta}\Gamma(\beta+1)} - \frac{\mu(\xi^{\rho}-a^{\rho})^{\beta+\gamma}}{\rho^{\beta+\gamma}\Gamma(\beta+\gamma+1)}\Big)\Big({}^{\rho}I_{a+}^{\alpha+\beta}h(\eta) - \lambda^{\rho}I_{a+}^{\beta}x(\eta)\Big)\Big\}. \end{aligned}$$

Inserting the values of c_1, c_2 and c_3 in (13) yields the solution (12). The converse of the Lemma 2, can be obtained by direct computation. This finishes the proof. □

3. Existence and Uniqueness Results

In view of Lemma 2, we introduce an operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by:

$$\begin{aligned} \mathcal{F}(x)(t) &= {}^{\rho}I_{a+}^{\alpha+\beta}f(t,x(t)) - \lambda^{\rho}I_{a+}^{\beta}x(t) + \frac{(t^{\rho}-a^{\rho})^{\beta}(\eta^{\rho}-t^{\rho})}{\rho^{\beta+1}\Gamma(\beta+2)\Omega}\Big\{{}^{\rho}I_{a+}^{\alpha+\beta}f(T,x(T)) - \lambda^{\rho}I_{a+}^{\beta}x(T) \\ &\quad - \mu^{\rho}I_{a+}^{\alpha+\beta+\gamma}f(\xi,x(\xi)) + \mu\lambda^{\rho}I_{a+}^{\beta+\gamma}x(\xi)\Big\} - \frac{(t^{\rho}-a^{\rho})^{\beta}}{\Omega(\eta^{\rho}-a^{\rho})^{\beta}}\Big[\frac{(T^{\rho}-a^{\rho})^{\beta}(T^{\rho}-t^{\rho})}{\rho^{\beta+1}\Gamma(\beta+2)} \\ &\quad - \frac{\mu(\xi^{\rho}-a^{\rho})^{\beta+\gamma}[(\beta+1)(\xi^{\rho}-t^{\rho})-\gamma(t^{\rho}-a^{\rho})]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big]\Big\{{}^{\rho}I_{a+}^{\alpha+\beta}f(\eta,x(\eta)) - \lambda^{\rho}I_{a+}^{\beta}x(\eta)\Big\}. \end{aligned} \tag{16}$$

Here, $\mathcal{C}$ denotes the Banach space of all continuous functions from $[a,T]$ to $\mathbb{R}$ equipped with the norm $\|x\| = \sup_{t\in[a,T]} |x(t)|$.

For the sake of computational convenience, we set:

$$\begin{aligned}\Lambda_1 &= \frac{(T^\rho - a^\rho)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Big[1+\frac{\zeta_1}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big] + \frac{|\mu|(\xi^\rho - a^\rho)^{\alpha+\beta+\gamma}\zeta_1}{\rho^{\alpha+2\beta+\gamma+1}\Gamma(\alpha+\beta+\gamma+1)\Gamma(\beta+2)|\Omega|} \\ &\quad + \frac{(\eta^\rho - a^\rho)^\alpha \zeta_2}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)|\Omega|},\end{aligned} \tag{17}$$

$$\begin{aligned}\Lambda_2 &= \frac{|\lambda|(T^\rho - a^\rho)^\beta}{\rho^\beta\Gamma(\beta+1)}\Big[1+\frac{\zeta_1}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big] + \frac{|\mu||\lambda|(\xi^\rho - a^\rho)^{\beta+\gamma}\zeta_1}{\rho^{2\beta+\gamma+1}\Gamma(\beta+\gamma+1)\Gamma(\beta+2)|\Omega|} \\ &\quad + \frac{|\lambda|\zeta_2}{\rho^\beta\Gamma(\beta+1)|\Omega|},\end{aligned} \tag{18}$$

where:

$$\zeta_1 := \max_{t\in[a,T]} \Big|(t^\rho - a^\rho)^\beta (\eta^\rho - t^\rho)\Big|, \tag{19}$$

$$\zeta_2 := \max_{t\in[a,T]} \Big|(t^\rho - a^\rho)^\beta \Big[\frac{(T^\rho - a^\rho)^\beta (T^\rho - t^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)} - \frac{\mu(\xi^\rho - a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho - t^\rho) - \gamma(t^\rho - a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big]\Big|. \tag{20}$$

Now, we are in a position to present our main results. Our first existence result for the problem (1) is based on Krasnoselskii's fixed point theorem [25], which is stated below.

Lemma 3. *(Krasnoselskii's fixed point theorem) Let $\mathcal{S}$ be a closed convex and non-empty subset of a Banach space E. Let $\mathcal{G}_1, \mathcal{G}_2$ be the operators from $\mathcal{S}$ to E such that (a) $\mathcal{G}_1 x + \mathcal{G}_2 y \in \mathcal{S}$ whenever $u, v \in \mathcal{S}$; (b) $\mathcal{G}_1$ is compact and continuous; and (c) $\mathcal{G}_2$ is a contraction mapping. Then, there exists a fixed point $\omega \in \mathcal{S}$ such that $\omega = \mathcal{G}_1\omega + \mathcal{G}_2\omega$.*

Theorem 1. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the following condition holds:*

(A_1) *There exists a continuous function $\phi \in C([a,T], \mathbb{R}^+)$ such that:*

$$|f(t,u)| \le \phi(t), \quad \forall (t,u) \in J \times \mathbb{R}.$$

Then, the problem (1) has at least one solution on J, provided that:

$$\Lambda_2 < 1. \tag{21}$$

Proof. Introduce a closed ball $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$, with $r > \frac{\|\phi\|\Lambda_1}{1-\Lambda_2}$, $\|\phi\| = \sup_{t\in[a,T]} |\phi(t)|$, where Λ_2 is given by (18). Then, we define operators $\mathcal{F}_1$ and $\mathcal{F}_2$ from B_r to $\mathcal{C}$ by:

$$\begin{aligned}\mathcal{F}_1(x)(t) &= {}^\rho I_{a+}^{\alpha+\beta} f(t,x(t)) + \frac{(t^\rho - a^\rho)^\beta(\eta^\rho - t^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)\Omega}\Big\{{}^\rho I_{a+}^{\alpha+\beta} f(T,x(T)) - \mu\, {}^\rho I_{a+}^{\alpha+\beta+\gamma} f(\xi,x(\xi))\Big\} \\ &\quad - \frac{(t^\rho - a^\rho)^\beta}{\Omega(\eta^\rho - a^\rho)^\beta}\Big[\frac{(T^\rho - a^\rho)^\beta(T^\rho - t^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)} - \frac{\mu(\xi^\rho - a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho - t^\rho) - \gamma(t^\rho - a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big] \\ &\quad \times {}^\rho I_{a+}^{\alpha+\beta} f(\eta,x(\eta)), \\ \mathcal{F}_2(x)(t) &= -\lambda\, {}^\rho I_{a+}^{\beta} x(t) - \frac{(t^\rho - a^\rho)^\beta(\eta^\rho - t^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)\Omega}\Big\{\lambda\, {}^\rho I_{a+}^{\beta} x(T) - \mu\lambda\, {}^\rho I_{a+}^{\beta+\gamma} x(\xi)\Big\} + \frac{\lambda(t^\rho - a^\rho)^\beta}{\Omega(\eta^\rho - a^\rho)^\beta} \times \\ &\quad \times \Big[\frac{(T^\rho - a^\rho)^\beta(T^\rho - t^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)} - \frac{\mu(\xi^\rho - a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho - t^\rho) - \gamma(t^\rho - a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big]\, {}^\rho I_{a+}^{\beta} x(\eta).\end{aligned}$$

Note that $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ on B_r. For $x, y \in B_r$, we find that:

$$\begin{aligned}
& \|\mathcal{F}_1 x + \mathcal{F}_2 y\| \\
\leq\ & \sup_{t\in J}\Big\{{}^{\rho}I_{a+}^{\alpha+\beta}|f(t,x(t))| + |\lambda|^{\rho}I_{a+}^{\beta}|y(t)| \\
& + \frac{(t^\rho - a^\rho)^\beta|(\eta^\rho - t^\rho)|}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big\{{}^{\rho}I_{a+}^{\alpha+\beta}|f(T,x(T))| + |\lambda|^{\rho}I_{a+}^{\beta}|y(T)| \\
& + |\mu|^{\rho}I_{a+}^{\alpha+\beta+\gamma}|f(\xi,x(\xi))| + |\mu||\lambda|^{\rho}I_{a+}^{\beta+\gamma}|x(\xi)|\Big\} + \frac{(t^\rho - a^\rho)^\beta}{|\Omega|(\eta^\rho - a^\rho)^\beta}\Big|\frac{(T^\rho - a^\rho)^\beta(T^\rho - t^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)} \\
& - \frac{\mu(\xi^\rho - a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho - t^\rho) - \gamma(t^\rho - a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big|\Big\{{}^{\rho}I_{a+}^{\alpha+\beta}|f(\eta,x(\eta))| + |\lambda|^{\rho}I_{a+}^{\beta}|y(\eta)|\Big\} \\
\leq\ & \|\phi\|\Bigg\{\frac{(T^\rho - a^\rho)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Big[1 + \frac{\zeta_1}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big] + \frac{|\mu|(\xi^\rho - a^\rho)^{\alpha+\beta+\gamma}\zeta_1}{\rho^{\alpha+2\beta+\gamma+1}\Gamma(\alpha+\beta+\gamma+1)\Gamma(\beta+2)|\Omega|} \\
& + \frac{(\eta^\rho - a^\rho)^\alpha\zeta_2}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)|\Omega|}\Bigg\} + \|x\|\Big\{\frac{|\lambda|(T^\rho - a^\rho)^\beta}{\rho^\beta\Gamma(\beta+1)}\Big[1 + \frac{\zeta_1}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big] \\
& + \frac{|\mu||\lambda|(\xi^\rho - a^\rho)^{\beta+\gamma}\zeta_1}{\rho^{2\beta+\gamma+1}\Gamma(\beta+\gamma+1)\Gamma(\beta+2)|\Omega|} + \frac{|\lambda|\zeta_2}{\rho^\beta\Gamma(\beta+1)|\Omega|}\Bigg\} \\
\leq\ & \|\phi\|\Lambda_1 + r\Lambda_2 < r.
\end{aligned}$$

Thus, $\mathcal{F}_1 x + \mathcal{F}_2 y \in B_r$.
Next, it will be shown that $\mathcal{F}_2$ is a contraction. For that, let $x, y \in \mathcal{C}$. Then:

$$\begin{aligned}
& \|\mathcal{F}_2 x - \mathcal{F}_2 y\| \\
\leq\ & \sup_{t\in J}\Bigg\{|\lambda|^{\rho}I_{a+}^{\beta}|x(t) - y(t)| + \frac{|(t^\rho - a^\rho)^\beta(\eta^\rho - t^\rho)|}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\times \\
& \times\Big\{|\lambda|^{\rho}I_{a+}^{\beta}|x(T) - y(T)| + |\mu||\lambda|^{\rho}I_{a+}^{\beta+\gamma}|x(\xi) - y(\xi)|\Big\} \\
& + \frac{|\lambda||(t^\rho - a^\rho)^\beta|}{|\Omega|(\eta^\rho - a^\rho)^\beta}\Big|\frac{(T^\rho - a^\rho)^\beta(T^\rho - t^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)} - \frac{\mu(\xi^\rho - a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho - t^\rho) - \gamma(t^\rho - a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big|\times \\
& \times{}^{\rho}I_{a+}^{\beta}|x(\eta) - y(\eta)|\Bigg\} \\
\leq\ & \Big\{\frac{|\lambda|(T^\rho - a^\rho)^\beta}{\rho^\beta\Gamma(\beta+1)}\Big[1 + \frac{\zeta_1}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big] + \frac{|\mu||\lambda|(\xi^\rho - a^\rho)^{\beta+\gamma}\zeta_1}{\rho^{2\beta+\gamma+1}\Gamma(\beta+\gamma+1)\Gamma(\beta+2)|\Omega|} \\
& + \frac{|\lambda|\zeta_2}{\rho^\beta\Gamma(\beta+1)|\Omega|}\Big\}\|x - y\| \\
=\ & \Lambda_2\|x - y\|,
\end{aligned}$$

which, by the condition (21), implies that $\mathcal{F}_2$ is a contraction. The continuity of the operator $\mathcal{F}_1$ follows from that of f. Furthermore, $\mathcal{F}_1$ is uniformly bounded on B_r as:

$$\|\mathcal{F}_1 x\| \leq \|\phi\|\Lambda_1.$$

Finally, we establish the compactness of the operator $\mathcal{F}_1$. Let us set $\sup_{(t,x)\in J\times B_r}|f(t,x)| = \bar{f} < \infty$. Then, for $t_1, t_2 \in J$, $t_1 < t_2$, we have:

$$|(\mathcal{F}_1 x)(t_2) - (\mathcal{F}_1 x)(t_1)|$$

$$
\begin{aligned}
= & \Big|\frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)}\Big[\int_0^{t_1} s^{\rho-1}[(t_2^\rho-s^\rho)^{\alpha+\beta-1}-(t_1^\rho-s^\rho)^{\alpha+\beta-1}]f(s,x(s))ds \\
& +\int_{t_1}^{t_2} s^{\rho-1}(t_2^\rho-s^\rho)^{\alpha+\beta-1}f(s,x(s))ds\Big] \\
& +\Big[\frac{(t_2^\rho-a^\rho)^\beta(\eta^\rho-t_2^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)\Omega}-\frac{(t_1^\rho-a^\rho)^\beta(\eta^\rho-t_1^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)\Omega}\Big]\Big\{{}^\rho I_{a+}^{\alpha+\beta}f(T,x(T))-\mu^\rho I_{a+}^{\alpha+\beta+\gamma}f(\xi,x(\xi))\Big\} \\
& -\Big[\frac{(t_2^\rho-a^\rho)^\beta}{\Omega(\eta^\rho-a^\rho)^\beta}\Big[\frac{(T^\rho-a^\rho)^\beta(T^\rho-t_2^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)}-\frac{\mu(\xi^\rho-a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho-t_2^\rho)-\gamma(t^\rho-a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big] \\
& -\frac{(t_1^\rho-a^\rho)^\beta}{\Omega(\eta^\rho-a^\rho)^\beta}\Big[\frac{(T^\rho-a^\rho)^\beta(T^\rho-t_1^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)}-\frac{\mu(\xi^\rho-a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho-t_1^\rho)-\gamma(t_1^\rho-a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big]\Big]\times \\
& \times{}^\rho I_{a+}^{\alpha+\beta}f(\eta,x(\eta))\Big| \\
\leq & \frac{\bar{f}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Big\{|t_2^{\rho(\alpha+\beta)}-t_1^{\rho(\alpha+\beta)}|+2(t_2^\rho-t_1^\rho)^{\alpha+\beta}\Big\} \\
& +\Big|\frac{(t_2^\rho-a^\rho)^\beta(\eta^\rho-t_2^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)\Omega}-\frac{(t_1^\rho-a^\rho)^\beta(\eta^\rho-t_1^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)\Omega}\Big|\Big\{{}^\rho I_{a+}^{\alpha+\beta}|f(T,x(T))|+\mu^\rho I_{a+}^{\alpha+\beta+\gamma}|f(\xi,x(\xi))|\Big\} \\
& +\Big|\frac{(t_2^\rho-a^\rho)^\beta}{\Omega(\eta^\rho-a^\rho)^\beta}\Big[\frac{(T^\rho-a^\rho)^\beta(T^\rho-t_2^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)}-\frac{\mu(\xi^\rho-a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho-t_2^\rho)-\gamma(t_2^\rho-a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big] \\
& -\frac{(t_1^\rho-a^\rho)^\beta}{\Omega(\eta^\rho-a^\rho)^\beta}\Big[\frac{(T^\rho-a^\rho)^\beta(T^\rho-t_1^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)}-\frac{\mu(\xi^\rho-a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho-t_1^\rho)-\gamma(t_1^\rho-a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big]\Big|\times \\
& \times{}^\rho I_{a+}^{\alpha+\beta}|f(\eta,x(\eta))|,
\end{aligned}
$$

which tends to zero as $t_2 \to t_1$, independently of $x \in B_r$. Thus, $\mathcal{F}_1$ is equicontinuous. Therefore, $\mathcal{F}_1$ is relatively compact on B_r. As a consequence, we deduce by the the Arzelá–Ascoli theorem that $\mathcal{F}_1$ is compact on B_r. Thus, the hypothesis of Lemma 3 is satisfied. Therefore, the conclusion of Lemma 3 applies, and hence, there exists at least one solution for the problem (1) on J. □

In the next result, the uniqueness of solutions for the problem (1) is shown by means of the Banach contraction mapping principle.

Theorem 2. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition:*

(A_2)

$$|f(t,u)-f(t,v)| \leq L|u-v|, L>0, /, \quad \text{for } t \in J \text{ and every } u,v \in \mathbb{R}.$$

Then, there exists a unique solution for the problem (1) on $[a,T]$, provided that:

$$L\Lambda_1+\Lambda_2<1, \tag{22}$$

where Λ_1 and Λ_2 are respectively given by (17) and (18).

Proof. In the first step, we show that $\mathcal{F}B_{\bar{r}} \subset B_{\bar{r}}$, where $B_{\bar{r}}=\{x \in C([a,T],\mathbb{R}) : \|x\| \leq \bar{r}\}$, $M=\sup_{t\in[a,T]}|f(t,0)|$, $\bar{r} \geq \frac{\Lambda_1 M}{1-L\Lambda_1-\Lambda_2}$, and the operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ is given by (16). For $x \in B_{\bar{r}}$, using (A_2), we get:

$$
\begin{aligned}
& |\mathcal{F}(x)(t)| \\
\leq & {}^\rho I_{a+}^{\alpha+\beta}[|f(t,x(t))-f(t,0)|+|f(t,0)|]+|\lambda|^\rho I_{a+}^{\beta}|x(t)| \\
& +\frac{(t^\rho-a^\rho)^\beta|(\eta^\rho-t^\rho)|}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big\{{}^\rho I_{a+}^{\alpha+\beta}[|f(T,x(T))-f(T,0)|+|f(T,0)|]+|\lambda|^\rho I_{a+}^{\beta}|x(T)| \\
& +|\mu|^\rho I_{a+}^{\alpha+\beta+\gamma}[|f(\xi,x(\xi))-f(\xi,0)|+|f(\xi,0)|]+|\mu||\lambda|^\rho I_{a+}^{\beta+\gamma}|x(\xi)|\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(t^\rho-a^\rho)^\beta}{|\Omega|(\eta^\rho-a^\rho)^\beta}\Big|\frac{(T^\rho-a^\rho)^\beta(T^\rho-t^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)}-\frac{\mu(\xi^\rho-a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho-t^\rho)-\gamma(t^\rho-a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big|\\
&\times\Big\{{}^\rho I_{a+}^{\alpha+\beta}[|f(\eta,x(\eta))-f(\eta,0)|+|f(\eta,0)|]+|\lambda|{}^\rho I_{a+}^{\beta}|x(\eta)|\Big\}\\
\leq\;&(L\bar r+M)\Big(\frac{(T^\rho-a^\rho)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Big[1+\frac{\zeta_1}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big]+\frac{|\mu|(\xi^\rho-a^\rho)^{\alpha+\beta+\gamma}\zeta_1}{\rho^{\alpha+2\beta+\gamma+1}\Gamma(\alpha+\beta+\gamma+1)\Gamma(\beta+2)|\Omega|}\\
&+\frac{(\eta^\rho-a^\rho)^\alpha\zeta_2}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)|\Omega|}\Big)\\
&+\bar r\Big(\frac{|\lambda|(T^\rho-a^\rho)^\beta}{\rho^\beta\Gamma(\beta+1)}\Big[1+\frac{\zeta_1}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big]+\frac{|\mu||\lambda|(\xi^\rho-a^\rho)^{\beta+\gamma}\zeta_1}{\rho^{2\beta+\gamma+1}\Gamma(\beta+\gamma+1)\Gamma(\beta+2)|\Omega|}+\frac{|\lambda|\zeta_2}{\rho^\beta\Gamma(\beta+1)|\Omega|}\Big)\\
=\;&(L\bar r+M)\Lambda_1+\Lambda_2\bar r\leq\bar r,
\end{aligned}
$$

which, on taking the norm for $t\in[a,T]$, implies that $\|\mathcal{F}(x)\|\leq\bar r$. Thus, the operator $\mathcal{F}$ maps $B_{\bar r}$ into itself. Now, we proceed to prove that the operator $\mathcal{F}$ is a contraction. For $x,y\in C([a,T],\mathbb{R})$ and $t\in[a,T]$, we have:

$$
\begin{aligned}
&|\mathcal{F}(x)(t)-\mathcal{F}(y)(t)|\\
\leq\;&{}^\rho I_{a+}^{\alpha+\beta}|f(t,x(t))-f(t,y(t))|+|\lambda|{}^\rho I_{a+}^{\beta}|x(t)-y(t)|\\
&+\frac{(t^\rho-a^\rho)^\beta|(\eta^\rho-t^\rho)|}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big\{{}^\rho I_{a+}^{\alpha+\beta}|f(T,x(T))-f(T,y(T))|+|\lambda|{}^\rho I_{a+}^{\beta}|x(T)-y(T)|\\
&+|\mu|{}^\rho I_{a+}^{\alpha+\beta+\gamma}|f(\xi,x(\xi))-f(\xi,y(\xi))|+|\mu||\lambda|{}^\rho I_{a+}^{\beta+\gamma}|x(\xi)-y(\xi)|\Big\}\\
&+\frac{(t^\rho-a^\rho)^\beta}{|\Omega|(\eta^\rho-a^\rho)^\beta}\Big|\frac{(T^\rho-a^\rho)^\beta(T^\rho-t^\rho)}{\rho^{\beta+1}\Gamma(\beta+2)}-\frac{\mu(\xi^\rho-a^\rho)^{\beta+\gamma}[(\beta+1)(\xi^\rho-t^\rho)-\gamma(t^\rho-a^\rho)]}{\rho^{\beta+\gamma+1}\Gamma(\beta+\gamma+2)(\beta+1)}\Big|\times\\
&\times\Big\{{}^\rho I_{a+}^{\alpha+\beta}|f(\eta,x(\eta))-f(\eta,y(\eta))|+|\lambda|{}^\rho I_{a+}^{\beta}|x(\eta)-y(\eta)|\Big\}\\
\leq\;&L\|x-y\|\Big(\frac{(T^\rho-a^\rho)^{\alpha+\beta}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)}\Big[1+\frac{\zeta_1}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big]\\
&+\frac{|\mu|(\xi^\rho-a^\rho)^{\alpha+\beta+\gamma}\zeta_1}{\rho^{\alpha+2\beta+\gamma+1}\Gamma(\alpha+\beta+\gamma+1)\Gamma(\beta+2)|\Omega|}\\
&+\frac{(\eta^\rho-a^\rho)^\alpha\zeta_2}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)|\Omega|}\Big)+\|x-y\|\Big(\frac{|\lambda|(T^\rho-a^\rho)^\beta}{\rho^\beta\Gamma(\beta+1)}\Big[1+\frac{\zeta_1}{\rho^{\beta+1}\Gamma(\beta+2)|\Omega|}\Big]\\
&+\frac{|\mu||\lambda|(\xi^\rho-a^\rho)^{\beta+\gamma}\zeta_1}{\rho^{2\beta+\gamma+1}\Gamma(\beta+\gamma+1)\Gamma(\beta+2)|\Omega|}+\frac{|\lambda|\zeta_2}{\rho^\beta\Gamma(\beta+1)|\Omega|}\Big)\\
=\;&(L\Lambda_1+\Lambda_2)\|x-y\|.
\end{aligned}
$$

Taking the norm of the above inequality for $t\in[a,T]$, we get:

$$\|\mathcal{F}(x)-\mathcal{F}(y)\|\leq(L\Lambda_1+\Lambda_2)\|x-y\|,$$

which implies that the operator $\mathcal{F}$ is a contraction on account of the condition (22). Thus, we deduce by the Banach contraction mapping principle that the operator $\mathcal{F}$ has a unique fixed point. Hence, there exists a unique solution for the problem (1). The proof is complete. □

Example 1. *Let us consider the following boundary value problem:*

$$
\begin{cases}
{}_c^{1/3}D^{5/4}\Big({}_c^{1/3}D^{1/4}+1/5\Big)x(t)=\dfrac{1}{\sqrt{400+t}}\Big(\dfrac{|x(t)|+2}{|x(t)|+1}+e^{-t}\Big),\;\; t\in J:=[1,2],\\
x(1)=0,\quad x(3/2)=0,\quad x(2)=2/7\,{}^{1/3}I^{3/4}x(7/4).
\end{cases}
\tag{23}
$$

Here, $\rho = 1/3, \alpha = 5/4$, $\beta = 1/4$, $\gamma = 3/4$, $\lambda = 1/5$, $\mu = 2/7$, $a = 1$, $\eta = 3/2$, $\xi = 7/4$, $T = 2$ and $f(t,x) = \frac{1}{\sqrt{400+t}}\left(\frac{|x|+2}{|x|+1} + e^{-t}\right)$. Using the given data, we find that $|\Omega| \approx 0.293634, \Lambda_1 \approx 1.336009, \Lambda_2 \approx 0.673563, \zeta_1 \approx 0.082260$, $\zeta_2 \approx 0.232036$, where $\Omega, \Lambda_1, \Lambda_2, \zeta_1$, and ζ_2 are given by (10), (17), (18), (19) and (20) respectively.

For illustrating Theorem 1, we show that all the conditions of Theorem 1 are satisfied. Clearly, $f(t,x)$ is continuous and satisfies the condition (A_1) with $\phi(t) = \frac{2+e^{-t}}{\sqrt{400+t}}$. Furthermore, $\Lambda_2 \approx 0.673563 < 1$. Thus, all the conditions of Theorem 1 are satisfied, and consequently, the problem (23) has at least one solution on $[1,2]$.

Furthermore, Theorem 2 is applicable to the problem (23) with $L = 1/20$ as $L\Lambda_1 + \Lambda_2 \approx 0.740363 < 1$. Thus, all the assumptions of Theorem 2 are satisfied. Therefore, the conclusion of Theorem 2 applies to the problem (23) on $[1,2]$.

4. Conclusions

We have introduced a new type of nonlinear Langevin equation in terms of Liouville-Caputo-type generalized fractional differential operators of different orders and solved it with nonlocal generalized integral boundary conditions. The existence result was obtained by applying the Krasnoselskii fixed point theorem without requiring the nonlinear function to be of the Lipschitz type, while the uniqueness of solutions for the given problem was based on a celebrated fixed point theorem due to Banach. Here, we remark that many known existence results, obtained by means of the Krasnoselskii fixed point theorem, demand the associated nonlinear function to satisfy the Lipschitz condition. Moreover, by fixing the parameters involved in the given problem, we can obtain some new results as special cases of the ones presented in this paper. For example, letting $\rho = 1$, $\mu = 0$, $a = 0$ and $T = 1$ in the results of Section 3, we get the ones derived in [15].

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