Three Results on the Nonlinear Differential Equations and Differential-Difference Equations

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Abstract: We mainly study the transcendental entire solutions of the differential equation

$$f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where $p_1$, $p_2$, $\alpha_1$ and $\alpha_2$ are nonzero constants satisfying $\alpha_1 \neq \alpha_2$ and $P(f)$ is a differential polynomial in $f$ of degree $n - 1$. We improve Chen and Gao’s results and partially answer a question proposed by Li (J. Math. Anal. Appl. 375 (2011), pp. 310–319).

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1. Introduction and Main Results

In the past several decades, a great deal of mathematical effort in complex analysis has been devoted to studying differential equations, differential-difference equations and difference equations. The essential reason is penetration and application of Nevanlinna theory for the difference operator, see [1–4]. In this study, we assume readers are familiar with the standard notations and fundamental results used in the theory such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the counting function $N(r, f)$, see [5–8]. Moreover, we use the notations $\rho(f)$ and $\rho_2(f)$ to denote the order and the hyper-order of $f$, respectively.

Many scholars recently have had tremendous interest in developing solvability and existence of solutions of non-linear differential equations and differential-difference equations in the complex plane, see [9–15].

In 2011, Li [16] considered to find all entire solutions of the following nonlinear differential equation

$$f^n(z) + P(f) = p_1 e^{\lambda z} + p_2 e^{-\lambda z}$$

and obtained the following result.

**Theorem 1.** (see [16]) Let $n \geq 2$ be an integer, $P(f)$ be a differential polynomial in $f$ of degree at most $n - 1$ and $\lambda$, $p_1$, $p_2$ be three nonzero constants. If $f$ is a meromorphic function of Equation (1) satisfying $N(r, f) = S(r, f)$, then there exist two nonzero constants $c_1$, $c_2$ ($c_1^{\#} = p_1$) and a small function $c_0$ of $f$ such that

$$f = c_0 + c_1 e^{\frac{\lambda z}{n}} + c_2 e^{-\frac{\lambda z}{n}}.$$

Li [16] also investigated $p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$ for two distinct constants $\alpha_1$ and $\alpha_2$ instead of $p_1 e^{\lambda z} + p_2 e^{-\lambda z}$ in the right side of Equation (1) and obtained the following results.
Theorem 2. (see [16]) Let \( n \geq 2 \) be an integer, \( P(f) \) be a differential polynomial in \( f(z) \) of degree at most \( n - 2 \) and \( \alpha_1, \alpha_2, p_1, p_2 \) be nonzero constants satisfying \( \alpha_1 \neq \alpha_2 \). If \( f(z) \) is a transcendental meromorphic solution of the following equation

\[
f''(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}
\]  

(2)

satisfying \( N(r, f) = S(r, f) \), then one of the following relations holds:

1. \( f(z) = c_0(z) + c_1 e^{\frac{\alpha_1}{2} z} \);
2. \( f(z) = c_0(z) + c_2 e^{\frac{\alpha_2}{2} z} \);
3. \( f(z) = c_1 e^{\frac{\alpha_1}{2} z} + c_2 e^{\frac{\alpha_2}{2} z} \) and \( \alpha_1 + \alpha_2 = 0 \),

where \( c_0(z) \) is a small function of \( f \) and constants \( c_1 \) and \( c_2 \) satisfy \( c_1^0 = p_1 \) and \( c_2^0 = p_2 \), respectively.

For further study, Li proposed a related question:

Question 1. How to find the solutions of Equation (2) if \( \deg P(f) = n - 1 \)?

The question was studied by Chen and Gao [17]. They partially answered it and obtained the following result.

Theorem 3. (see [17]) Let \( a(z) \) be a nonzero polynomial and \( p_1, p_2, \alpha_1, \alpha_2 \) be nonzero constants such that \( \alpha_1 \neq \alpha_2 \). If \( f(z) \) is a transcendental entire solution of finite order of the differential equation

\[
f^2(z) + a(z)f'(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}
\]  

(3)

satisfying \( N(r, \frac{1}{f}) = S(r, f) \), then \( a(z) \) must be a constant and one of the following relations holds:

1. \( f(z) = c_1 e^{\frac{\alpha_1}{2} z}, ac_1 \alpha_1 = 2p_2 \) and \( \alpha_1 = 2\alpha_2 \);
2. \( f(z) = c_2 e^{\frac{\alpha_2}{2} z}, ac_2 \alpha_2 = 2p_1 \) and \( \alpha_2 = 2\alpha_1 \),

where \( c_1 \) and \( c_2 \) are constants satisfying \( c_1^2 = p_1 \) and \( c_2^2 = p_2 \), respectively.

Now, we remove the condition that \( f(z) \) is a finite-order function, improve Theorem 3 and obtain the following result.

Theorem 4. Let \( a(z) \) be a nonzero polynomial and \( p_1, p_2, \alpha_1, \alpha_2 \) be nonzero constants such that \( \alpha_1 \neq \alpha_2 \). Suppose that \( f(z) \) is a transcendental entire solution of the differential Equation (3) satisfying \( N(r, \frac{1}{f}) = S(r, f) \). Then \( a(z) \) must be a constant and one of the following relations holds:

1. \( f(z) = c_1 e^{\frac{\alpha_1}{2} z}, ac_1 \alpha_1 = 2p_2 \) and \( \alpha_1 = 2\alpha_2 \);
2. \( f(z) = c_2 e^{\frac{\alpha_2}{2} z}, ac_2 \alpha_2 = 2p_1 \) and \( \alpha_2 = 2\alpha_1 \),

where \( c_1 \) and \( c_2 \) are constants satisfying \( c_1^2 = p_1 \) and \( c_2^2 = p_2 \), respectively.

Next we consider the general case in Question 1 and obtain the following theorem.

Theorem 5. Let \( n \geq 2 \) be an integer. Suppose that \( P(f) \) is a differential polynomial in \( f(z) \) of degree \( n - 1 \) and that \( \alpha_1, \alpha_2, p_1 \) and \( p_2 \) are nonzero constants such that \( \alpha_1 \neq \alpha_2 \). If \( f(z) \) is a transcendental meromorphic solution of the differential Equation (2) satisfying \( N(r, f) = S(r, f) \), then \( \rho(f) = 1 \) and one of the following relations holds:

1. \( f(z) = c_1 e^{\frac{\alpha_1}{2} z} \) and \( c_1^0 = p_1 \);
2. \( f(z) = c_2 e^{\frac{\alpha_2}{2} z} \) and \( c_2^0 = p_2 \), where \( c_1 \) and \( c_2 \) are constants;
We can easily verify the inequality $T(r, f) \leq N_1(r, \frac{1}{f}) + T(r, \varphi) + S(r, f)$, where $N_1(r, \frac{1}{f})$ denotes the counting function corresponding to simple zeros of $f$ and $\varphi(\neq 0)$ is equal to $\alpha_1 + \alpha_2 f^2 - n(a_1 + a_2)ff' + n(n - 1)(f')^2 + nf''$.

Three examples are shown to illustrate the cases (1)–(3) of Theorem 5.

**Example 1.** Let $f(z) = e^z$ be an entire solution of the differential equation

$$f^2(z) + f'(z) = e^{2z} + e^z,$$

where $c_1 = 1$ and $p_1 = 1$. It implies the case (1) occurs.

**Example 2.** Let $f(z) = 2e^{2z}$ be an entire solution of the differential equation

$$f^2(z) + \frac{1}{2}f''(z) = e^{2z} + 4e^{4z},$$

where $c_2 = 2$ and $p_2 = 4$. It implies case (2) occurs.

**Example 3.** Let $f(z) = e^z - 1$ be an entire solution of the differential equation

$$f^2(z) + (f' - 1) = e^{2z} - e^z.$$

We can easily verify the inequality $T(r, f) \leq N_1(r, \frac{1}{f}) + T(r, \varphi) + S(r, f)$, where $\varphi = 2f^2 - 6ff' + 2(f')^2 + 2f'' = 2$. It implies case (3) occurs.

**Remark 1.** From Theorem 4 and Example 3, we conjecture that case (3) in Theorem 5 can be removed if $N(r, 1/f) = S(r, f)$.

In [18], Wang and Li investigated the following differential-difference equation

$$f^n(z) + q(z)f^{(k)}(z + c) = ae^{ibz} + de^{-ibz}$$

and obtained the existence of entire solutions when $n \geq 3$.

In 2018, Chen and Gao went far to study Equation (4) with $n = 2$. They obtained the following theorem.

**Theorem 6.** (see [17]) Let $a(z)$ be a nonzero polynomial, $k \geq 0$ be an integer and $p_1$, $p_2$, $\lambda$, $c$ be nonzero constants. If $f(z)$ is a transcendental entire solution of finite order of the differential-difference equation

$$f^2(z) + a(z)f^{(k)}(z + c) = p_1 e^{\lambda z} + p_2 e^{-\lambda z},$$

then $a(z)$ must be a constant and one of the following relations holds:

1. $f(z) = \pm \frac{1}{2}a\left(\frac{1}{2}\right)^k + c_1 e^{\lambda z} + c_2 e^{-\lambda z}$ and $e^{\alpha c} = -1$, when $k$ is odd;
2. $f(z) = \pm \frac{1}{2}a\left(\frac{1}{2}\right)^k + c_1 e^{\lambda z} + c_2 e^{-\lambda z}$ and $e^{\alpha c} = 1$, when $k$ is even and $k > 0$, where $a$, $c_1$ and $c_2$ are constants with $\frac{1}{2}\alpha^4 d^4(\lambda z)^k = p_1 p_2$ and $c_1^2 = p_1 (i = 1, 2)$;
3. $f(z) = \pm \frac{1}{2}a + c_1 e^{\lambda z} + c_2 e^{-\lambda z}$ and $e^{\alpha c} = 1$, when $k = 0$, where $a$, $c_1$ and $c_2$ are constants with $\frac{1}{2}\alpha^4 d^4 = p_1 p_2$ or $\frac{2}{2}\alpha^2 d^4 = p_1 p_2$ and $c_1^2 = p_1 (i = 1, 2)$.

For the right side of Equations (4) and (5), a question to be raised is how to find the existence of solutions if $e^{\lambda z}$ and $e^{-\lambda z}$ can be replaced by a linear combination of $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ for two distinct constants $\alpha_1$ and $\alpha_2$. We consider the question and obtain the following result.
**Theorem 7.** Let $\alpha_1, \alpha_2, p_1, p_2$ and $h$ be nonzero constants satisfying $\alpha_1 \neq \alpha_2$. Suppose that $k \geq 0$ and $n \geq 2$ are integers and that $q(z)$ is a nonzero polynomial. If $f(z)$ is a transcendental entire solution with $p_2(f) < 1$ of the differential-difference equation

$$f''(z) + q(z)f^{(k)}(z + h) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

then we have $\rho(f) = 1$, $q(z)$ must be a constant and one of the following relations holds:

1. $f(z) = c_1 e^{\alpha_1 z}, \quad q c_1 \left(\frac{\alpha_1}{n}\right)^k e^{\alpha_1 h} = p_2, \quad \alpha_1 = n \alpha_2$ and $c_1 = p_1$;
2. $f(z) = c_2 e^{\alpha_2 z}, \quad q c_2 \left(\frac{\alpha_2}{n}\right)^k e^{\alpha_2 h} = p_1, \quad \alpha_2 = n \alpha_1$ and $c_2^2 = p_2$;
3. If $n = 2$, we have $T(r, f) \leq N_{11}(r, 1/f) + T(r, q) + S(r, f)$, where $N_{11}(r, 1/f)$ and $q$ are the same as defined in Theorem 5. If $n = 3$, we have $T(r, f) = N_{11}(r, 1/f) + S(r, f)$. If $n \geq 4$, we only have the cases (1) and (2).

Next we give three examples to show existence of solutions of Equation (6).

**Example 4.** Let $f(z) = e^z$. Then $f$ is a transcendental entire solution of the following differential-difference equation

$$f^3(z) + f'(z + 2\pi i) = e^{2z} + e^z,$$

where $\alpha_1 = 3 = 3\alpha_2$, $c_1 = 1$, $q = 1$ and $p_1 = p_2 = 1$. Thus, case (1) occurs.

**Example 5.** Let $f(z) = \sqrt{2} e^z$. Then $f$ is a transcendental entire solution of the following differential-difference equation

$$f^2(z) + \sqrt{2} f^{(3)}(z + 2\pi i) = 2e^{2z} + 2e^{2z},$$

where $\alpha_2 = 2 = 2 \alpha_1$, $c_2 = \sqrt{2}$, $q = \sqrt{2}$ and $p_1 = p_2 = 2$. Thus, case (2) occurs.

**Example 6.** Let $f(z) = e^z - 1$. Then $f$ is a transcendental entire solution of the following equation

$$f^2(z) + f(z + \pi i) = e^{2z} - 3e^z.$$

A routine computation yields $T(r, f) \leq N_{11}(r, \frac{1}{f}) + T(r, q) + S(r, f)$, where $q = 2f^2 - 6ff' + 2(f')^2 + 2ff'' = 2$. Thus, case (3) occurs.

**Example 7.** Let $f(z) = e^z + e^{-z}$. Then $f$ is a transcendental entire solution of the following differential-difference equation

$$f^3(z) + f''(z + \pi i) = e^{3z} + e^{-3z}.$$

A routine computation yields $T(r, f) = N_{11}(r, \frac{1}{f}) + S(r, f)$.

**Remark 2.** From Examples 6 and 7, we conjecture that case (3) in Theorem 7 can be removed if $N(r, 1/f) = S(r, f)$ for $n = 2, 3$.

**Remark 3.** In Theorem 3, our result holds for $\alpha_1 \neq \alpha_2$. However, if $\alpha_1 + \alpha_2 = 0$, we just know the solutions satisfy case (3) for $n = 2, 3$. The expression of solutions can be obtained when $n = 2$ in Theorem 6.

2. Some Lemmas

In this section, we introduce several lemmas to prove three theorems.
Lemma 1. (see [5]) Let $f(z)$ be an entire function and $k$ be a positive integer. Then

$$m\left(r, \frac{f^k(z)}{f(z)}\right) = S(r, f).$$

Lemma 2. (see [3]) Let $c \in \mathbb{C}\setminus\{0\}$, $\epsilon > 0$ and $f(z)$ be a meromorphic function of $\rho_2(f) < 1$. Then

$$m\left(r, \frac{f(z + c)}{f(z)}\right) = o\left(\frac{T(r, f)}{1 - \rho_2(f)^{-\epsilon}}\right)$$

outside of an exceptional set of finite logarithmic measures.

Lemma 3. (see [8]) Suppose that $f_1(z), f_2(z), \ldots, f_n(z) (n \geq 2)$ are meromorphic functions and that $g_1(z), g_2(z), \ldots, g_n(z) (n \geq 2)$ are entire functions satisfying the following conditions:

1. $f_1(z)e^{g_1(z)} + f_2(z)e^{g_2(z)} + \cdots + f_n(z)e^{g_n(z)} \equiv 0$;
2. $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;
3. For $1 \leq j \leq n$ and $1 \leq h < k \leq n$, $T(r, f_j(z)) = o(T(r, e^{g_h(z) - g_k(z)}))(r \to \infty, r \notin E)$, where $E \subset [1, \infty)$ is a finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0$ ($j = 1, 2, \ldots, n$).

Applying Lemmas 1 and 2 to Theorem 2.3 of [19], we get the following lemma, which is a version of the difference analogue of the Clunie lemma.

Lemma 4. Let $f$ be a transcendental meromorphic solution of $\rho_2(f) < 1$ of a difference equation of the form

$$H(z, f)P(z, f) = Q(z, f),$$

where $H(z, f)$, $P(z, f)$, $Q(z, f)$ are difference polynomials in $f$ such that the total degree of $H(z, f)$ in $f$ and its shifts is $n$, and that the corresponding total degree of $Q(z, f)$ is $\leq n$. If $H(z, f)$ contains just one term of maximal total degree, then for any $\epsilon > 0$

$$m(r, P(z, f)) = S(r, f)$$

possibly outside of an exceptional set of finite logarithmic measure.

3. Proof of Theorem 4

Proof. Denote $P_1(f) := a(z)f'(z)$. Suppose $f(z)$ be a transcendental entire solution of Equation (3). Differentiating Equation (3), we obtain

$$2ff'' + P_1' = a_1p_1e^{a_1z} + a_2p_2e^{a_2z}.$$  \hfill (7)

Eliminating $e^{a_2z}$ from Equations (3) and (7) gives

$$a_2f^2 - 2ff' + a_2P_1 - P_1' = (a_2 - a_1)p_1e^{a_1z}.$$  \hfill (8)

Differentiating Equation (8) yields

$$2a_2ff' - 2(f')^2 - 2ff'' + a_2P_1' - P_1'' = a_1(a_2 - a_1)p_1e^{a_1z}.$$  \hfill (9)

It follows from Equations (8) and (9) that

$$\varphi = Q.$$
where
\[
\varphi = a_1a_2f^2 - 2(a_1 + a_2)ff' + 2(f')^2 + 2ff''
\]
and
\[
Q = -a_1a_2P_1 + (a_1 + a_2)P_1' - P_1''.
\]

Here we distinguish two cases below.

**Case 1.** \( \varphi \neq 0 \).

Similar to the proof of Theorem 3 [17], we can obtain a contradiction.

**Case 2.** \( \varphi \equiv 0 \).

By taking \( n = 2 \), we use the method of Case 1 of Theorem 5 to obtain \( t_1 = \frac{a_1}{2} \) and \( t_2 = \frac{a_2}{2} \), where \( t_i = \frac{f}{r} \) (\( i = 1, 2 \)).

Now if \( t_1 = \frac{a_1}{2} \), then \( f(z) = c_1e^{\frac{a_1}{2}z} \), where \( c_1 \) is a constant satisfying \( c_1^2 = P_1 \). Substituting these formulas into Equation (3), we have \( a(z)c_1a_1 = 2P_2 \) and \( a_1 = 2a_2 \), where \( a(z) \) must be a constant. Let \( a := a(z) \).

Similarly, if \( t_2 = \frac{a_2}{2} \), then we have \( f(z) = c_2e^{\frac{a_2}{2}z} \), \( a_2a_2 = 2P_1 \) and \( a_2 = 2a_1 \), where \( c_2 \) is a constant satisfying \( c_2^2 = P_2 \).

### 4. Proof of Theorem 5

**Proof.** Assume that \( f(z) \) is a transcendental meromorphic solution of Equation (2) with \( N(r,f) = S(r,f) \).

A differential polynomial \( P(f) \) with \( \deg P(f) = n - 1 \) can be written in the following form
\[
P(f) = \sum_{i=1}^{n-1} a_iM_i(f) = a_1M_1(f) + a_2M_2(f) + \cdots + a_{n-1}M_{n-1}(f),
\]
where \( a_i \) are the small functions of \( f \) and \( M_i(f) = f^{n_0}(f')^{n_1} \cdots (f^{(k)})^{n_k} \) are the differential monomials such that \( \deg M_i(f) = n_0 + n_1 + \cdots + n_k = i \leq n - 1 \).

We can represent \( P(f) \) as
\[
P(f) = \frac{a_1M_1(f)}{f}f + \frac{a_2M_2(f)}{f^2}f^2 + \cdots + \frac{a_{n-1}M_{n-1}(f)}{f^{n-1}}f^{n-1}.
\]

By Lemma 1, we derive
\[
m\left( r, \frac{a_iM_i(f)}{f^i} \right) = m\left( r, \frac{a_if^{n_0}(f')^{n_1} \cdots (f^{(k)})^{n_k}}{f^i} \right) = S(r,f)
\]
for \( 1 \leq i \leq n - 1 \). Furthermore, we have
\[
m(r, P(f)) \leq (n - 1)m(r, f) + S(r, f).
\]

Since \( N(r, f) = S(r, f) \)
\[
T(r, P(f)) \leq (n - 1)T(r, f) + S(r, f)
\]
holds.

By Equation (10), we obtain
\[ T(r, p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}) = T(r, f^n(z) + P(f)) \]
\[ \leq T(r, f^n(z)) + T(r, P(f)) + O(1) \]
\[ \leq nT(r, f) + (n - 1)T(r, f) + S(r, f) \]
\[ = (2n - 1)T(r, f) + S(r, f) \] (11)

and

\[ T(r, p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}) = T(r, f^n(z) + P(f)) \]
\[ \geq T(r, f^n(z)) - T(r, P(f)) + O(1) \]
\[ \geq nT(r, f) - (n - 1)T(r, f) + S(r, f) \]
\[ = T(r, f) + S(r, f). \] (12)

It follows from Equations (11) and (12) that

\[ T(r, f) + S(r, f) \leq T(r, p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}) \leq (2n - 1)T(r, f) + S(r, f), \]

which implies \( \rho(f) = 1 \).

We next turn to proving conclusions (1)–(3).

Differentiating Equation (2), we have

\[ nf^{n-1}f' + P' = \alpha_1p_1e^{\alpha_1 z} + \alpha_2p_2e^{\alpha_2 z}. \] (13)

Eliminating \( e^{\alpha_2 z} \) from Equations (2) and (13) gives

\[ \alpha_2f^n - nf^{n-1}f' + \alpha_2P - P' = (\alpha_2 - \alpha_1)p_1e^{\alpha_1 z}. \] (14)

Differentiating Equation (14) yields

\[ n\alpha_2f^{n-1}f' - n(n - 1)f^{n-2}(f')^2 - nf^{n-1}f'' + \alpha_2P' - P'' = \alpha_1(\alpha_2 - \alpha_1)p_1e^{\alpha_1 z}. \] (15)

By Equations (14) and (15), we have

\[ f^{n-2}\varphi = Q, \]

where

\[ \varphi = \alpha_1\alpha_2f^2 - n(\alpha_1 + \alpha_2)ff' + n(n - 1)(f')^2 + nf'' \] (16)

and

\[ Q = -\alpha_1\alpha_2P + (\alpha_1 + \alpha_2)P' - P''. \]

We still consider two cases below.

**Case 1.** \( \varphi \equiv 0 \).

Dividing with \( f^2 \) on both sides in Equation (16) and recalling \( \frac{f''}{f} = \left( \frac{f}{f'} \right)' + \left( \frac{f}{f'} \right)^2 \), we get a Riccati equation

\[ t' + nt^2 - (\alpha_1 + \alpha_2)t + \frac{\alpha_1\alpha_2}{n} = 0 \]

where \( t = \frac{f}{f'} \). A routine computation yields two constant solutions \( t_1 = \frac{\alpha_1}{n} \) and \( t_2 = \frac{\alpha_2}{n} \).

Given that \( t \neq t_1 \) and \( t \neq t_2 \) hold, we have

\[ \frac{1}{t_1 - t_2} \left( \frac{t'}{t - t_1} - \frac{t'}{t - t_2} \right) = -n. \]

Integrating it on both sides gives

\[ \ln \frac{t - t_1}{t - t_2} = n(t_2 - t_1)z + C, \quad C \in \mathbb{C}, \]
which is equivalent to
\[ \frac{t - t_1}{t - t_2} = e^{n(t_2 - t_1)z + C}. \]

It immediately yields
\[ t = t_2 + \frac{t_2 - t_1}{e^{n(t_2 - t_1)z + C} - 1} = \frac{f'}{f}. \]

Note that zeros of \( e^{n(t_2 - t_1)z + C} - 1 \) are the zeros of \( f \). If \( z_0 \) is a zero of \( f \) with multiplicity \( k \), then
\[ k = \text{Res} \left[ \frac{f'}{f}, z_0 \right] = \text{Res} \left[ t_2 + \frac{t_2 - t_1}{e^{n(t_2 - t_1)z + C} - 1}, z_0 \right] = \frac{1}{n} \]
is a contradiction.
If \( t_1 = \frac{\alpha_1}{2} \), then \( f(z) = c_1 e^{\frac{\alpha_1 z}{2}} \), where \( c_1 \) is a constant satisfying \( c_1^2 = p_1 \).
Similarly, if \( t_2 = \frac{\alpha_2}{2} \), then we have \( f(z) = c_2 e^{\frac{\alpha_2 z}{2}} \), where \( c_2 \) is a constant satisfying \( c_2^2 = p_2 \).

**Case 2.** \( \varphi \not\equiv 0 \).

Equation (16) can be written as
\[ \frac{1}{f^2} = \frac{1}{\varphi} \left[ \alpha_1 \alpha_2 - n(\alpha_1 + \alpha_2) \left( \frac{f'}{f} \right) + n(n - 1) \left( \frac{f'}{f} \right)^2 + n \left( \frac{f''}{f} \right) \right]. \]

Using Lemma 1, we have
\[ 2m \left( r, \frac{1}{f} \right) = m \left( r, \frac{1}{\varphi} \right) \leq m \left( r, \frac{1}{f} \right) + S(r, f). \tag{17} \]

From Equation (16), if \( z_0 \) is a multiple zero of \( f \), then \( z_0 \) must be a zero of \( \varphi \). Thus, it follows that
\[ N_{f_2} \left( r, \frac{1}{f} \right) \leq N \left( r, \frac{1}{\varphi} \right) + S(r, f), \tag{18} \]
where \( N_{f_2} \left( r, \frac{1}{f} \right) \) denotes the counting function of multiple zeros of \( f \). Equations (17) and (18) and the first fundamental theorem give
\[ T(r, f) \leq N_{f_1} \left( r, \frac{1}{f} \right) + T(r, \varphi) + S(r, f). \tag{19} \]

\[ \square \]

**5. Proof of Theorem 7**

**Proof.** Assume that \( f(z) \) is a transcendental entire solution with \( \rho_2(f) < 1 \) of Equation (6). Applying Lemmas 1 and 2 to Equation (6), we have
\begin{align*}
T(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}) &= T(r, f^n(z) + q(z) f^k(z + h)) \\
&\leq T(r, f^n) + T(r, q(z) f^k(z + h)) + O(1) \\
&\leq T(r, f^n) + m \left( r, \frac{f(z + h)}{f(z)} \right) + m(r, f) + O(1) \\
&\leq T(r, f^n) + m \left( r, q(z) \frac{f(z + h)}{f(z)} \right) + m \left( r, \frac{f(z + h)}{f(z)} \right) + m(r, f) + O(1) \\
&\leq (n + 1) T(r, f) + S(r, f). \tag{20}
\end{align*}
On the other hand, we deduce
\[
T(r, p_1 e^{a_1 z} + p_2 e^{a_2 z}) = T(r, f^n(z) + q(z) f^{(k)}(z + h)) 
\]
\[\geq T(r, f^n) - T(r, q(z) f^{(k)}(z + h)) + O(1)\]
\[\geq nT(r, f) - m \left( r, \frac{q(z) f^{(k)}(z + h)}{f(z)} \right) - m(r, f) + O(1)\]
\[\geq nT(r, f) - m \left( r, \frac{q(z) f^{(k)}(z + h)}{f(z)} \right) - m(r, f) + O(1)\]
\[\geq nT(r, f) - T(r, f) + S(r, f)\]
\[= (n - 1)T(r, f) + S(r, f) .\]

Combining Equations (20) and (21), it follows that
\[ (n - 1)T(r, f) + S(r, f) \leq T(r, p_1 e^{a_1 z} + p_2 e^{a_2 z}) \leq (n + 1)T(r, f) + S(r, f), \]
which implies \( \rho(f) = 1 \).

Denoting \( P_2(f) := q(z) f^{(k)}(z + h) \) and differentiating Equation (6), we have
\[ nf^{n-1} f' + P_2' = a_1 p_1 e^{a_1 z} + a_2 p_2 e^{a_2 z}. \]  
(22)

Eliminating \( e^{a_2 z} \) from Equations (6) and (22) gives
\[ a_2 f^n - n f^{n-1} f' + a_2 P_2 - P_2' = (a_2 - a_1) p_1 e^{a_1 z}. \]
(23)

Differentiating Equation (23) yields
\[ na_2 f^{n-1} f' - n(n - 1) f^{n-2} (f')^2 - n f^{n-1} f'' + a_2 P_2' - P_2'' = a_1 (a_2 - a_1) p_1 e^{a_1 z}. \]
(24)

It follows from Equations (23) and (24) that
\[ f^{n-2} \varphi = Q , \]
where
\[ \varphi = a_1 a_2 f^2 - n(a_1 + a_2) f f' + n(n - 1) (f')^2 + n f f'' \]
and
\[ Q = -a_1 a_2 P_2 + (a_1 + a_2) P_2' - P_2''. \]

Next we discuss two cases below.

**Case 1.** \( \varphi \equiv 0 \).

This case can be completed by the same method as employed in Case 1 of Theorem 5. We obtain \( f(z) = c_2 e^{\frac{a_2 z}{1}} \), where \( c_2 \) is a constant satisfying \( c_2^n = p_2 \). Substituting these formulas into Equation (6), we have
\[ q(z) c_2 \left( \frac{a_2 z}{n} \right)^k e^{\frac{a_2 k}{n}} e^{\frac{a_2 z}{n}} - p_1 e^{a_1 z} = 0. \]

According to \( a_1 \neq a_2 \) and Lemma 3, we have
\[ a_2 = n a_1 \text{ and } q(z) c_2 \left( \frac{a_2 z}{n} \right)^k e^{\frac{a_2 k}{n}} = p_1. \]
which implies that \( q(z) \) is a constant. Set \( q := q(z) \).

Similarly, we proceed to obtain \( f(z) = c_1 e^{\frac{a_1 z}{1}} \), \( q c_1 \left( \frac{a_1 z}{n} \right)^k e^{\frac{a_1 k}{n}} = p_2 \). Substituting these formulas into Equation (6), we have
\[ a_1 = n a_2 \text{ and } c_1^n = p_1. \]

**Case 2.** \( \varphi \neq 0 \).
For $n \geq 4$, we shall derive a contradiction. In fact, $Q$ is a difference-differential polynomial in $f$ and its degree at most is 1. By Equation (25) and Lemma 4, we have $m(r, \varphi) = S(r, f)$ and $T(r, \varphi) = S(r, f)$. On the other hand, we can rewrite Equation (25) as $f^{n-3}(f\varphi) = Q$, which implies $m(r, f\varphi) = S(r, f)$ and $T(r, f\varphi) = S(r, f)$. If $\varphi \neq 0$, then $T(r, f) = T(r, \frac{Le}{\varphi}) = S(r, f)$ and this is impossible.

For $n = 3$, since $Q$ is a difference-differential polynomial in $f$ and its degree at most is 1, it follows from Equation (25) and Lemma 4 that $m(r, \varphi) = S(r, f)$ and

$$T(r, \varphi) = S(r, f).$$  (26)

We still use the same method in Case 2 of Theorem 5 to obtain the inequality of Equation (19). Equations (19) and (26) and the first fundamental theorem result in

$$T(r, f) = N_{1}(r, \frac{1}{f}) + S(r, f).$$

For $n = 2$, we just obtain the inequality of Equation (19). $\square$

6. Conclusions

In this study, we consider two questions. Firstly, the first question posed by Li in [16] is how to find the solutions of Equation (2) if $\deg P(f) = n - 1$. Since the degree of $P(f)$ is bigger than $n - 2$, one cannot use Clunie’s lemma which is a key in the proof in Theorem 2. It is very difficult to resolve the question. Chen and Gao considered the entire solution $f$ of Equation (2) with the order $\rho(f) < \infty$ and $N(r, 1/f) = S(r, f)$ when $n = 2$ and partially answered the question. We remove the condition that the order $\rho(f) < \infty$ by a different method and improve the result of Chen and Gao in Theorem 4. For the general case of Li’s question, we use the method of Theorem 4 and give a partial answer in Theorem 5.

Secondly, motivated by Theorem 2, a question to be raised is how to find the existence of solutions to Equation (5) if $e^{\lambda z}$ and $e^{-\lambda z}$ can be replaced by a linear combination of $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ for two distinct constants $\alpha_1$ and $\alpha_2$. We consider the general case by the similar method with Theorem 5 and give the partial solutions of Equation (6).

For further study, we conjecture that the inequality $T(r, f) \leq N_{1}(r, \frac{1}{f}) + T(r, \varphi) + S(r, f)$ or $T(r, f) = N_{1}(r, \frac{1}{f}) + S(r, f)$ can be removed if $N(r, 1/f) = S(r, f)$ in Theorems 5 and 7.

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