New Hybrid Contractions on b -Metric Spaces

Erdal Karapınar ^{1,*} and Andreea Fulga ² ¹ Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan² Department of Mathematics and Computer Sciences, Universitatea Transilvania Brasov, 500036 Brasov, Romania

* Correspondence: karapinar@mail.cmuh.org.tw

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Abstract: In this manuscript, we introduce the notion of b -hybrid contraction in the setting of b -metric space. We investigate the existence and uniqueness of a fixed point for this contraction. Our results combine and merge several existing results in the corresponding literature and we list some of them as corollaries. Finally, we consider an Ulam's type stability for an application.

Keywords: hybrid contraction; b -metric space; fixed point; metric space

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1. Introduction and Preliminaries

The notion of b -metric, also known as quasi-metric [1], is one of the genuine extensions of the concept of metric that was observed by changing the triangle inequality by a weighted-triangle inequality (see (b3) below). On account of the fact that b -metric is a real generalization of the standard metric, we consider a notion of b -hybrid contraction in the setting of b -metric space. Recently, hybrid contraction has attracted attention from researchers since such contractions not only improve the nonlinear functional analysis, in particular, metric fixed point theory, but also, organize the untidiness of the metric fixed point theory.

We recall here the notion of b -metric. Let $\mathcal{M}$ be a nonempty set. A function $b : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ is called a b -metric if there is a constant $s \geq 1$ and the following conditions holds true for all $x, y, z \in \mathcal{M}$:

- (b1) $b(x, y) = 0$ if and only if $x = y$ (self-distance).
- (b2) $b(x, y) = b(y, x)$ (symmetry)
- (b3) $b(x, y) \leq s[b(x, z) + b(z, y)]$ (weighted-triangle inequality).

In addition, the tripled $(\mathcal{M}, b, s)$ is called a b -metric space. We mention that for $s = 1$, the b -metric becomes a usual metric and also, in general, (b) -metrics are not necessarily continuous.

An non-decreasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ is named as

- (c) a comparison function [1,2] if $\phi^n(t) \rightarrow 0$ as $n \rightarrow \infty$ for every $t \in [0, \infty)$;
- (b_c) a (b) -comparison function if there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leq as^k\phi^k(t) + v_k$, for $k \geq k_0$ and any $t \geq 0$,

where by ϕ^n is denote n -th iterate of ϕ .

Denote by

$$\Phi = \{ \phi : [0, \infty) \rightarrow [0, \infty) | \phi \text{ is a comparison function} \}$$

and

$$\Psi = \{ \psi : [0, \infty) \rightarrow [0, \infty) | \psi \text{ is a } (b)\text{-comparison function} \}$$

Remark 1. A (b) -comparison function is called a (c) -comparison function when $s = 1$. Furthermore, it can be shown that a (c) -comparison function is a comparison function, but the converse is not necessary true. For further properties and examples of comparison functions see [1–3].

We recall here the following two important Lemma.

Lemma 1 ([1,2]). For a comparison function $\phi : [0, \infty) \rightarrow [0, \infty)$ the following properties hold:

1. each iterate ϕ^i of ϕ , $i \geq 1$ is also a comparison function;
2. ϕ is continuous at 0;
3. $\phi(t) < t$ for all $t > 0$.

Lemma 2 ([2]). Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a (b) -comparison function. Then,

1. the series $\sum_{k=0}^{\infty} s^k \phi^k(t)$ converges for any $t \in [0, \infty)$;
2. the function $b_s : [0, \infty) \rightarrow [0, \infty)$ defined as $b_s = \sum_{k=0}^{\infty} s^k \phi^k(t)$ is increasing and continuous at $t = 0$.

Remark 2. In Lemma 2, every (b) -comparison function is a comparison function and hence, in Lemma 1, any (b) -comparison function ψ satisfies $\psi(t) < t$.

In this manuscript, we aim to introduce the notion of a (b) -hybrid contraction that not only extends several existing contraction definitions but also merges a number of existing contractions. In particular, hybrid contractions aim to combine linear contractions [4–11] and nonlinear contractions [12–17] in the fixed point theory literature. After we investigate the existence of a fixed point for this new type contraction, we state some consequences. Besides on these corollaries, to underline the novelty of the given results, we consider examples (see Examples 1–3). In addition, as a possible application, we consider an Ulam stability of our result.

2. Main Results

Definition 1. A self-mapping T on a complete (b) -metric space $(\mathcal{M}, b, s)$ is said to be a (b) -hybrid contraction, if there is $\psi \in \Psi$ so that

$$b(Tx, Ty) \leq \psi(\mathcal{P}_T^r(x, y)), \quad (1)$$

where $r \geq 0$ and $\kappa_i \geq 0, i = 1, 2, 3, 4$, such that $\sum_{i=1}^4 \kappa_i = 1$ and

$$\mathcal{P}_T^r(x, y) = \begin{cases} [\kappa_1(b(x, y))^r + \kappa_2(b(x, Tx))^r + \kappa_3(b(y, Ty))^r + \kappa_4 \left(\frac{b(y, Tx) + b(x, Ty)}{2s} \right)^r]^{1/r}, & \text{for } r > 0, \quad x, y \in \mathcal{M} \\ (b(x, y))^{\kappa_1} (b(x, Tx))^{\kappa_2} (b(y, Ty))^{\kappa_3} \left(\frac{b(x, Ty) + b(y, Tx)}{2s} \right)^{\kappa_4}, & \text{for } r = 0, \quad x, y \in \mathcal{M} \setminus F_T(\mathcal{M}), \end{cases} \quad (2)$$

where $F_T(\mathcal{M}) = \{\omega \in \mathcal{M} : T\omega = \omega\}$.

Theorem 1. Suppose that $(\mathcal{M}, b, s)$ is a complete (b) -metric space and $T : \mathcal{M} \rightarrow \mathcal{M}$ is a (b) -hybrid contraction. Then T has a fixed point q and for any $x_0 \in \mathcal{M}$, the sequence $\{T^n x_0\}$ converges to q if either

- (a₁) T is continuous at q ;
- (a₂) or T^2 is continuous at q ;
- (a₃) or $\kappa_1 + \kappa_3 > 0$, (or $\kappa_2 + \kappa_3 > 0$).

Proof. We consider that the sequence $\{x_n\}$ is defined by the relation $x_{n+1} = Tx_n, n \geq 0$, by taking an arbitrary point $x_0 \in \mathcal{M}$. Under these considerations, it is easy to see that in case that there is n_0 so that

$x_{n_0} = x_{n_0+1} = Tx_{n_0}$, so x_{n_0} turns to be a fixed point of T . For this reason, we shall presume that for all $n \in \mathbb{N}_0$,

$$x_{n+1} \neq x_n \Leftrightarrow b(x_{n+1}, x_n) > 0.$$

On account of the (15), setting $x = x_n$ and $y = x_n$, we find that we have that

$$\mathcal{P}_T^r(x_n, x_{n-1}) = \begin{cases} [\kappa_1(b(x_n, x_{n-1}))^r + \kappa_2(b(x_n, Tx_n))^r + \kappa_3(b(x_{n-1}, Tx_{n-1}))^r + \kappa_4\left(\frac{b(x_n, Tx_{n-1}) + b(x_{n-1}, Tx_n)}{2s}\right)^r]^{1/r}, & \text{for } r > 0, \\ (b(x_n, x_{n-1}))^{\kappa_1} (b(x_n, Tx_n))^{\kappa_2} (b(x_{n-1}, Tx_{n-1}))^{\kappa_3} \left(\frac{b(x_n, Tx_{n-1}) + b(x_{n-1}, Tx_n)}{2s}\right)^{\kappa_4}, & \text{for } r = 0 \end{cases} \quad (3)$$

Now, we consider first the case $r > 0$. If we suppose that $b(x_{n+1}, x_n) \geq b(x_{n+1}, x_n)$ then by an elementary estimation, we obtain that

$$\begin{aligned} \mathcal{P}_T^r(x_n, x_{n-1}) &= [\kappa_1(b(x_n, x_{n-1}))^r + \kappa_2(b(x_n, x_{n+1}))^r + \kappa_3(b(x_{n-1}, x_n))^r + \kappa_4\left(\frac{b(x_n, x_{n+1}) + b(x_{n-1}, x_{n+1})}{2s}\right)^r]^{1/r} \\ &\leq [\kappa_1(b(x_n, x_{n-1}))^r + \kappa_2(b(x_n, x_{n+1}))^r + \kappa_3(b(x_{n-1}, x_n))^r + \kappa_4\left(\frac{s(b(x_{n-1}, x_n) + b(x_n, x_{n+1}))}{2s}\right)^r]^{1/r} \\ &= b(x_{n+1}, x_n) \sqrt[r]{\sum_{i=1}^4 \kappa_i} = b(x_{n+1}, x_n), \end{aligned}$$

since $\sum_{i=1}^4 \kappa_i = 1$.

Thus, since ψ is non decreasing, from condition (1), we get that

$$b(x_{n+1}, x_n) \leq \psi(\mathcal{P}_T^r(x_n, x) n - 1) \leq \psi(b(x_{n+1}, x_n)) < b(x_{n+1}, x_n), \quad (4)$$

which is a contradiction. Attendantly, we can conclude that $b(x_{n+1}, x_n) < b(x_{n+1}, x_n)$ and from (4) that

$$b(x_{n+1}, x_n) \leq \psi(b(x_n, x_{n-1})). \quad (5)$$

Inductively, from the inequalities above, we deduce

$$b(x_{n+1}, x_n) \leq \psi^n(b(x_1, x_0)), \text{ for all } n \in \mathbb{N}. \quad (6)$$

From (6), using the triangular inequality, for all $p \geq 1$, we have

$$\begin{aligned} b(x_{n+p}, x_n) &\leq s(b(x_{n+p}, x_{n+1}) + b(x_{n+1}, x_n)) \\ &\leq \frac{1}{s^{n-1}} \sum_{k=n}^{n+p-1} s^k b(x_k, x_{k+1}) \\ &\leq \frac{1}{s^{n-1}} \sum_{k=n}^{n+p-1} s^k \psi^k(b(x_1, x_0)) \\ &\leq \frac{1}{s^{n-1}} \sum_{k=n}^{+\infty} \psi^k(b(x_1, x_0)) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

(We used here Lemma 2). Thus, we get that the sequence $\{x_n\}$ is Cauchy in $(\mathcal{M}, b, s)$. Since this space is complete we conclude the existence of $\varrho \in \mathcal{M}$ such that

$$\lim_{n \rightarrow \infty} b(x_n, \varrho) = 0. \quad (7)$$

We claim that ϱ is the requested fixed point of T under the given presumptions.

If (a_1) holds, that is, T is continuous, we have

$$T\varrho = T(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = \varrho.$$

Under the second assumption, (a_2) , due the continuity of T^2 , $T^2\varrho = \lim_{n \rightarrow \infty} T^2\chi_n = \varrho$. We claim that $T\varrho = \varrho$. On the contrary, if $T\varrho \neq \varrho$, we have

$$\begin{aligned}\mathcal{P}_T^r(\varrho, T\varrho) &= [\kappa_1(\mathfrak{b}(\varrho, T\varrho))^r + \kappa_2(\mathfrak{b}(\varrho, T\varrho))^r + \kappa_3(\mathfrak{b}(T\varrho, T^2\varrho))^r \\ &\quad + \kappa_4 \left(\frac{\mathfrak{b}(T\varrho, T\varrho) + \mathfrak{b}(\varrho, T^2\varrho)}{2} \right)^r]^{1/r} \\ &= \mathfrak{b}(\varrho, T\varrho) (\kappa_1 + \kappa_2 + \kappa_3)^{1/p} \leq \mathfrak{b}(\varrho, T\varrho).\end{aligned}$$

Thus,

$$0 < \mathfrak{b}(T\varrho, \varrho) = \mathfrak{b}(T\varrho, T^2\varrho) \leq \psi(\mathcal{P}_T^r(\varrho, T\varrho)) < \mathcal{P}_T^r(\varrho, T\varrho) \leq d(\varrho, T\varrho), \quad (8)$$

which is a contradiction. Thus, $\mathfrak{b}(T\varrho, \varrho)$, that is ϱ is a fixed point of T . On the other hand,

$$\begin{aligned}0 < \mathfrak{b}(T\varrho, \varrho) &\leq \mathfrak{b}(T\varrho, \chi_{n+1}) + \mathfrak{b}(\chi_{n+1}, \varrho) \\ &= \mathfrak{b}(T\varrho, T\chi_{n+1}) + \mathfrak{b}(\chi_{n+1}, \varrho) \\ &\leq \psi(\mathcal{P}_T^r(\varrho, \chi_n)) + \mathfrak{b}(\chi_{n+1}, \varrho) \\ &< \mathcal{P}_T^r(\varrho, \chi_n) + \mathfrak{b}(\chi_{n+1}, \rho) \\ &= \left[\kappa_1(\mathfrak{b}(\varrho, \chi_n))^r + \kappa_2(\mathfrak{b}(\varrho, T\varrho))^r + \kappa_3(\mathfrak{b}(\chi_n, \chi_{n+1}))^r + \kappa_4 \left(\frac{\mathfrak{b}(\chi_n, T\varrho) + \mathfrak{b}(\varrho, \chi_{n+1})}{2} \right)^r \right]^{1/r} \\ &\quad + \mathfrak{b}(\chi_{n+1}, \rho).\end{aligned}$$

Thus, taking $n \rightarrow \infty$ in (9), we get

$$0 < \mathfrak{b}(T\varrho, \varrho) \leq \Lambda \mathfrak{b}(T\varrho, \varrho),$$

where $\Lambda := [\kappa_2 + \frac{\kappa_4}{2}]^{1/p}$.

If we suppose that (a_3) holds, that is, κ_1, κ_3 are not simultaneous 0, he have that $\Lambda < 1$ and we obtain a contradiction, that is, $T\varrho = \varrho$.

We skip the details of second assumption of the case (a_3) since it is verbatim of the above lines.

We shall consider now the case $r = 0$. Since in this case,

$$\begin{aligned}\mathcal{P}_T^r(\chi_n, \chi_{n-1}) &= (\mathfrak{b}(\chi_n, \chi_{n-1}))^{\kappa_1} (\mathfrak{b}(\chi_n, \chi_{n+1}))^{\kappa_2} (\mathfrak{b}(\chi_{n-1}, \chi_n))^{\kappa_3} \left(\frac{\mathfrak{b}(\chi_n, \chi_n) + \mathfrak{b}(\chi_{n-1}, \chi_{n+1})}{2s} \right)^{\kappa_4} \\ &\leq (\mathfrak{b}(\chi_n, \chi_{n-1}))^{\kappa_1} (\mathfrak{b}(\chi_n, \chi_{n+1}))^{\kappa_2} (\mathfrak{b}(\chi_{n-1}, \chi_n))^{\kappa_3} \left(\frac{s(\mathfrak{b}(\chi_{n-1}, \chi_n) + \mathfrak{b}(\chi_n, \chi_{n+1}))}{2s} \right)^{\kappa_4},\end{aligned}$$

the inequality (1) becomes

$$\begin{aligned}\mathfrak{b}(\chi_{n+1}, \chi_n) &= \mathfrak{b}(T\chi_n, T\chi_{n-1}) \leq \psi(\mathcal{P}_T^r(\chi_n, \chi_{n-1})) \leq \mathcal{P}_T^r(\chi_n, \chi_{n-1}) \\ &= (\mathfrak{b}(\chi_n, \chi_{n-1}))^{\kappa_1 + \kappa_3} (\mathfrak{b}(\chi_{n+1}, \chi_n))^{\kappa_2} \left(\frac{\mathfrak{b}(\chi_n, \chi_{n-1}) + \mathfrak{b}(\chi_{n+1}, \chi_n)}{2} \right)^{1 - \kappa_1 - \kappa_2 - \kappa_3}\end{aligned} \quad (9)$$

In case that $\mathfrak{b}(\chi_n, \chi_{n-1}) \leq \mathfrak{b}(\chi_{n+1}, \chi_n)$ we have

$$\frac{\mathfrak{b}(\chi_n, \chi_{n-1}) + \mathfrak{b}(\chi_{n+1}, \chi_n)}{2} \leq \mathfrak{b}(\chi_{n+1}, \chi_n)$$

and the inequality (12) yields that

$$[\mathfrak{b}(\chi_{n+1}, \chi_n)]^{\kappa_1 + \kappa_3} < [\mathfrak{b}(\chi_n, \chi_{n-1})]^{\kappa_1 + \kappa_3}, \quad (10)$$

which contradict our assumption. Thus, $b(x_n, x_{n-1}) > b(x_{n+1}, x_n)$ for all $n \geq 1$ and the sequence $\{b(x_n, x_{n-1})\}$ is a non-increasing. On account of the observation below, we have

$$\mathcal{P}_T^r(x_n, x_{n-1}) < b(x_n, x_{n-1})$$

and then

$$b(x_{n+1}, x_n) \leq \psi(\mathcal{P}_T^r(b(x_n, x_{n-1}))) < \psi(b(x_n, x_{n-1})) < b(x_n, x_{n-1}) \quad (11)$$

for all $n \in \mathbb{N}$. However, the above inequality is equivalent to (5), so we derive that the iterated sequence $\{x_n\}$ is Cauchy and there is $q \in \mathcal{M}$ such that $\lim_{n \rightarrow \infty} b(x_n, q) = 0$. We claim that this point q is a fixed point for T . Indeed, if T is continuous, following the proof for the case $r = 0$ we find that $Tq = q$. Suppose now that $q \neq Tq$. Taking into account the assumption (a_2) , that T^2 is continuous, as previous we get that $T^2q = q$ and from (15) with $x = q$ and $y = Tq$ we have

$$\begin{aligned} b(q, Tq) &= b(T^2q, Tq) \leq \psi(\mathcal{P}_T^r(Tq, q)) \\ &= \psi\left([b(Tq, q)]^{\kappa_1} [b(Tq, T^2q)]^{\kappa_2} (b(q, Tq))^{\kappa_3} \left[\frac{(b(q, T^2q) + b(Tq, Tq))}{2}\right]^{\kappa_4}\right) \\ &< [b(Tq, q)]^{\kappa_1} [b(Tq, q)]^{\kappa_2} [b(q, Tq)]^{\kappa_3} \left[\frac{(b(q, q) + b(Tq, Tq))}{2s}\right]^{\kappa_4} = 0, \end{aligned} \quad (12)$$

which is a contradiction.

For the last presumption, (a_3) , since $x_n \neq x_{n+1} = Tx_n$ for each $n \geq 0$, by letting $x = x_n$ and $y = q$ in (1), we have

$$\begin{aligned} b(x_{n+1}, Tq) &= b(Tx_n, Tq) \\ &\leq \psi\left([b(x_n, q)]^{\kappa_1} \cdot [b(x_n, Tx_n)]^{\kappa_2} \cdot [b(q, Tq)]^{\kappa_3} \cdot \left[\frac{1}{2}(b(x_n, Tq) + b(q, Tx_n))\right]^{\kappa_4}\right) \\ &< [b(x_n, q)]^{\kappa_1} \cdot [b(x_n, x_{n+1})]^{\kappa_2} \cdot [b(q, Tq)]^{\kappa_3} \\ &\quad \cdot \left[\frac{b(x_n, Tq) + b(q, x_{n+1})}{2s}\right]^{\kappa_4} \end{aligned} \quad (13)$$

Letting $n \rightarrow \infty$ in the inequality (13), we get $b(q, Tq) = 0$, which is a contradiction. Thus, $Tq = q$. $\square$

Example 1. On set $\mathcal{M} = \{1, 2, 5, 8\}$, endowed with the b -metric $b(x, y) = (x - y)^2$, we consider the mapping $T : \mathcal{M} \rightarrow \mathcal{M}$ given as follows:

$$T1 = 1, T2 = 2, T5 = 2, T8 = 1.$$

Undoubtedly, taking $\kappa_1 = \kappa_4 = \frac{1}{4}, \kappa_3 = \frac{1}{2}, \kappa_2 = 0$ and for example, $\psi(t) = \frac{t}{2}$ we have

$$b(T5, T8) = 1 < \frac{1}{2} \sqrt[4]{9} \cdot \sqrt{49} \cdot \sqrt[4]{13} = \psi(\mathcal{P}_T(5, 8)).$$

Hence, (1) is satisfied for every $x, y \in \mathcal{M} \setminus F_T(\mathcal{M})$ and the mapping T has two fixed points; these are 1 and 2.

Taking $s = 1$ in the above theorem we find the following corollary.

Corollary 1. Let T be a self-mapping on a complete metric space $(\mathcal{M}, b)$ such that

$$b(Tx, Ty) \leq \psi(\mathcal{P}_T^r(x, y)), \quad (14)$$

where $\psi \in \Psi$, $r \geq 0$ and $\kappa_i \geq 0, i = 1, 2, 3, 4$, such that $\sum_{i=1}^4 \kappa_i = 1$ and

$$\mathcal{P}_T^r(x, y) = \begin{cases} [\kappa_1(b(x, y))^r + \kappa_2(b(x, Tx))^r + \kappa_3(b(y, Ty))^r + \kappa_4 \left(\frac{b(y, Tx) + b(x, Ty)}{2} \right)^r]^{1/r}, & \text{for } r > 0, \quad x, y \in \mathcal{M} \\ (b(x, y))^{\kappa_1} (b(x, Tx))^{\kappa_2} (b(y, Ty))^{\kappa_3} \left(\frac{b(x, Ty) + b(y, Tx)}{2} \right)^{\kappa_4}, & \text{for } r = 0, \quad x, y \in \mathcal{M} \setminus F_T(\mathcal{M}), \end{cases} \quad (15)$$

where $F_T(\mathcal{M}) = \{\omega \in \mathcal{M} : T\omega = \omega\}$. Then T has a fixed point $q \in \mathcal{M}$, if any of the following statements hold

- (a₁) T is continuous at q ;
- (a₂) or T^2 is continuous at q ;
- (a₃) or $\kappa_1 + \kappa_3 > 0$, (or $\kappa_2 + \kappa_3 > 0$).

Corollary 2. Let $(\mathcal{M}, b, s)$ be a complete (b) -metric space and $T : \mathcal{M} \rightarrow \mathcal{M}$ is a mapping such that there exists $\lambda \in [0, 1)$ such that

$$b(Tx, Ty) \leq \frac{\lambda}{4} \left(b(x, y) + b(x, Tx) + b(y, Ty) + \frac{b(x, Ty) + b(Tx, y)}{2s} \right). \quad (16)$$

If T is continuous or T^2 is continuous then T has a fixed point q and for any $x_0 \in \mathcal{M}$, the sequence $\{T^n x_0\}$ converges to q .

Corollary 3. Let $(\mathcal{M}, b, s)$ be a complete (b) -metric space and $T : \mathcal{M} \rightarrow \mathcal{M}$ is a mapping such that there exists $\lambda \in [0, 1)$ such that

$$b(Tx, Ty) \leq \frac{\lambda}{2} \left(b^2(x, y) + b^2(x, Tx) + b^2(y, Ty) + \left(\frac{b(x, Ty) + b(Tx, y)}{2s} \right)^2 \right)^{1/2}. \quad (17)$$

If T is continuous or T^2 is continuous then T has a fixed point q and for any $x_0 \in \mathcal{M}$, the sequence $\{T^n x_0\}$ converges to q .

Proof. Let $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \frac{1}{4}$, $\psi(t) = \frac{\lambda}{2}t$, $t \geq 0$ and $r = 2$ in Theorem 1. $\square$

Example 2. Define a self-mapping T on $\mathcal{M}$ by $Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1) \\ 0, & \text{if } x = 1 \end{cases}$, where $\mathcal{M} = [0, 1]$. On the set $\mathcal{M}$ we consider the b -metric $b(x, y) = (x - y)^2$ with $s = 2$. Since $T^2x = \frac{x}{4}$ for all $x \in \mathcal{M}$ we have that T^2 is continuous, while T is not continuous. Letting $\lambda = \frac{3}{4}$, we have:

(i) For $x, y \in [0, 1)$:

$$\begin{aligned} b(Tx, Ty) &= b\left(\frac{x}{2}, \frac{y}{2}\right) = \frac{(x-y)^2}{4} \leq \frac{3}{8}(x-y)^2 \leq \frac{3}{8} \left((x-y)^4 + \frac{x^4}{16} + \frac{y^4}{16} \right)^{1/2} \\ &\leq \frac{\lambda}{2} \left(b^2(x, y) + b^2(x, Tx) + b^2(y, Ty) + \left(\frac{b(x, Ty) + b(Tx, y)}{2s} \right)^2 \right)^{1/2}. \end{aligned}$$

(ii) For $x \in [0, 1)$ and $y = 1$:

$$\begin{aligned} b(Tx, T1) &= b\left(\frac{x}{2}, 0\right) = \frac{x^2}{4} \leq \frac{3}{8} \leq \frac{3}{8} \left((x-1)^4 + \frac{x^4}{16} + 1 + \left(\frac{x^2 + (\frac{x}{2}-1)^2}{4} \right)^2 \right)^{1/2} \\ &\leq \frac{\lambda}{2} \left(b^2(x, 1) + b^2(x, Tx) + b^2(1, T1) + \left(\frac{b(x, T1) + b(Tx, 1)}{2s} \right)^2 \right)^{1/2}. \end{aligned}$$

Consequently, the assumptions of Corollary 3 are satisfied and $q = 0$ is a fixed point of T .

Corollary 4. Let $(\mathcal{M}, b, s)$ be a complete (b) -metric space. Let $T : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping such that

$$b(Tx, Ty) \leq \lambda \sqrt[4]{b(x, y) \cdot b(x, Tx) \cdot b(y, Ty) \cdot \frac{b(x, Ty) + b(Tx, y)}{2s}} \quad (18)$$

for any $x, y \in \mathcal{M} \setminus F_T(\mathcal{M})$ and where $\lambda \in [0, 1)$. If T is continuous or T^2 is continuous then T has a fixed point $q \in \mathcal{M}$.

Proof. Let $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \frac{1}{4}$, $\psi(t) = \lambda t$, $t \geq 0$ and $r = 0$ in Theorem 1. $\square$

Of course, any fixed point of a map T is a fixed point for T^n , for any $n \geq 1$, but the converse is not generally true. A mapping who has the property that $F_T(\mathcal{M}) = F_{T^n}(\mathcal{M})$ is said that satisfies the property P ([8,10]). Regarding the maps that satisfy the property P , in a b -metric space, it is known the following result.

Theorem 2 ([9]). Let $(\mathcal{M}, b, s)$ be a complete (b) -metric space and $T : \mathcal{M} \rightarrow \mathcal{M}$ such that $F_T(\mathcal{M}) \neq \emptyset$. The mapping T satisfies the property P if for every x

$$b(Tx, T^2x) \leq \lambda b(x, Tx), \quad (19)$$

where $\lambda \in [0, 1)$.

We give now a generalization of this result.

Theorem 3. Let $(\mathcal{M}, b, s)$ be a complete (b) -metric space and $T : \mathcal{M} \rightarrow \mathcal{M}$ such that $F_T(\mathcal{M}) \neq \emptyset$. The mapping T satisfies the property P if for every x

$$b(Tx, T^2x) \leq \psi(b(x, Tx)), \quad (20)$$

where $\psi \in \Psi$.

Proof. Let $n > 1$ and $u \in F_T^n(\mathcal{M})$. By (20), for $x = T^{n-1}u$ we have

$$b(T^n u, T^{n+1} u) = b(T(T^{n-1} u), T^2(T^{n-1} u)) \leq \psi(b(T^{n-1} u, T^n u)).$$

Since ψ is a b -comparison function, is non-decreasing and from the previous inequality we have that

$$b(u, Tu) = b(T^n u, T^{n+1} u) \leq \psi(b(T^{n-1} u, T^n u)) \leq \dots \leq \psi^n(b(u, Tu)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, $b(u, Tu) = 0$, so that $u \in F_T(\mathcal{M})$. $\square$

Remark 3. Taking $\psi(t) = \lambda t$, for any $t \geq 0$ and $0 \leq \lambda < 1$ in Theorem 3, this is reduced to the Theorem 2.

Theorem 4. Under the presumptions of Theorem 1, any (b) -hybrid contraction T satisfies the property P .

Proof. Since T is a (b) -hybrid contraction, the inequality Equation (1) is satisfies for any $x, y \in \mathcal{M}$, so that in particular we can consider $y = Tx$ and keeping in mind the properties of function ψ we get

$$b(Tx, T^2x) \leq \psi(\mathcal{P}_T^r(x, Tx)) < \mathcal{P}_T^r(x, Tx), \quad (21)$$

where $\psi \in \Psi$. We will discuss the two separate cases, namely, $r > 0$ respectively $r = 0$.

In the first case, for $r > 0$ we have

$$\begin{aligned}
 \mathcal{P}_T^r(\chi, T\chi) &= [\kappa_1(\mathfrak{b}(\chi, T\chi))^r + \kappa_2(\mathfrak{b}(\chi, T\chi))^r + \kappa_3(\mathfrak{b}(T\chi, T^2\chi))^r + \kappa_4 \left(\frac{\mathfrak{b}(T\chi, T\chi) + \mathfrak{b}(\chi, T^2\chi)}{2} \right)^r]^{1/r}, \\
 &\leq [\kappa_1(\mathfrak{b}(\chi, T\chi))^r + \kappa_2(\mathfrak{b}(\chi, T\chi))^r + \kappa_3(\mathfrak{b}(T\chi, T^2\chi))^r + \kappa_4 \left(\frac{\mathfrak{b}(\chi, T\chi) + \mathfrak{b}(T\chi, T^2\chi)}{2} \right)^r]^{1/r}, \\
 &\leq [\kappa_1(\mathfrak{b}(\chi, T\chi))^r + \kappa_2(\mathfrak{b}(\chi, T\chi))^r + \kappa_3(\mathfrak{b}(T\chi, T^2\chi))^r + \frac{\kappa_4}{2} ((\mathfrak{b}(\chi, T\chi))^r + (\mathfrak{b}(T\chi, T^2\chi))^r)]^{1/r}, \\
 &= [(\kappa_1 + \kappa_2 + \kappa_4/2)(\mathfrak{b}(\chi, T\chi))^r + (\kappa_3 + \kappa_4/2)(\mathfrak{b}(T\chi, T^2\chi))^r]^{1/r},
 \end{aligned}$$

and replacing in (21) we find that

$$\mathfrak{b}(T\chi, T^2\chi) < [(\kappa_1 + \kappa_2 + \kappa_4/2)(\mathfrak{b}(\chi, T\chi))^r + (\kappa_3 + \kappa_4/2)(\mathfrak{b}(T\chi, T^2\chi))^r]^{1/r}$$

or, equivalent

$$(\mathfrak{b}(T\chi, T^2\chi))^r < \left((\kappa_1 + \kappa_2 + \kappa_4/2)(\mathfrak{b}(\chi, T\chi))^r + (\kappa_3 + \kappa_4/2)(\mathfrak{b}(T\chi, T^2\chi))^r \right)^r.$$

If $\mathfrak{b}(\chi, T\chi) < \mathfrak{b}(T\chi, T^2\chi)$, the above inequality becomes

$$(\mathfrak{b}(T\chi, T^2\chi))^r < (\mathfrak{b}(\chi, T\chi))^r$$

which is a contradiction. Thus, $\mathfrak{b}(T\chi, T^2\chi) \leq \mathfrak{b}(\chi, T\chi)$ and then $\mathcal{P}_T^r(\chi, T\chi) < \mathfrak{b}(\chi, T\chi)$. Since $\psi \in \Psi$,

$$\mathfrak{b}(T\chi, T^2\chi) \leq \psi(\mathcal{P}_T^r(\chi, T\chi) \leq \psi(\mathfrak{b}(\chi, T\chi)).$$

Therefore, from Theorem 3 we obtain that T satisfies the property P .

For the case $r = 0$, since

$$\begin{aligned}
 \mathcal{P}_T^r(\chi, T\chi) &= (\mathfrak{b}(\chi, T\chi))^{\kappa_1} (\mathfrak{b}(\chi, T\chi))^{\kappa_2} (\mathfrak{b}(T\chi, T^2\chi))^{\kappa_3} \left(\frac{(\mathfrak{b}(\chi, T^2\chi) + \mathfrak{b}(T\chi, T\chi))}{2} \right)^{\kappa_4} \\
 &\leq (\mathfrak{b}(\chi, T\chi))^{\kappa_1 + \kappa_2} (\mathfrak{b}(T\chi, T^2\chi))^{\kappa_3} \left(\frac{(\mathfrak{b}(\chi, T\chi))^{\kappa_4} + (\mathfrak{b}(T\chi, T^2\chi))^{\kappa_4}}{2} \right).
 \end{aligned}$$

If we suppose, by reductio ad absurdum that $\mathfrak{b}(\chi, T\chi) < \mathfrak{b}(T\chi, T^2\chi)$ for $\chi \in \mathcal{M}$, from the inequality (21) and taking into account the above inequality we get

$$\begin{aligned}
 \mathfrak{b}(T\chi, T^2\chi) &\leq \psi(\mathcal{P}_T^r(\chi, T\chi)) \\
 &\leq \psi((\mathfrak{b}(\chi, T\chi))^{\kappa_1 + \kappa_2} (\mathfrak{b}(T\chi, T^2\chi))^{\kappa_3} \left(\frac{(\mathfrak{b}(\chi, T\chi))^{\kappa_4} + (\mathfrak{b}(T\chi, T^2\chi))^{\kappa_4}}{2} \right)) \\
 &< \psi((\mathfrak{b}(T\chi, T^2\chi))^{\kappa_1 + \kappa_2} (\mathfrak{b}(T\chi, T^2\chi))^{\kappa_3} (\mathfrak{b}(T\chi, T^2\chi))^{\kappa_4}) \\
 &= \psi((\mathfrak{b}(T\chi, T^2\chi))^{\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4}) \\
 &= \psi(\mathfrak{b}(T\chi, T^2\chi)) < \mathfrak{b}(T\chi, T^2\chi),
 \end{aligned}$$

which is a contradiction. Consequently, $\mathfrak{b}(\chi, T\chi) \geq \mathfrak{b}(T\chi, T^2\chi)$ and we obtain

$$\begin{aligned}
 \mathfrak{b}(T\chi, T^2\chi) &\leq \psi(\mathcal{P}_T^r(\chi, T\chi) = \psi((\mathfrak{b}(\chi, T\chi))^{\kappa_1 + \kappa_2} (\mathfrak{b}(T\chi, T^2\chi))^{\kappa_3} \left(\frac{(\mathfrak{b}(\chi, T\chi))^{\kappa_4} + (\mathfrak{b}(T\chi, T^2\chi))^{\kappa_4}}{2} \right)) \\
 &\leq \psi((\mathfrak{b}(\chi, T\chi))).
 \end{aligned}$$

□

Theorem 5. Let $(\mathcal{M}, b, s)$ be a complete b -metric space, $a_1, a_2 \in [0, 1)$. Let $T : \mathcal{M} \rightarrow \mathcal{M}$ be a mapping such that either T or T^2 is continuous. If there exist $a_1, a_2 \in [0, 1)$ such that $a_1 + a_2 < \frac{1}{s^2}$ such that

$$b(T^2\chi, T^2y) \leq \frac{a_1}{4} \cdot \left(b(\chi, y) + b(\chi, T\chi) + b(y, Ty) + \frac{b(\chi, Ty) + b(y, T\chi)}{2s} \right) + \frac{a_2}{4} \cdot \left(b(T\chi, Ty) + b(T\chi, T^2\chi) + b(Ty, T^2y) + \frac{b(T\chi, T^2y) + b(Ty, T^2\chi)}{2s} \right), \quad (22)$$

then T has a unique fixed point.

Proof. Let $z_0 \in \mathcal{M}$. Replacing χ by χ_0 and y by Ty_0 in (22) we have

$$\begin{aligned} b(T^2\chi_0, T^3\chi_0) &\leq \frac{a_1}{4} \cdot \left(b(\chi_0, T\chi_0) + b(\chi_0, T\chi_0) + b(T\chi_0, T^2\chi_0) + \frac{b(\chi_0, T^2\chi_0) + b(T\chi_0, T\chi_0)}{2s} \right) + \\ &\quad + \frac{a_2}{4} \cdot \left(b(T\chi_0, T^2\chi_0) + b(T\chi_0, T^2\chi_0) + b(T^2\chi_0, T^3\chi_0) + \frac{b(T\chi_0, T^3\chi_0) + b(T^2\chi_0, T^2\chi_0)}{2s} \right) \\ &\leq \frac{a_1}{4} \cdot \left(b(\chi_0, T\chi_0) + b(\chi_0, T\chi_0) + b(T\chi_0, T^2\chi_0) + \frac{b(\chi_0, T\chi_0) + b(T\chi_0, T^2\chi_0)}{2} \right) + \\ &\quad + \frac{a_2}{4} \cdot \left(b(T\chi_0, T^2\chi_0) + b(T\chi_0, T^2\chi_0) + b(T^2\chi_0, T^3\chi_0) + \frac{b(T\chi_0, T^2\chi_0) + b(T^2\chi_0, T^3\chi_0)}{2} \right) \end{aligned} \quad (23)$$

which is equivalent with

$$\begin{aligned} (8 - 3a_2)b(T^2\chi_0, T^3\chi_0) &\leq 5a_1b(\chi_0, T\chi_0) + (3a_1 + 5a_2)b(T\chi_0, T^2\chi_0) \\ &\leq (8a_1 + 5a_2) \max \{ b(\chi_0, T\chi_0), b(T\chi_0, T^2\chi_0) \}, \end{aligned}$$

or

$$b(T^2\chi_0, T^3\chi_0) \leq \frac{\alpha}{\beta} \cdot m \quad (24)$$

where $m = \max \{ b(\chi_0, T\chi_0), b(T\chi_0, T^2\chi_0) \}$, $\alpha = 8a_1 + 5a_2$, $\beta = 8 - 3a_2$. Of course, how long $a_1 + a_2 < \frac{1}{s^2}$, we have $\frac{\alpha}{\beta} < 1$. Again, for $\chi = T\chi_0$, $y = T^2\chi_0$, we have

$$\begin{aligned} b(T^3\chi_0, T^4\chi_0) &\leq \frac{a_1}{4} \cdot \left(b(T\chi_0, T^2\chi_0) + b(T\chi_0, T^2\chi_0) + b(T^2\chi_0, T^3\chi_0) + \frac{b(T\chi_0, T^3\chi_0) + b(T^2\chi_0, T^2\chi_0)}{2s} \right) + \\ &\quad + \frac{a_2}{4} \cdot \left(b(T^2\chi_0, T^3\chi_0) + b(T^2\chi_0, T^3\chi_0), b(T^3\chi_0, T^4\chi_0) + \frac{b(T^2\chi_0, T^4\chi_0) + b(T^3\chi_0, T^3\chi_0)}{2s} \right) \\ &\leq \frac{a_1}{4} \cdot \left(b(T\chi_0, T^2\chi_0) + b(T\chi_0, T^2\chi_0) + b(T^2\chi_0, T^3\chi_0) + \frac{b(T\chi_0, T^2\chi_0) + b(T^2\chi_0, T^3\chi_0)}{2} \right) + \\ &\quad + \frac{a_2}{4} \cdot \left(b(T^2\chi_0, T^3\chi_0) + b(T^2\chi_0, T^3\chi_0), b(T^3\chi_0, T^4\chi_0) + \frac{b(T^2\chi_0, T^3\chi_0) + b(T^3\chi_0, T^4\chi_0)}{2} \right) \\ &= \frac{a_1}{8} \cdot (5b(T\chi_0, T^2\chi_0) + 3b(T^2\chi_0, T^3\chi_0)) + \frac{a_2}{8} \cdot (5b(T^2\chi_0, T^3\chi_0) + 3b(T^3\chi_0, T^4\chi_0)) \end{aligned}$$

or

$$\begin{aligned} (8 - 3a_2)b(T^3\chi_0, T^4\chi_0) &\leq 5a_1b(T\chi_0, T^2\chi_0) + (3a_1 + 5a_2)b(T^2\chi_0, T^3\chi_0) \\ &\leq 5a_1 \cdot m + (3a_1 + 5a_2) \cdot \frac{\alpha}{\beta} \cdot m < (8a_1 + 5a_2) \cdot m, \end{aligned}$$

which gives us that

$$b(T^3\chi_0, T^4\chi_0) \leq \frac{8a_1 + 5a_2}{8 - 3a_2} m = \frac{\alpha}{\beta} m. \quad (25)$$

Analogously, taking $\chi = T^2\chi_0$, $y = T^3\chi_0$ we get

$$\begin{aligned} b(T^4\chi_0, T^5\chi_0) &\leq \frac{a_1}{4} \cdot \left(b(T\chi_0, T^2\chi_0) + b(T\chi_0, T^2\chi_0) + b(T^2\chi_0, T^3\chi_0) + \frac{b(T\chi_0, T^2\chi_0) + b(T^2\chi_0, T^3\chi_0)}{2} \right) + \\ &\quad + \frac{a_2}{4} \cdot \left(b(T^2\chi_0, T^3\chi_0) + b(T^2\chi_0, T^3\chi_0), b(T^3\chi_0, T^4\chi_0) + \frac{b(T^2\chi_0, T^3\chi_0) + b(T^3\chi_0, T^4\chi_0)}{2} \right) \\ &= \frac{a_1}{8} \cdot (5b(T^2\chi_0, T^3\chi_0) + 3b(T^3\chi_0, T^4\chi_0)) + \frac{a_2}{8} \cdot (5b(T^3\chi_0, T^4\chi_0) + 3b(T^4\chi_0, T^5\chi_0)) \end{aligned} \quad (26)$$

or, taking into account (24) and (25),

$$\begin{aligned} (8 - 3a_2)b(T^4\chi_0, T^5\chi_0) &\leq 5a_1b(T^2\chi_0, T^3\chi_0) + (3a_1 + 5a_2)b(T^3\chi_0, T^4\chi_0) \\ &\leq 5a_1 \frac{\alpha}{\beta} m + (3a_1 + 5a_2) \frac{\alpha}{\beta} m < (8a_1 + 5a_2) \cdot m \leq \frac{\alpha}{\beta} (8a_1 + 5a_2) m. \end{aligned}$$

In this way, we find that

$$b(T^4 x_0, T^5 x_0) \leq \left(\frac{\alpha}{\beta}\right)^2 m \quad (27)$$

and similar

$$b(T^5 x_0, T^6 x_0) \leq \left(\frac{\alpha}{\beta}\right)^2 m. \quad (28)$$

If we build the sequence $\{x_n\}$ as follows

$$x_{n+1} = T^n x_0, \quad n \geq 0,$$

following the same reasoning we get that

$$b(x_{n+1}, x_{n+2}) = b(T^n x_0, T^{n+1} x_0) \leq \left(\frac{\alpha}{\beta}\right)^p m, \quad (29)$$

for the both cases $n = 2p$ and $n = 2p + 1$, where $p \in \mathbb{N}$

In the next step, we want to demonstrate that the sequence defined above is Cauchy. Let for this, $n, m \in \mathbb{N}$, with $m > n$. We will consider four cases:

Case 1. If $n = 2p$ and $m = 2k$, where $p \geq 1$ and $k \geq 2$, then by triangle inequality we have

$$\begin{aligned} b(x_n, x_m) &= b(x_{2p}, x_{2k}) \leq s[b(x_{2p}, x_{2p+1}) + b(x_{2p+1}, x_{2k})] \\ &\leq s \cdot b(x_{2p}, x_{2p+1}) + s^2 \cdot b(x_{2p+1}, x_{2p+2}) + \dots + s^{2k-2p} \cdot b(x_{2k-1}, x_{2k}) \\ &\leq (s + s^2) \left(\frac{\alpha}{\beta}\right)^p m + (s^3 + s^4) \left(\frac{\alpha}{\beta}\right)^{p+1} m + \dots + (s^{2k-2p-1} + s^{2k-2p}) \left(\frac{\alpha}{\beta}\right)^{k-1} m \\ &= m(s + s^2) \left(\frac{\alpha}{\beta}\right)^p \left(1 + \frac{\alpha}{\beta} s^2 + \left(\frac{\alpha}{\beta} s^2\right)^2 + \dots + \left(\frac{\alpha}{\beta} s^2\right)^{k-p-1}\right) \\ &= m(s + s^2) \left(\frac{\alpha}{\beta}\right)^p \frac{1 - \left(\frac{\alpha}{\beta} s^2\right)^{k-p+1}}{1 - \frac{\alpha}{\beta} s^2}. \end{aligned}$$

So,

$$b(x_{2p}, x_{2k}) \leq m(s + s^2) \frac{\left(\frac{\alpha}{\beta}\right)^p}{1 - \frac{\alpha}{\beta} s^2}, \quad (30)$$

for any $p \geq 1$ and $k \geq 2$, since $\frac{\alpha}{\beta} s^2 < 1$.

Case 2. If $n = 2p$ and $m = 2k + 1$, where $p \geq 1$ and $k \geq 1$, then

$$b(x_{2p}, x_{2k+1}) \leq s[b(x_{2p}, x_{2k}) + b(x_{2k}, x_{2k+1})]$$

and together with (29) and (30) we get

$$b(x_{2p}, x_{2k+1}) \leq m(s^2 + s^3) \frac{\left(\frac{\alpha}{\beta}\right)^p}{1 - \frac{\alpha}{\beta} s^2} + s \left(\frac{\alpha}{\beta}\right)^k m,$$

which shows us that

$$b(x_{2p}, x_{2k+1}) \leq m(s^2 + s^3) \frac{\left(\frac{\alpha}{\beta}\right)^p}{1 - \frac{\alpha}{\beta} s^2}, \quad (31)$$

for any $p \geq 1$ and $k \geq 1$.

Case 3. If $n = 2p + 1$ and $m = 2k + 1$, where $p \geq 1$ and $k \geq 2$, then

$$b(\chi_{2p+1}, \chi_{2k+1}) \leq s[b(\chi_{2p+1}, \chi_{2p+2}) + b(\chi_{2p+2}, \chi_{2k+1})].$$

Taking into account (29) and (31), we obtain

$$b(\chi_{2p+1}, \chi_{2k+1}) \leq m(s^2 + s^3) \frac{\left(\frac{\alpha}{\beta}\right)^p}{1 - \frac{\alpha}{\beta}s^2} \quad (32)$$

Case 4. If $n = 2p + 1$ and $m = 2k$, where $p \geq 1$ and $k \geq 2$, similarly we get

$$b(\chi_{2p+1}, \chi_{2k}) \leq m(s^2 + s^3) \frac{\left(\frac{\alpha}{\beta}\right)^p}{1 - \frac{\alpha}{\beta}s^2}. \quad (33)$$

We conclude that, for any $n, m \in \mathbb{N}$, with $m > n$, the following inequality holds:

$$b(\chi_n, \chi_m) \leq m(s^2 + s^3) \frac{\left(\frac{\alpha}{\beta}\right)^p}{1 - \frac{\alpha}{\beta}s^2} \quad (34)$$

and letting $p \rightarrow \infty$ in (34),

$$b(\chi_n, \chi_m) \rightarrow 0.$$

Consequently, the sequence $\{\chi_n\}$ is Cauchy in a b -complete metric space, so that there exists $\varrho \in \mathcal{M}$ such that $\lim_{n \rightarrow \infty} b(\chi_n, \varrho) = 0$. We state that the point ϱ is a fixed point of T . Indeed,

1. If T is continuous then $T\varrho = \lim_{n \rightarrow \infty} T\chi_n = \lim_{n \rightarrow \infty} \chi_{n+1} = \varrho$ and then $T^2\varrho = T(T\varrho) = T\varrho = \varrho$;
2. If T^2 is continuous then $T^2\varrho = \lim_{n \rightarrow \infty} T^2\chi_n = \lim_{n \rightarrow \infty} \chi_{n+2} = \varrho$. In order to prove that ϱ is also, a fixed point of T , we proceed by *reductio ad absurdum*. For this, we assume that $\varrho \neq T\varrho$. Consequently, from (22) we have

$$\begin{aligned} 0 < b(\varrho, T\varrho) &= b(T^2\varrho, T^2(T\varrho)) \leq \\ &\leq \frac{a_1}{4} \cdot \left(b(\varrho, T\varrho) + b(\varrho, T\varrho) + b(T\varrho, T^2\varrho) + \frac{b(\varrho, T^2\varrho) + b(T\varrho, T\varrho)}{2s} \right) + \\ &\quad + \frac{a_2}{4} \cdot \left(b(T\varrho, T^2\varrho) + b(T\varrho, T^2\varrho) + b(T^2\varrho, T^3\varrho) + \frac{b(T\varrho, T^3\varrho) + b(T^2\varrho, T^2\varrho)}{2s} \right) \\ &= 3\frac{a_1}{4} b(\varrho, T\varrho) + 3\frac{a_2}{4} \cdot b(T\varrho, \varrho) < b(T\varrho, \varrho) \end{aligned}$$

which is a contradiction, so that $T^2\varrho = \varrho = T\varrho$.

To prove the uniqueness, let $\varrho, \omega \in \mathcal{M}$ such that $T\varrho = \varrho \neq \omega = T\omega$. We have then

$$\begin{aligned} 0 < b(\varrho, \omega) &= b(T^2\varrho, T^2\omega) \leq \frac{a_1}{4} \cdot \left(b(\varrho, \omega) + b(\varrho, T\varrho) + b(\omega, T\omega) + \frac{b(\varrho, T\omega) + b(\omega, T\varrho)}{2s} \right) + \\ &\quad + \frac{a_2}{4} \cdot \left(b(T\varrho, T\omega) + b(T\varrho, T^2\varrho), b(T\omega, T^2\omega) + \frac{b(T\varrho, T^2\omega) + b(T\omega, T^2\varrho)}{2s} \right) \\ &= \frac{a_1}{4} \cdot \left(b(\varrho, \omega) + \frac{b(\varrho, \omega)}{s} \right) + \frac{a_2}{4} \cdot \left(b(\varrho, \omega) + \frac{b(\varrho, \omega)}{s} \right) \\ &\leq \frac{(a_1 + a_2)}{4} b(\varrho, \omega) < b(\varrho, \omega) \end{aligned}$$

thus, ϱ, ω , which is a contradiction. $\square$

Example 3. Let $\mathcal{M} = [0, 2]$ and $b : \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$, where $b(x, y) = (x - y)^2$ is a b -metric with $s = 2$. Let the mapping $T : \mathcal{M} \rightarrow \mathcal{M}$ defined as $T\chi = \begin{cases} \frac{1}{4}, & \text{if } \chi \in [0, 1] \\ \frac{\chi+1}{2\sqrt{2\chi-1}}, & \text{if } \chi \in (1, 2] \end{cases}$. The mapping T is not continuous,

but on the other hand T^2 is continuous, since $T^2\chi = \frac{1}{4}$. Moreover, for any $\chi, y \in \mathcal{M}$ we have $b(T^2\chi, T^2y) = 0$ so that T fulfills the presumptions of Theorem 5 for any $a_1, a_2 \in [0, 1)$ with $a_1, a_2 < \frac{1}{4}$ and $q = \frac{1}{4}$ is the fixed point of T .

3. An Application: Ulam's Type Stability

In much recent research, the issue of stability (of the fixed point problem or the iteration procedure) occupies an important place. Let $(\mathcal{M}, b, s)$ be a complete b -metric space, a mapping $T : \mathcal{M} \rightarrow \mathcal{M}$ and the fixed point problem

$$\chi = T\chi. \quad (35)$$

The problem (35) is said to be

1. Ulam type stable if and only if there exist $a > 0$ such that for every $\varepsilon > 0$ and for each $y^* \in \mathcal{M}$ which satisfies the inequality

$$b(y^*, Ty^*) \leq \varepsilon \quad (36)$$

there exists a solution $q \in \mathcal{X}$ of (35) such that

$$b(q, y^*) \leq a \cdot \varepsilon. \quad (37)$$

2. well posed if the following assumptions are satisfy:

- (a) T has a unique fixed point q in $\mathcal{X}$;
- (b) $\lim_{n \rightarrow \infty} b(z_n, q) = 0$ for each sequence $\{z_n\} \in \mathcal{X}$ such that $\lim_{n \rightarrow \infty} b(z_n, Tz_n) = 0$.

Definition 2 ([9]). Let $(\mathcal{M}, b, s)$ be a complete b -metric space and $T : \mathcal{M} \rightarrow \mathcal{M}$ a mapping such that $F_{T^n}(\mathcal{M}) \neq \emptyset$. The Picard's iteration $\chi_{n+1} = T\chi_n$ started from a point $\chi_0 \in \mathcal{M}$ is called T -stable if $\chi_n \rightarrow q$ and for a sequence $\{z\}_n \in \mathcal{M}$ such that $\lim_{n \rightarrow \infty} b(z_{n+1}, Tz_n) = 0$ we have $\lim_{n \rightarrow \infty} b(z_n, q) = 0$

Before presenting the main result of this section, we recall here the next useful Lemma.

Lemma 3. Let $\{\xi\}_n, \{\eta_n\}$ be two sequences of nonnegative real numbers such that $\xi_{n+1} \leq \lambda\xi_n + \eta_n$ for $n \geq 0$ and $\lambda \in [0, 1)$. If $\lim_{n \rightarrow \infty} \eta_n = 0$ then $\lim_{n \rightarrow \infty} \xi_n = 0$.

Theorem 6. Let $(\mathcal{M}, b, s)$ be a b -complete metric space. and T be a b -hybrid contraction. Under the presumptions of Theorem 1:

- (i) the fixed point problem (35) is Ulam type stable if $r > 0$ and $c(r)[s^r\kappa_1 + \frac{\kappa_4}{2}(c(r) + 1)] < 1$, where $c(r) = \max\{1, 2^{r-1}\}$;
- (ii) the Picard's iteration is T -stable if $s^r\kappa_1 + \kappa_i s^r c(r)(s^r + 1) + \frac{\kappa_4}{2}(s^r + 1) < 1$ where $i = 1$ or $i = 2$.

Proof. Taking into account Theorem 1, we know that $F_T(\mathcal{M}) \neq \emptyset$, so let be $q \in F_T(\mathcal{M})$.

(i) Suppose that y^* is a solution of (36), which means that $b(y^*, Ty^*) \leq \varepsilon$ for $\varepsilon > 0$. We have

$$b(y^*, q) = b(y^*, Tq) \leq s[b(y^*, Ty^*) + b(Ty^*, Tq)]$$

and since T is a b -hybrid contraction, keeping in mind the property of function ψ we get

$$\begin{aligned} b(q, y^*) &\leq s[b(y^*, Ty^*) + \psi(\mathcal{P}_T^r(y^*, q))] < s[b(y^*, Ty^*) + \mathcal{P}_T^r(y^*, q)] = sb(y^*, Ty^*) + \\ &\quad + s[\kappa_1(b(y^*, q))^r + \kappa_2(b(y^*, Ty^*))^r + \kappa_3(b(q, Tq))^r + \kappa_4\left(\frac{b(q, Ty^*) + b(y^*, Tq)}{2s}\right)^r]^{1/r}, \\ &= sb(y^*, Ty^*) + s[\kappa_1(b(y^*, q))^r + \kappa_2(b(y^*, Ty^*))^r + \kappa_4\left(\frac{b(q, Ty^*) + b(y^*, Tq)}{2s}\right)^r]^{1/r}. \end{aligned}$$

By elementary calculations, the above inequality becomes

$$\begin{aligned} (b(q, y^*))^r &< c(r) \left[s^r (b(y^*, Ty^*))^r + s^r [\kappa_1 (b(y^*, q))^r + \kappa_2 (b(y^*, Ty^*))^r + \kappa_4 \left(\frac{b(q, Ty^*) + b(y^*, q)}{2s} \right)^r] \right] \\ &\leq c(r) \left[s^r (b(y^*, Ty^*))^r + s^r [\kappa_1 (b(y^*, q))^r + \kappa_2 (b(y^*, Ty^*))^r + \frac{\kappa_4}{s^r} \frac{(b(q, Ty^*))^r + (b(y^*, q))^r}{2}] \right] \\ &\leq c(r) \left[s^r (b(y^*, Ty^*))^r + s^r [\kappa_1 (b(y^*, q))^r + \kappa_2 (b(y^*, Ty^*))^r + \frac{\kappa_4}{s^r} \frac{s^r (b(q, y^*) + b(y^*, Ty^*))^r + (b(y^*, q))^r}{2}] \right] \\ &\leq c(r) \left[s^r (b(y^*, Ty^*))^r + s^r [\kappa_1 (b(y^*, q))^r + \kappa_2 (b(y^*, Ty^*))^r + \frac{\kappa_4}{s^r} \frac{s^r c(r) ((b(q, y^*))^r + (b(y^*, Ty^*))^r) + (b(y^*, q))^r}{2}] \right] \end{aligned}$$

Thus,

$$(b(q, y^*))^r (1 - s^r \kappa_1 \cdot c(r) - \frac{\kappa_4 (s^r c(r) + 1)}{2} c(r)) \leq c(r) (s^r + s^r \kappa_2 + \frac{s^r c(r) \cdot \kappa_4}{2}) (b(y^*, Ty^*))^r,$$

or,

$$b(q, y^*) < \left[\frac{c(r) (s^r + s^r \kappa_2 + \frac{s^r c(r) \cdot \kappa_4}{2})}{1 - s^r \kappa_1 \cdot c(r) - \frac{\kappa_4 (s^r c(r) + 1)}{2} c(r)} \right]^{1/r} b(y^*, Ty^*) \leq a \cdot \varepsilon$$

where $a = \left[\frac{c(r) (s^r + s^r \kappa_2 + \frac{s^r c(r) \cdot \kappa_4}{2})}{1 - s^r \kappa_1 \cdot c(r) - \frac{\kappa_4 (s^r c(r) + 1)}{2} c(r)} \right]^{1/r}$, wherefore, the fixed point problem (29) is (35) is Ulam's type stable.

(ii) Let $\{z_n\}$ be a sequence in $\mathcal{M}$ such that $\lim_{n \rightarrow \infty} b(z_{n+1}, Tz_n) = 0$. Inasmuch as T is a b -hybrid contraction we have

$$b(Tz_n, q) = b(Tz_n, Tq) \leq \psi(\mathcal{P}_T^r(z_n, q)), \quad (38)$$

where, for $r > 0$,

$$\begin{aligned} \mathcal{P}_T^r(z_n, q) &= [\kappa_1 (b(z_n, q))^r + \kappa_2 (b(z_n, Tz_n))^r + \kappa_3 (b(q, Tq))^r + \kappa_4 \left(\frac{b(q, Tz_n) + b(z_n, Tq)}{2s} \right)^r]^{1/r} \\ &= [\kappa_1 (b(z_n, q))^r + \kappa_2 (b(z_n, Tz_n))^r + \kappa_4 \left(\frac{b(q, Tz_n) + b(z_n, q)}{2s} \right)^r]^{1/r} \\ &\leq [\kappa_1 (b(z_n, q))^r + \kappa_2 (s(b(z_n, q) + b(q, Tz_n)))^r + \kappa_4 \left(\frac{b(q, Tz_n) + b(z_n, q)}{2s} \right)^r]^{1/r} \\ &\leq [\kappa_1 (b(z_n, q))^r + \kappa_2 s^r c(r) ((b(z_n, q))^r + (b(q, Tz_n))^r) + \kappa_4 \frac{(b(q, Tz_n))^r + (b(z_n, q))^r}{2s}]^{1/r}. \end{aligned}$$

Returning to (38) we get

$$(b(Tz_n, q))^r (1 - \kappa_2 s^r c(r) - \frac{\kappa_4}{2}) < (\kappa_1 + \kappa_2 s^r c(r) + \frac{\kappa_4}{2}) (b(z_n, q))^r$$

or,

$$(b(Tz_n, q))^r < \frac{\kappa_1 + \kappa_2 s^r c(r) + \frac{\kappa_4}{2}}{1 - \kappa_2 s^r c(r) - \frac{\kappa_4}{2}} (b(z_n, q))^r.$$

If we denote by $\lambda = \left(\frac{\kappa_1 + s^r c(r) \kappa_2 + \frac{\kappa_4}{2}}{1 - s^r c(r) \kappa_2 - \frac{\kappa_4}{2}} \right)^{1/r}$, as long as $\kappa_1 + 2s^r c(r) \kappa_2 + \kappa_4 \leq 1$ we have $0 \leq \lambda < 1$ and then

$$b(Tz_n, q) < \lambda b(z_n, q). \quad (39)$$

Of course, since $b(Tz_n, q) = b(q, Tz_n)$ using the alternative condition $\kappa_1 + 2s^r c(r) \kappa_3 + \kappa_4 \leq 1$ we obtain again (39). By using the triangular inequality,

$$b(z_{n+1}, q) \leq s [b(z_{n+1}, Tz_n) + b(Tz_n, q)] \leq s [b(z_{n+1}, Tz_n) + \lambda b(z_n, q)]$$

Denoting $\xi_n = b(z_n, \varrho)$, $\eta_n = sb(z_{n+1}, Tz_n)$, the above inequality becomes

$$\xi_{n+1} < c \cdot \xi_n + \eta_n.$$

Thus, since $\lim_{n \rightarrow \infty} \eta_n = 0$ and $c = s\lambda < 1$, taking into account Lemma 3 we get that $\lim_{n \rightarrow \infty} \xi_n = \lim_{n \rightarrow \infty} b(z_n, \varrho) = 0$ which proves that the Picard's iteration is T -stable. $\square$

An immediate consequence is presented in the following theorem (it is enough to take $s = 1$ in Theorem 6).

Theorem 7. Let $(\mathcal{M}, d)$ be complete metric space and $T : \mathcal{M} \rightarrow \mathcal{M}$. Under the presumptions of Corollary 1:

- (i) the fixed point problem (35) is Ullam's type stable if $r > 0$ and $c(r)(\kappa_1 + \frac{\kappa_4}{2}(c(r) + 1)) < 1$, where $c(r) = \max \{1, 2^{r-1}\}$;
- (ii) the Picard's iteration is T -stable if $\kappa_1 + 2c(r)\kappa_i + \kappa_4 < 1$ where $i = 1$ or $i = 2$.

4. Conclusions

The obtained results unify several existing results in a single theorem. We list some of the consequences but, it is clear that there are more consequences of our main results. Due to the length of the paper, we skip them.

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