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Linear Maps in Minimal Free Resolutions of Stanley-Reisner Rings

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Abstract: In this short note we give an elementary description of the linear part of the minimal free resolution of a Stanley-Reisner ring of a simplicial complex Δ. Indeed, the differentials in the linear part are simply a compilation of restriction maps in the simplicial cohomology of induced subcomplexes of Δ. Along the way, we also show that if a monomial ideal has at least one generator of degree 2, then the linear strand of its minimal free resolution can be written using only ± 1 coefficients.

Keywords: monomial ideal; Stanley-Reisner ring; linear part

MSC: Primary: 05E40; Secondary: 13D02, 13F55

1. Introduction

Let $\Bbbk$ be a field and $S = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring over it. Consider a finitely generated graded S-module M, and its minimal free resolution $\mathbb{F}_\bullet$. The linear part [1] $\operatorname{lin}(\mathbb{F}_\bullet)$ of $\mathbb{F}_\bullet$ has the same modules as $\mathbb{F}_\bullet$, and its differential d^{lin} is obtained from the differential d of $\mathbb{F}_\bullet$ by deleting all non-linear entries in the matrices representing d in some basis of $\mathbb{F}_\bullet$.

The main result of this short note is an explicit description of $\operatorname{lin}(\mathbb{F}_\bullet)$ in the case where $M = \Bbbk[\Delta]$ is the Stanley-Reisner ring of a simplicial complex Δ. It is well-known that $\mathbb{F}_\bullet$ is multigraded and generated as S-module in squarefree multidegrees. By Hochster's formula, it holds that

$$\operatorname{Tor}_i^S(\Bbbk[\Delta], \Bbbk)_U \cong \widetilde{H}^{\#U-i-1}(\Delta_U; \Bbbk),$$

where $U \subseteq \{1, \ldots, n\}$ is a squarefree multidgree and $\Delta_U := \{F \in \Delta : F \subseteq U\}$ is the restriction of Δ. Therefore, $\operatorname{lin}(\mathbb{F}_i)$ is isomorphic to the direct sum of modules of the form $\widetilde{H}^{\#U-i-1}(\Delta_U) \otimes_\Bbbk S(-U)$. The differential d^{lin} turns out to be simply a compilation of all the restriction maps $\widetilde{H}^i(\Delta_U) \to \widetilde{H}^i(\Delta_{U \setminus u}), \omega \mapsto \omega|_{U \setminus u}$, induced by the inclusions $\Delta_{U \setminus u} \subset \Delta_U$.

Theorem 1. *Let $\Bbbk[\Delta]$ be the Stanley-Reisner ring of a simplicial complex Δ and let $\mathbb{F}_\bullet$ denote its minimal free resolution. The linear part $\operatorname{lin}(\mathbb{F}_\bullet)$ of $\mathbb{F}_\bullet$ is isomorphic to the complex with modules*

$$\operatorname{lin}(\mathbb{F}_i) = \bigoplus_{U \subseteq [n]} \widetilde{H}^{\#U-i-1}(\Delta_U) \otimes_\Bbbk S(-U),$$

and the components of the differential are given by

$$\begin{array}{ccc} \widetilde{H}^j(\Delta_U) \otimes_\Bbbk S(-U) & \longrightarrow & \widetilde{H}^j(\Delta_{U \setminus u}) \otimes_\Bbbk S(-U \setminus u) \\ \omega \otimes s & \longmapsto & (-1)^{\alpha(u,U)} \omega|_{U \setminus u} \otimes x_u s \end{array}$$

This extends the result of Reiner and Welker ([2], Theorem 3.2), which describes the maps in the linear strand of $\mathbb{F}_\bullet$. An alternative description of $\mathrm{lin}(\mathbb{F}_\bullet)$ in terms of the Alexander dual of Δ was given by Yanagawa ([3], Theorem 4.1).

Example 1. *Let Δ be the simplicial complex with vertex set $\{a, b, c, d, e\}$ and facets $\{a, c, d\}$, $\{b, d, e\}$, $\{c, d, e\}$ and $\{b, c\}$. Its Stanley-Reisner ideal is $I_\Delta = \langle ab, ac, bcd, bce \rangle$. A minimal free resolution $\mathbb{F}_\bullet$ is given by the following complex:*

$$0 \leftarrow S \xleftarrow{\begin{array}{cc} & \begin{array}{cccc} ab & ac & bcd & bce \end{array} \\ 1 & \begin{pmatrix} ab & ac & bcd & bce \end{pmatrix} \end{array}} S^4 \xleftarrow{\begin{array}{cc} & \begin{array}{cccc} abc & abcd & abce & bcde \end{array} \\ \begin{array}{c} ab \\ ac \\ bcd \\ bce \end{array} & \begin{pmatrix} -\mathbf{c} & cd & ce & 0 \\ \mathbf{d} & 0 & 0 & 0 \\ 0 & -\mathbf{a} & \mathbf{0} & \mathbf{e} \\ 0 & \mathbf{0} & -\mathbf{a} & -\mathbf{d} \end{pmatrix} \end{array}} S^4 \xleftarrow{\begin{array}{cc} & abcde \\ \begin{array}{c} abc \\ abcd \\ abce \\ bcde \end{array} & \begin{pmatrix} 0 \\ \mathbf{e} \\ -\mathbf{d} \\ \mathbf{a} \end{pmatrix} \end{array}} S \leftarrow 0$$

The linear entries are marked in boldface. We indicate the relevant induced subcomplexes of Δ in Figure 1. There, the arrows indicate non-zero linear entries in the matrices of $\mathbb{F}_\bullet$. They correspond to non-zero restriction maps in the zero- or one-dimensional cohomology.

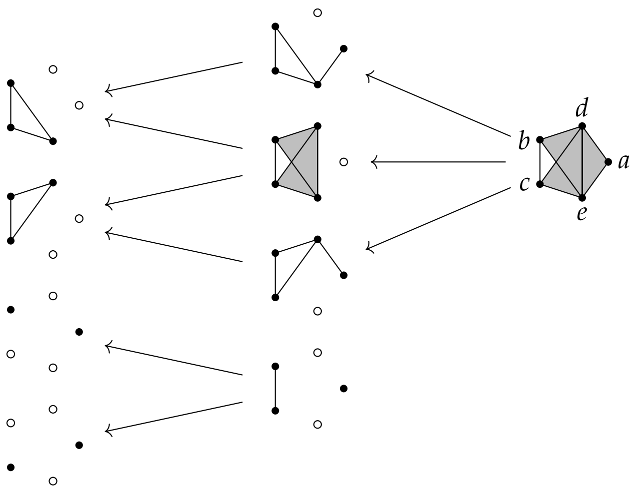

Figure 1. The induced subcomplexes of Δ from Example 1. The arrows indicate non-zero linear coefficients.

As a special case of Theorem 1, we obtain a very simple and explicit description of the 1-linear strand of $\mathbb{F}_\bullet$ (this is the strand containing the quadratic generators of I_Δ). In particular, we show that the maps in the 1-linear strand can always be written using only ± 1 coefficients, see Corollary 1. This extends and simplifies the results of Horwitz [4] and Chen [5], who constructed the minimal free resolution of I_Δ under the assumption that I_Δ is generated by quadrics and has a linear resolution.

This article is structured as follows. In Section 2 we set up notational conventions and recall various preliminaries. In the subsequent Section 3 we prove our main result. In the last section, we ask several open questions and pose a conjecture.

2. Notation and Preliminaries

For $n \in \mathbb{N}$ we write $[n] := \{1, \ldots, n\}$. To simplify the notation, we set $U \setminus u := U \setminus \{u\}$ and $U \cup u := U \cup \{u\}$ for $U \subseteq [n]$ and $u \in [n]$.

Throughout the paper let $\Bbbk$ denote a fixed field and $S = \Bbbk[x_1, \ldots, x_n]$. Further, we write

$$\mathfrak{m} := \langle x_1, \ldots, x_n \rangle$$

for the unique maximal graded ideal in S. We only consider the fine $\mathbb{Z}^n$-grading on S, i.e., the degree of x_i is the i-th unit vector in $\mathbb{Z}^n$. Squarefree multidegrees are identified with subsets of $[n]$. In particular, for $U \subseteq [n]$, we write $S(-U)$ for the free cyclic S-module whose generator is in degree U.

2.1. The Linear Part

Let M be a finitely generated graded S-module. We consider its minimal free resolution

$$\mathbb{F}_\bullet : 0 \longleftarrow M \longleftarrow \mathbb{F}_0 \xleftarrow{\mathrm{d}_1} \cdots \xleftarrow{\mathrm{d}_n} \mathbb{F}_n \longleftarrow 0.$$

There is a natural filtration on $\mathbb{F}_\bullet$, which is given by

$$\mathcal{F}^j(\mathbb{F}_i) := \mathfrak{m}^{j-i}\mathbb{F}_i.$$

The associated graded complex $\operatorname{lin}(\mathbb{F}_\bullet)$ is called the linear part of $\mathbb{F}_\bullet$. It was introduced in [1], but see also ([6], Chapter 5). Note that $\operatorname{lin}(\mathbb{F}_i) \cong \mathbb{F}_i$ as S-modules, but the differentials on the complexes are different. Indeed, $\operatorname{lin}(\mathbb{F}_\bullet)$ can be constructed alternatively by choosing a basis for $\mathbb{F}_\bullet$, representing its differential in this basis by matrices, and deleting all non-linear entries, that is, entries in $\mathfrak{m}^2$.

2.2. Simplicial Chains and Cochains

Let Δ be a simplicial complex with vertex set $[n]$. For the convenience of the reader, we recall the definitions of the chain and cochain complexes of Δ. For keeping track of the signs, we use the notation

$$\alpha(A, B) := \#\{(a, b) \in A \times B : \ a > b\}$$

for subsets $A, B \subseteq [n]$. We further set $\alpha(a, B) = \alpha(\{a\}, B)$. The (augmented oriented) chain complex of Δ is the complex of $\Bbbk$-vector spaces $\widetilde{C}_\bullet(\Delta)$, where $\widetilde{C}_d(\Delta)$ is the $\Bbbk$-vector space spanned by the d-faces of Δ, and the differential is given by

$$\partial(F) = \sum_{i \in F} (-1)^{\alpha(i,F)} F \setminus i.$$

Here, we consider the empty set as the unique face of dimension -1. Note that the definition of $\alpha(i, F)$ depends on the ordering of $[n]$. The *(augmented oriented) cochain complex* of Δ is the dual complex $\widetilde{C}^\bullet(\Delta) := \hom_\Bbbk(\widetilde{C}_\bullet(\Delta), \Bbbk)$. We write $F^* \in \widetilde{C}^d(\Delta)$ for the basis element dual to a d-face $F \in \Delta$. In this basis, the differential on $\widetilde{C}^\bullet(\Delta)$ can be written as

$$\partial(F^*) = \sum_{i \in [n] \setminus F} (-1)^{\alpha(i,F)} (F \cup i)^*.$$

Here, we adopt the convention that $(F \cup i)^* = 0$ if $F \cup i \notin \Delta$. The (reduced) simplicial cohomology of Δ is $\widetilde{H}^*(\Delta) := \widetilde{H}^*(\Delta; \Bbbk) := H^*(\widetilde{C}^\bullet(\Delta))$.

For a subcomplex $\Gamma \subseteq \Delta$, there is a restriction map $\widetilde{C}^\bullet(\Delta) \to \widetilde{C}^\bullet(\Gamma)$. If $\omega \in \widetilde{C}^\bullet(\Delta)$ is a cochain and $U \subseteq [n]$, then we write $\omega|_U$ for the restriction of ω to Δ_U.

3. Proof of the Main Result

Let Δ be a simplicial complex with vertex set $[n]$. Recall that the Stanley-Reisner ideal of Δ is defined as $I_\Delta := \langle \mathbf{x}^U : \ U \subseteq [n], U \notin \Delta \rangle$, where $\mathbf{x}^U := \prod_{i \in U} x_i$. Further, the Stanley-Reisner ring is $\Bbbk[\Delta] := S/I_\Delta$. Every squarefree monomial ideal arises as the Stanley-Reisner ideal of some simplicial complex ([7], Theorem 1.7).

We are going to need an explicit version of Hochster's formula. It is of course well known, but we give the details for the convenience of the reader. Let $V = \operatorname{span}_\Bbbk\{e_1, \ldots, e_n\}$ be an n-dimensional

$\Bbbk$-vector space and let $\Lambda^\bullet V$ denote the exterior algebra over it. For $F = \{i_1, \dots, i_r\} \subseteq [n]$ with $i_1 < \cdots < i_r$, we set $\mathbf{e}_F := e_{i_1} \wedge \cdots \wedge e_{i_r}$. Then $\Bbbk[\Delta] \otimes_\Bbbk \Lambda^\bullet V$ is the Koszul complex of $\Bbbk[\Delta]$.

Proposition 1 ([8]). *For each squarefree multidegree $U \subseteq [n]$, there is an isomorphism of complexes $(\Bbbk[\Delta] \otimes_\Bbbk \Lambda^\bullet V)_U \longrightarrow \widetilde{C}^{\#U-1-\bullet}(\Delta_U)$, given by $\boldsymbol{x}^F \otimes \mathbf{e}_{U\setminus F} \mapsto (-1)^{\alpha(F,U)} F^*$.*

Proof. It suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{x}^F \otimes \mathbf{e}_{U\setminus F} & \longrightarrow & \sum_{i\in U\setminus F} (-1)^{\alpha(i,U\setminus F)} \boldsymbol{x}^F x_i \otimes \mathbf{e}_{U\setminus(F\cup i)} \\ \downarrow & & \downarrow \\ (-1)^{\alpha(F,U)} F^* & \longrightarrow & (-1)^{\alpha(F,U)} \sum_{i\in U\setminus F} (-1)^{\alpha(i,F)} (F\cup i)^* \end{array}$$

We only need to show that $\alpha(F,U) + \alpha(i,F) \equiv \alpha(i, U\setminus F) + \alpha(F\cup i, U)$ modulo 2. This follows from the following computation:

$$\alpha(F\cup i, U) - \alpha(F,U) = \alpha(i,U) = \alpha(i,F) + \alpha(i, U\setminus F) \quad \square$$

Now we turn to the proof of Theorem 1, which we restate for convenience.

Theorem 2. *Let $\Bbbk[\Delta]$ be the Stanley-Reisner ring of a simplicial complex Δ and let $\mathbb{F}_\bullet$ denote its minimal free resolution. The linear part $\operatorname{lin}(\mathbb{F}_\bullet)$ of $\mathbb{F}_\bullet$ is isomorphic to the complex with modules*

$$\operatorname{lin}(\mathbb{F}_i) = \bigoplus_{U\subseteq[n]} \widetilde{H}^{\#U-i-1}(\Delta_U) \otimes_\Bbbk S(-U),$$

and the components of the differential are given by

$$\begin{array}{ccc} \widetilde{H}^j(\Delta_U) \otimes_\Bbbk S(-U) & \longrightarrow & \widetilde{H}^j(\Delta_{U\setminus u}) \otimes_\Bbbk S(-U\setminus u) \\ \omega \otimes s & \longmapsto & (-1)^{\alpha(u,U)} \omega|_{U\setminus u} \otimes x_u s \end{array}$$

Proof of Theorem 1. We follow the arguments of the proof of ([3], Theorem 4.1). Following [6] and ([1], pp. 107–109), we consider the double complex $(\mathcal{L}_{\bullet,\bullet}, \partial, \partial')$, whose modules are given by $\mathcal{L}_{a,b} := \Bbbk[\Delta] \otimes_\Bbbk \Lambda^a V \otimes_\Bbbk S_b$ and the differentials are:

$$\begin{aligned} \partial(s_1 \otimes \mathbf{e}_F \otimes s_2) &:= \sum_{i\in F} (-1)^{\alpha(i,F)} s_1 x_i \otimes \mathbf{e}_{F\setminus i} \otimes \quad s_2 \\ \partial'(s_1 \otimes \mathbf{e}_F \otimes s_2) &:= \sum_{i\in F} (-1)^{\alpha(i,F)} s_1 \quad \otimes \mathbf{e}_{F\setminus i} \otimes x_i s_2 \end{aligned}$$

It is not difficult to see that the homology of $(\mathcal{L}_{\bullet,\bullet}, \partial)$ is isomorphic to $\operatorname{Tor}^S_\bullet(\Bbbk[\Delta], \Bbbk) \otimes_\Bbbk S$. By ([6], Theorem 5.1), the linear part of the minimal free resolution is induced by ∂'.

Consider that the sub-double complex $\mathcal{L}'_{a,b} := \bigoplus_{\sigma\in[n]} (\Bbbk[\Delta] \otimes_\Bbbk \Lambda^a V \otimes_\Bbbk S_b)_\sigma$ of $\mathcal{L}_{\bullet,\bullet}$. As $\operatorname{Tor}^S_\bullet(\Bbbk[\Delta], \Bbbk)$ is non-zero in squarefree degrees only ([7], Corollary 1.40), both $\mathcal{L}'_{\bullet,\bullet}$ and $\mathcal{L}_{\bullet,\bullet}$ have the same homology with respect to ∂.

By Proposition 1, $(\mathcal{L}'_{\bullet,\bullet}, \partial)$ is isomorphic to $\bigoplus_{U\subseteq[n]} \widetilde{C}^{\#U-1-\bullet}(\Delta_U) \otimes_\Bbbk S(-U)$, where ∂' translates to the map:

$$\begin{aligned} \widetilde{C}^j(\Delta_U) \otimes_\Bbbk S(-U) &\longrightarrow \bigoplus_{u\in U} \widetilde{C}^j(\Delta_{U\setminus u}) \otimes_\Bbbk S(-U\setminus u) \\ F^* \otimes s &\longmapsto \sum_{u\in U} (-1)^{\alpha(u,F)} F^*|_{U\setminus u} \otimes x_u s \end{aligned}$$

Now the claim follows by taking homology with respect to ∂ and applying ([6], Theorem 5.1). □

A particularly simple case of Theorem 1 is the following. See Conjecture 1 for a conjectural improvement of this result.

Corollary 1. *Let $I \subseteq S$ be a monomial ideal and let $\mathbb{F}_\bullet$ be its minimal free resolution. Then one can choose a basis of $\mathbb{F}_\bullet$ such that the maps in its 2-linear strand have only coefficients in $\{-1, 0, 1\}$.*

Proof. We may assume that I is squarefree by replacing it with its polarization ([7], p. 44). So it is the Stanley-Reisner ideal of some simplicial complex Δ. By Theorem 1, maps in the 2-linear strand of its minimal free resolution are induced by the restriction maps $\widetilde{H}^0(\Delta_U) \to \widetilde{H}^0(\Delta_{U\setminus u})$ for each $u \in U$.

For each subset $U \subseteq [n]$ we choose a distinguished connected component $C_{U,0}$ of Δ_U. For each other connected component $C_{U,i}$ of it, let $e_{U,i} : U \to \Bbbk$ the function which is 1 on the vertices of $C_{U,i}$ and 0 on the others. It is clear that the set $\{e_{U,i} : i > 0\}$ forms a basis of $\widetilde{H}^0(\Delta_U)$.

We claim that in this basis, the differential has coefficients ± 1. For $i > 0$ there are the following cases:

1. $C_{U,i} = C_{U\setminus u,j}$ for some $j > 0$,
2. $C_{U,i} = C_{U\setminus u,0}$,
3. $C_{U,i}$ splits into several connected components $C_{U\setminus u,j_1}, \dots, C_{U\setminus u,j_r}$ of $\Delta_{U\setminus u}$ with $j_1, \dots, j_r > 0$,
4. same as (3), with $j_1 = 0$,
5. $C_{U,i}$ is the isolated vertex u.

In each case, it is easy to see that $e_{U,i}$ is mapped to a linear combination of the $e_{U\setminus u,j}$ with coefficients in $\{-1, 0, 1\}$. □

4. Questions and Open Problems

4.1. Affine Monoid Algebras

Recall that a (positive) affine monoid $Q \subseteq \mathbb{N}^n$ is a finitely generated submonoid of $\mathbb{N}^n$. The monoid algebras $\Bbbk[Q]$ of affine monoids form a well-studied class of algebras. We refer the reader to [7] or ([9], Chapter 6) for more information on these rings. Each positive affine monoid has a unique minimal generating set, which is called its Hilbert basis. It yields a set of generators for $\Bbbk[Q]$ and thus a surjection $S \to \Bbbk[Q]$ from a polynomial ring S. Moreover, $\Bbbk[Q]$ carries a natural $\mathbb{N}^n$-multigrading. There is a combinatorial interpretation of the multigraded Betti numbers of $\Bbbk[Q]$, namely $\mathrm{Tor}_i^S(\Bbbk[Q], \Bbbk)_{\mathbf{a}} \cong \widetilde{H}_i(\Delta_{\mathbf{a}})$, for a certain simplicial complex $\Delta_{\mathbf{a}}$, see ([7], Theorem 9.2).

Question 1. Is there a topological interpretation of the linear part of the minimal free resolution of $\Bbbk[Q]$ over S?

In this situation, a description along the lines of Theorem 1 would require a map $\widetilde{H}_i(\Delta_{\mathbf{a}}) \to \widetilde{H}_{i-1}(\Delta_{\mathbf{a}-\mathbf{b}})$, where $\mathbf{b}$ is an element of the Hilbert basis such that $\mathbf{a} - \mathbf{b} \in Q$. Here, $\Delta_{\mathbf{a}-\mathbf{b}}$ is a subcomplex of $\Delta_{\mathbf{a}}$, but in general it is neither a restriction nor a link.

4.2. Approximations of Resolutions

Let $I_\Delta \subseteq S$ be the Stanley-Reisner ideal of some simplicial complex Δ and let $\mathbb{F}_\bullet$ denote the minimal free resolution of I_Δ. Hochster's formula can be interpreted as giving a description of the complex $\mathbb{F}_\bullet/\mathfrak{m}\mathbb{F}_\bullet$ (with trivial differential). Our Theorem 1 extends this by (essentially) describing $\mathbb{F}_\bullet/\mathfrak{m}^2\mathbb{F}_\bullet$. These results can be considered as successive approximations of $\mathbb{F}_\bullet$, so the following question seems natural:

Question 2. Is there a combinatorial or topological description of $\mathbb{F}_\bullet/\mathfrak{m}^3\mathbb{F}_\bullet$?

This seems to be substantially more difficult than describing $\mathbb{F}_\bullet/\mathfrak{m}^2\mathbb{F}_\bullet$. One reason for this is the following. Even though a minimal free resolution is unique up to isomorphism, if one wants to write it down explicitly one needs to choose an S-basis for $\mathbb{F}_\bullet$. This choice can be done in two steps. First choose a $\mathbb{k}$-basis for $\mathbb{F}_\bullet/\mathfrak{m}\mathbb{F}_\bullet = \operatorname{Tor}_*(S/I_\Delta, \mathbb{k})$, and then choose a lifting of these elements to $\mathbb{F}_\bullet$ (any such lifting works due to Nakayama's lemma). Hochster's formula is a convenient tool for the first choice. Theorem 1 implies that the differential of $\mathbb{F}_\bullet/\mathfrak{m}^2\mathbb{F}_\bullet$ does not depend on the second choice, but this is no longer true for $\mathbb{F}_\bullet/\mathfrak{m}^3\mathbb{F}_\bullet$.

4.3. Coefficients in Resolutions

Let $I \subseteq S$ be a monomial ideal containing no variables, and let $\mathbb{F}_\bullet$ denote it with minimal free resolution. We saw in Corollary 1 that the differential in the 2-linear strand of $\mathbb{F}_\bullet$ can be written using only coefficients ± 1. On the other hand, in ([2], Section 5) Reiner and Welker gave an example where the differential on the 4-linear strand cannot be written using only coefficients ± 1. We believe that their example is optimal in that sense, and hence offer the following conjecture.

Conjecture 1. Let $I \subseteq S$ be a monomial ideal. Then it is possible to choose a basis for its minimal free resolution $\mathbb{F}_\bullet$, such that the differential on the 3-linear strand can be written using only coefficients ± 1.

Note that the first map in $\mathbb{F}_\bullet$, $\mathrm{d} : \mathbb{F}_1 \to \mathbb{F}_0$, can always be written using coefficients from $\{-1, 0, 1\}$. This is easily seen by considering the Taylor resolution. Further, it is not difficult to explicitly give a basis for $\mathbb{F}_2$ such that the differential $\mathrm{d} : \mathbb{F}_2 \to \mathbb{F}_1$ has coefficients ± 1.

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