Article

On \(q\)-Hermite-Hadamard Inequalities for Differentiable Convex Functions

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Abstract: In this paper, we establish some new results on the left-hand side of the \(q\)-Hermite–Hadamard inequality for differentiable convex functions with a critical point. Our work extends the results of Alp et. al (\(q\)-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, J. King Saud Univ. Sci., 2018, 30, 193-203), by considering the critical point-type inequalities.

Keywords: Hermite-Hadamard inequalities; \(q\)-derivative; \(q\)-integral; convex functions

1. Introduction

Quantum calculus (also known as \(q\)-calculus) is the study of calculus without limits, where classical mathematical formulas are obtained as \(q \to 1\). Firstly introduced by Euler (1707–1783) in the tracks of Newton’s infinite series, the study of \(q\)-calculus was established in the early Twentieth Century after the work of Jackson (1910) on defining an integral later known as the \(q\)-Jackson integral; see [1–4]. In \(q\)-calculus, the classical derivative is replaced by the \(q\)-difference operator in order to deal with non-differentiable functions; see [5,6] for more details. Applications of \(q\)-calculus can be found in various fields of mathematics and physics, and the interested readers are referred to [7–10].

The theory of convex functions has been widely studied and applied to various fields of science. Due to its close relation to the theory of inequalities, a rich literature on inequalities can be found in the study of convex functions; see [11–18]. This includes the Hermite–Hadamard inequality, introduced by Hermite and Hadamard independently, which has been studied extensively in recent years.

Let \(J \subseteq \mathbb{R}\) be an interval and \(f : J \to \mathbb{R}\) be a function from \(J\) to \(\mathbb{R}\). Recall that \(f\) is said to be a convex function if it satisfies the inequality:

\[ f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad (1) \]

for all \(x, y \in J\) and \(\lambda \in [0, 1]\). In addition, if an equality holds for all \(x, y \in J\) and \(\lambda \in [0, 1]\), then \(f\) is said to be affine.

It is also well known that \(f\) is convex if and only if it satisfies the Hermite–Hadamard inequality, which is defined by:

\[ f(\frac{a + b}{2}) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}, \quad (2) \]
for all \( a, b \in I \) and \( a < b \). One can estimate by the right-hand side of (2) by using Iyengar’s inequality, which is defined by:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{M(b - a)}{4} - \frac{1}{4M(b - a)}(f(b) - f(a))^2,
\]

where \( M \) denotes the Lipschitz constant, that is \( M = \sup \{ \frac{|f(x) - f(y)|}{|x - y|}; x \neq y \} \).

This fundamental result of Hermite and Hadamard has attracted many mathematicians, and consequently, this inequality has been generalized and extended in many directions; see [19–31] and the references cited therein.

In 2018, Alp et al. [32] studied the \( q \)-analogue of Hermite–Hadamard’s inequality for increasing functions, that is,

\[
f \left( \frac{qa + b}{1 + q} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dq x \leq \frac{qf(a) + f(b)}{1 + q},
\]

where \( q \) is a constant with \( 0 < q < 1 \). Moreover, they studied the generalized \( q \)-Hermite–Hadamard inequality for differentiable convex functions, that is,

\[
\max\{ I_1, I_2, I_3 \} \leq \frac{1}{b - a} \int_a^b f(x) \, dq x \leq \frac{qf(a) + f(b)}{1 + q},
\]

where:

\[
I_1 = f \left( \frac{qa + b}{1 + q} \right),
\]

\[
I_2 = f \left( \frac{a + qb}{1 + q} \right) + \frac{(1 - q)(b - a)}{1 + q} f' \left( \frac{a + qb}{1 + q} \right),
\]

\[
I_3 = f \left( \frac{a + b}{2} \right) + \frac{(1 - q)(b - a)}{2(1 + q)} f' \left( \frac{a + b}{2} \right).
\]

This paper aims to establish the generalized \( q \)-Hermite–Hadamard inequality for differentiable convex functions with a critical point.

The paper is organized as follows. Some basic concepts are recalled in Section 2. Section 3 contains the main results, while conclusions are given in Section 4.

2. Preliminaries

In this section, some basic results are mentioned. Throughout this section, we let \( I = [a, b] \subseteq \mathbb{R} \) be an interval and \( q \) be a constant with \( 0 < q < 1 \).

**Definition 1.** [33] The \( q \)-derivative of a continuous function \( f : I \to \mathbb{R} \) at \( x \) is defined as:

\[
a D_q f(x) = \frac{f(x) - f(qx + (1 - q)a)}{(1 - q)(x - a)}, \text{ for } x \neq a.
\]

For \( x = a \), we define \( a D_q f(a) = \lim_{x \to a} a D_q f(x) \).

If \( a D_q f(x) \) exists for all \( x \in J \), then \( f \) is \( q \)-differentiable on \( J \). Moreover, if \( a = 0 \), then (5) reduces to \( 0 D_q f = D_q f \), where \( D_q \) is the \( q \)-derivative of \( f \), which is defined by:

\[
D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}.
\]

For more details, see [4].

The higher-order \( q \)-derivatives of functions on \( J \) are also defined.
Theorem 1. Assume that the function $f : J \to \mathbb{R}$ is $q$-differentiable on $J$, denoted by $aD_q f$ and defined by:

$$aD_q^2 f = aD_q(aD_q f).$$

Similarly, provided that $aD_q^{n-1} f$ is the $q$-derivative on $J$ for some integer $n > 2$, the $n^{th}$-order $q$-derivative of $f$ on $J$ is the function from $J \to \mathbb{R}$ defined by:

$$aD_q^n f = aD_q(aD_q^{n-1} f).$$

Example 1. Let $f : J \to \mathbb{R}$ with $f(x) = x^2 + 1$. Let $q$ be a constant with $0 < q < 1$. Then, for $x \neq a$, we have:

$$aD_q(x^2 + 1) = \frac{(x^2+1)-[(qx+(1-q)a)^2+1]}{(1-q)(x-a)}$$

$$= \frac{(1+q)x^2-2qa(x-(1-q)a)}{(x-a)}$$

$$= (1+q)x + (1-q)a. \quad (6)$$

For $x = a$, $aD_q f(a) = \lim_{x \to a} aD_q f(x) = 2a$.

Definition 3. [33] The $q$-integral of a continuous function $f : J \to \mathbb{R}$ is defined as:

$$\int_a^x f(t) a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \quad (7)$$

for $x \in J$.

Note that if $a = 0$, then (7) becomes the classical $q$-integral of $f$, that is,

$$\int_a^x f(t) a d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x)$$

for $x \in [0, \infty)$; see [4] for more details.

Example 2. Let $f : [a, b] \to \mathbb{R}$ with $f(x) = 2x$. Let $q$ be a constant with $0 < q < 1$. Then, we have:

$$\int_a^b f(x) a d_q x = \int_a^b 2x a d_q x$$

$$= 2(1-q)(b-a) \sum_{n=0}^{\infty} q^n (q^n b + (1-q^n)a)$$

$$= \frac{2(b-a)(b+qa)}{1+q}.$$

Note that if $q \to 1$, we obtain the classical integration:

$$\int_a^b f(x) dx = \int_a^b 2x dx = b^2 - a^2.$$

Theorem 1. Assume that the function $f : J \to \mathbb{R}$ is continuous. Then, we have the following:

(i) $aD_q \int_a^x f(t) a d_q t = f(x) - f(a)$;

(ii) $\int_a^x aD_q f(t) a d_q t = f(x) - f(c)$ for $c \in (a, x)$.

Theorem 2. Assume that the functions $f, g : J \to \mathbb{R}$ are continuous and $a \in \mathbb{R}$. Then, we have the following:

(i) $\int_a^x [f(t) + g(t)] a d_q t = \int_a^x f(t) a d_q t + \int_a^x g(t) a d_q t$;

(ii) $\int_a^x (af)(t) a d_q t = a \int_a^x f(t) a d_q t$;
(iii)  \[ \int_{c}^{x} f(t) \, aD_{q} g(t) \, adt = (fg)|_{c}^{x} - \int_{c}^{x} g(qt + (1 - q)a) \, aD_{q} f(t) \, adt \] for \( c \in (a, x) \).

For the proofs of the properties in Theorems 1 and 2, see [34].

3. Main Results

In this section, we present our main results on the left-hand side of the \( q \)-Hermite–Hadamard inequality for differentiable convex functions with a critical point.

**Theorem 3.** Suppose that \( f : [a, b] \to \mathbb{R} \) is a differentiable convex function on \( (a, b) \) such that \( f'(c) = 0 \) for \( c \in (a, b) \), and let \( q \) be a constant with \( 0 < q < 1 \). Then, we have:

\[
f \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) + f' \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) \left( x - \frac{q(a + c) + (1 - q)b}{1 + q} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, adx.
\]

**Proof.** Since the function \( f \) is differentiable on \( (a, b) \), there exists a tangent line at the point \( \left( \frac{q(a + c) + (1 - q)b}{1 + q}, f \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) \right) \in (a, b) \), given by:

\[
h(x) = f \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) + f' \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) \left( x - \frac{q(a + c) + (1 - q)b}{1 + q} \right).
\]

Since \( f \) is a convex function on \( [a, b] \), it follows that \( h(x) \leq f(x) \) for all \( x \in [a, b] \). After \( q \)-integrating of (9) on \( [a, b] \), we have:

\[
\int_{a}^{b} h(x) \, adx = \int_{a}^{b} \left[ f \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) + f' \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) \left( x - \frac{q(a + c) + (1 - q)b}{1 + q} \right) \right] \, adx
\]

\[
= \left( b - a \right) f \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) + f' \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) \left( \int_{a}^{b} x \, adx - \left( b - a \right) \frac{q(a + c) + (1 - q)b}{1 + q} \right)
\]

\[
= \left( b - a \right) f \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) + f' \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) \left( \frac{b^{2} - a^{2}}{2} - \left( b - a \right) \frac{q(a + c) + (1 - q)b}{1 + q} \right)
\]

\[
= \left( b - a \right) f \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) + f' \left( \frac{q(a + c) + (1 - q)b}{1 + q} \right) \left( \frac{b - c}{1 + q} \right)
\]

\[
\leq \frac{1}{b - a} \int_{a}^{b} f(x) \, adx.
\]

On the other hand, since \( f \) is a convex function, we obtain:

\[
\frac{1}{b - a} \int_{a}^{b} f(x) \, adx = \left( 1 - q \right) \sum_{n=0}^{\infty} q^{n} f(q^{n}b + (1 - q^{n})a)
\]

\[
= (1 - q) \sum_{n=0}^{\infty} q^{n} f(q^{n}b + (1 - q^{n})a)
\]

\[
\leq (1 - q) \sum_{n=0}^{\infty} q^{n} \left[ q^{n} f(b) + (1 - q^{n}) f(a) \right]
\]

\[
= (1 - q) f(b) + f(a) - f(a) \sum_{n=0}^{\infty} q^{n}
\]

\[
= q f(a) + f(b) \frac{1}{1 + q}.
\]

The proof is complete. \( \Box \)
Theorem 3 as:

Let \( f \) be a differentiable convex function on \( (a, b) \) such that \( f' \left( \frac{a+b}{2} \right) = 0 \), for \( 0 < q < 1 \). Then, we have:

\[
f \left( \frac{q(a+c)+(1-q)b}{1+q} \right) + f' \left( \frac{q(a+c)+(1-q)b}{1+q} \right) \frac{q(b-a)}{2(1+q)} \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{qf(a) + f(b)}{1+q},
\]

since \( f' \left( \frac{q(a+c)+(1-q)b}{1+q} \right) \leq 0 \).

Remark 1. In Theorem 3, if \( q \in (0, \frac{c-b}{a-b}] \), then \( \frac{q(a+c)+(1-q)b}{1+q} \in [c, b) \). We can reduce the left-hand side of Theorem 3 as:

\[
f \left( \frac{q(a+c)+(1-q)b}{1+q} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{qf(a) + f(b)}{1+q},
\]

since \( f' \left( \frac{q(a+c)+(1-q)b}{1+q} \right) \geq 0 \).

\[
\begin{align*}
\text{Corollary 1.} & \text{ Assume that } f : [a, b] \to \mathbb{R} \text{ is a differentiable convex function on } (a, b) \text{ such that } f' \left( \frac{a+b}{2} \right) = 0, \\
& \text{for } 0 < q < 1. \text{ Then, we have:}
\end{align*}
\]

\[
f \left( \frac{q(a+a)+b(1-q)b}{1+q} \right) + f' \left( \frac{q(a+a)+(1-q)b}{1+q} \right) \frac{q(b-a)}{2(1+q)} \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{qf(a) + f(b)}{1+q}.
\]

\[
\begin{align*}
\text{Corollary 2.} & \text{ Assume that } f : [a, b] \to \mathbb{R} \text{ is a differentiable convex function on } (a, b) \text{ such that } f' (0) = 0, \\
& \text{for } 0 \in (a, b) \text{ and } 0 < q < 1. \text{ Then, we have:}
\end{align*}
\]

\[
f \left( \frac{q(a+1)b}{1+q} \right) + f' \left( \frac{q(a+1)b}{1+q} \right) \frac{qb}{(1+q)} \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{qf(a) + f(b)}{1+q}.
\]

\[
\begin{align*}
\text{Theorem 4.} & \text{ Let } f : [a, b] \to \mathbb{R} \text{ be a differentiable convex function on } (a, b) \text{ such that } f' (c) = 0 \text{ for } c \in (a, b) \text{ and } 0 < q < 1. \text{ Then, we have:}
\end{align*}
\]

\[
f \left( \frac{(1-q)a+q(c+b)}{1+q} \right) + f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( \frac{q(b-a-c)+b-a}{1+q} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{qf(a) + f(b)}{1+q}.
\]

\[
\text{Proof.} \text{ Since the function } f \text{ is differentiable on } (a, b), \text{ there exists a tangent line at the point}
\]

\[
\frac{(1-q)a+q(c+b)}{1+q} \in (a, b), \text{ which is given by:}
\]

\[
k(x) = f \left( \frac{(1-q)a+q(c+b)}{1+q} \right) + f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( x - \frac{(1-q)a+q(c+b)}{1+q} \right).
\]
Since $f$ is convex on $[a, b]$, it follows that $k(x) \leq f(x)$ for all $x \in [a, b]$. After $q$-integrating (13), we obtain:

$$\int_{a}^{b} k(x) \, a_{d}q_{x} = \int_{a}^{b} [f \left( \frac{(1-q)a+q(c+b)}{1+q} \right) + f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( x - \frac{(1-q)a+q(c+b)}{1+q} \right)] \, a_{d}q_{x}$$

$$= (b - a) f \left( \frac{(1-q)a+q(c+b)}{1+q} \right) + f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( \int_{a}^{b} x \, a_{d}q_{x} - (b - a) \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \right)$$

$$= (b - a) f \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( \frac{b}{1+q} - \frac{(1-q)a+q(c+b)}{1+q} \right)$$

$$+ f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( \frac{b}{1+q} \right) - f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( \frac{a}{1+q} \right) - (b - a) \left( \frac{(1-q)a+q(c+b)}{1+q} \right)$$

$$\leq \int_{a}^{b} f(x) \, a_{d}q_{x}.$$

The proof is complete. \(\square\)

**Remark 3.** In Theorem 4, if $q \in (\frac{1}{2}, \frac{a-b}{b-a}]$, then $f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \leq 0$ and $q(2a-b-c)+b-a < 0$. We can reduce the left-hand side of Theorem 4 as:

$$f \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \leq \frac{1}{(b-a)} \int_{a}^{b} f(x) \, a_{d}q_{x} \leq \frac{q f(a) + f(b)}{1+q},$$

since $f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( \frac{q(2a-b-c)+b-a}{1+q} \right) \geq 0$.

**Remark 4.** In Remark 3, if $c \to b$, then $q \to 1^-$. Since $q \in (\frac{1}{3}, 1)$, we have $\frac{(1-q)a+q(b+c)}{1+q} \in (\frac{a+2b}{3}, b)$. We can reduce the left-hand side of Theorem 4 as:

$$f \left( (1-q) \left( \frac{2a+b}{3} \right) + q(b) \right) \leq \frac{1}{(b-a)} \int_{a}^{b} f(x) \, a_{d}q_{x} \leq \frac{q f(a) + f(b)}{1+q},$$

since $f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( \frac{q(2a-b-c)+b-a}{1+q} \right) \geq 0$.

**Theorem 5.** [Generalized $q$-Hermite–Hadamard inequality for convex differentiable functions]. Let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function on $(a, b)$ such that $f'(c) = 0$ for $c \in (a, b)$ and $0 < q < 1$. Then, we have:

$$\max \{ I_1, I_2 \} \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, a_{d}q_{x} \leq \frac{q f(a) + f(b)}{1+q}, \quad (15)$$

where

$$I_1 = f \left( \frac{q(a+c)+(1-q)b}{1+q} \right) + f' \left( \frac{q(a+c)+(1-q)b}{1+q} \right) \left( \frac{q(b-c)}{1+q} \right),$$

$$I_2 = f \left( \frac{(1-q)a+q(c+b)}{1+q} \right) + f' \left( \frac{(1-q)a+q(c+b)}{1+q} \right) \left( \frac{q(2a-b-c)+b-a}{1+q} \right).$$

**Proof.** A combination of (8) and (12) yields (15). Thus, the proof is complete. \(\square\)
Example 3. Define the function \( f(x) = x^2 \) on \([-1,3]\), and let \( q \in (0,1) \). Applying Theorem 3 with \( a = -1 \), \( b = 3 \), and \( c = 0 \), the left-hand side becomes:

\[
\begin{align*}
  f \left( \frac{q(a+c) + (1-q)b}{1+q} \right) + f' \left( \frac{q(a+c) + (1-q)b}{1+q} \right) \frac{q(b-c)}{1+q} - \frac{1}{b-a} \int_a^b f(x) \, dqx \\
  = f \left( \frac{3-4q}{1+q} \right) + f' \left( \frac{3-4q}{1+q} \right) \left( \frac{3q}{1+q} \right) - \frac{1}{4} \left[ 4(1-q) \sum_{n=0}^{\infty} q^n f(3q^n - (1-q^n)) \right] \\
  = -9q^4 - 9q^3 - 9q^2 - 16q \\
  (1+q)^2(1+q+q^2) \\
  \leq 0.
\end{align*}
\]

For the right-hand side, we have:

\[
\begin{align*}
  \frac{1}{3-(-1)} \int_{-1}^3 x^2 \, dqx - qf(-1) + f(3) = \frac{16}{1+q+q^2} - \frac{8}{1+q} + 1 - \frac{9+q}{1+q} \leq 0.
\end{align*}
\]

Example 4. Define function \( f(x) = x^2 \) on \([-1,1]\), and let \( q \in (0,1) \). Applying Corollary 2 with \( a, b = -1 \) and \( c = 0 \), the left-hand side becomes:

\[
\begin{align*}
  f \left( \frac{qa + (1-q)b}{1+q} \right) + f' \left( \frac{qa + (1-q)b}{1+q} \right) \frac{q(b-c)}{1+q} - \frac{1}{b-a} \int_a^b f(x) \, dqx \\
  = f \left( \frac{1-2q}{1+q} \right) + f' \left( \frac{1-2q}{1+q} \right) \left( \frac{q}{1+q} \right) - (1-q) \sum_{n=0}^{\infty} q^n f(2q^n - 1) \\
  = \frac{4q^2 - 4q + 1}{(1+q)^2} + \frac{2q(1-2q)}{(1+q)^2} - \frac{1+2q-2q^2+q^3}{(1+q+q^2)(1+q)} \leq 0.
\end{align*}
\]

For the right-hand side, we have:

\[
\begin{align*}
  \frac{1}{1-(-1)} \int_{-1}^1 x^2 \, dqx - qf(-1) + f(1) = \frac{1}{2} \left[ (1-q)^2 \sum_{n=0}^{\infty} q^n f(q^n - (1-q^n)) \right] - \frac{1+q}{1+q} \\
  = \frac{4}{1+q+q^2} - \frac{4}{1+q} + 1 - 1 \leq 0.
\end{align*}
\]

Example 5. Define functions \( f(x) = x^2 \) on \([-3,1]\), and let \( q \in (0,1) \). Applying Theorem 4 with \( a = -3 \), \( b = 1 \), and \( c = 0 \), the left-hand side becomes:

\[
\begin{align*}
  f \left( \frac{(1-q)a + q(c+b)}{1+q} \right) + f' \left( \frac{(1-q)a + q(c+b)}{1+q} \right) \left( \frac{q(2a-b-c) + b-a}{1+q} \right) - \frac{1}{b-a} \int_a^b f(x) \, dqx \\
  = f \left( \frac{4q-3}{1+q} \right) + f' \left( \frac{4q-3}{1+q} \right) \left( \frac{4-7q}{1+q} \right) - \frac{1}{4} \left[ 4(1-q) \sum_{n=0}^{\infty} q^n f(4q^n - 3) \right] \\
  = \frac{16q^2 - 24q + 9}{(1+q)^2} + \frac{-56q^2 + 74q - 24}{(1+q)^2} - \frac{16}{1+q+q^2} + \frac{24}{1+q} - 9 \leq 0.
\end{align*}
\]
For the right-hand side, we have:

\[
\frac{1}{3 - (-1)} \int_{-1}^{-3} x^2 d_q x - \frac{q f(-3) + f(1)}{1 + q} = \frac{16}{1 + q + q^2} - \frac{24}{1 + q} + 9 - \frac{9q + 1}{1 + q} \leq 0.
\]

4. Conclusions

In this paper, we considered and investigated the class of differentiable convex functions, which has a critical point in the setting of \(q\)-calculus. We used the approach of \(q\)-calculus to derive some new results on the left-hand side of \(q\)-Hermite–Hadamard inequalities. It is expected that the ideas and techniques presented in this paper will stimulate further research in this field.

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