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Compatible Algebras with Straightening Laws on Distributive Lattices

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Abstract: We characterize the finite distributive lattices on which there exists a unique compatible algebra with straightening laws.

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1. Introduction

Let P be a finite partially ordered set (poset for short) and $\mathcal{I}(P)$ the distributive lattice of the poset ideals of P. A subset α of P is a poset ideal of P if it satisfies the following condition: for every $x \in \alpha$ and $y \in P$, if $y \leq x$, then $y \in \alpha$. By a famous theorem of Birkhoff [1], for every finite distributive lattice L, there exists a unique subposet P of L such that $L \cong \mathcal{I}(P)$. The order polytope $\mathcal{O}(P)$ and the chain polytope $\mathcal{C}(P)$ were introduced in [2]. In [3], it was shown that the toric ring $K[\mathcal{O}(P)]$ over a field K is an algebra with straightening laws (ASL in brief) on the distributive lattice $\mathcal{I}(P)$ over the field K. In [4], it was shown that the ring $K[\mathcal{C}(P)]$ associated with the chain polytope shares the same property.

Let $S = K[x_1, \ldots, x_n, t]$ be the polynomial ring over a field K and $\{w_\alpha\}_{\alpha \in \mathcal{I}(P)}$ be an arbitrary set of monomials in $x_1, \ldots, x_n$ indexed by $\mathcal{I}(P)$. Let $K[\Omega] \subset S$ be the toric ring generated over K by the set of monomials $\Omega = \{\omega_\alpha\}_{\alpha \in \mathcal{I}(P)}$ where $\omega_\alpha = w_\alpha t$ for all $\alpha \in \mathcal{I}(P)$. Clearly, $K[\Omega]$ is a graded algebra if we set $\deg(\omega_\alpha) = 1$ for all $\alpha \in \mathcal{I}(P)$. Let $\varphi : \mathcal{I}(P) \to K[\Omega]$ be the injective map defined by $\varphi(\alpha) = \omega_\alpha$ for all $\alpha \in \mathcal{I}(P)$. Assume that $K[\Omega]$ is an ASL on $\mathcal{I}(P)$ over K. According to [4], $K[\Omega]$ is a *compatible* ASL if each of its straightening relations is of the form $\varphi(\alpha)\varphi(\alpha') = \varphi(\beta)\varphi(\beta')$ with $\beta \subseteq \alpha \cap \alpha'$ and $\beta' \supseteq \alpha \cup \alpha'$, where α, α' are incomparable elements in $\mathcal{I}(P)$. If $K[\Omega]$ and $K[\Omega']$ are compatible ASL on $\mathcal{I}(P)$ over K, we identify them if they have the same straightening relations. In this case, we write $K[\Omega] \equiv K[\Omega']$.

In ([4], Question 5.1), Hibi and Li asked the following questions:

(a) Given a finite poset P, find all possible compatible algebras with straightening laws on $\mathcal{I}(P)$ over K.
(b) For which posets P, does there exist a unique compatible ASL on $\mathcal{I}(P)$ over K?

In this note, we give a complete answer to question (b). Namely, we prove the following:

Theorem 1. *Let P be a finite poset. Then, the following statements are equivalent:*

(i) *There exists a unique compatible ASL on $\mathcal{I}(P)$ over K.*
(ii) $K[\mathcal{O}(P)] \equiv K[\mathcal{C}(P)] \equiv K[\mathcal{C}(P^*)]$, *where P^* denotes the dual poset of P.*
(iii) *Each connected component of P is a chain, that is, P is a direct sum of chains.*

An answer to question (a) seems to be quite difficult. In ([4], Example 5.2), it was observed that, if one considers the poset $P = \{a, b, c, d, e\}$ with $a < c < e$ and $b < c < d$, then there exist nine compatible ASL structures on $\mathcal{I}(P)$ over K, while if one considers $P = \{a, b, c, d\}$ with $a < c, b < c, b < d$, then there are three compatible ASL structures on $\mathcal{I}(P)$ over K, namely, $K[\mathcal{O}(P)]$, $K[\mathcal{C}(P)]$, and $K[\mathcal{C}(P^*)]$.

2. Order Polytopes, Chain Polytopes, and Their Associated Toric Rings

Let $P = \{p_1, \ldots, p_n\}$ be a finite poset. For the basic terminology regarding posets used in this paper, we refer to [1] and ([5], Chapter 3). The order polytope $\mathcal{O}(P)$ is defined as

$$\mathcal{O}(P) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, 1 \leq i \leq n, \text{ and } x_i \geq x_j \text{ if } p_i \leq p_j \text{ in } P\}.$$

In ([2], Corollary 1.3), it was shown that the vertices of $\mathcal{O}(P)$ are $\sum_{p_i \in \alpha} \mathbf{e}_i, \alpha \in \mathcal{I}(P)$. Here, $\mathbf{e}_i$ denotes the unit coordinate vector in $\mathbb{R}^n$. If $\alpha = \emptyset$, then the corresponding vertex in $\mathcal{O}(P)$ is the origin of $\mathbb{R}^n$.

The chain polytope $\mathcal{C}(P)$ is defined as

$$\mathcal{C}(P) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n,$$

$$x_{i_1} + \cdots + x_{i_r} \leq 1 \text{ if } p_{i_1} < \cdots < p_{i_r} \text{is a maximal chain in } P\}.$$

In ([2], Theorem 2.2), it was proved that the vertices of $\mathcal{C}(P)$ are $\sum_{p_i \in A} \mathbf{e}_i$, where A is an antichain in P. Recall that an antichain in P is a subset of P such that any two distinct elements in the subset are incomparable. Since every poset ideal is uniquely determined by its antichain of maximal elements, it follows that $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have the same number of vertices. However, as it was observed in [2], $\mathcal{O}(P)$ and $\mathcal{C}(P)$ need not have the same number of i-dimensional faces for $i > 0$. Therefore, in general, they are not combinatorial equivalent. Combinatorially, equivalence of order and chain polytopes are studied in [6].

The Toric Rings $K[\mathcal{O}(P)]$ and $K[\mathcal{C}(P)]$

To each subset $W \subset P$, we attach the squarefree monomial $u_W \in K[x_1, \ldots, x_n]$, $u_W = \prod_{p_i \in W} x_i$. If $W = \emptyset$, then $u_W = 1$. The toric ring $K[\mathcal{O}(P)]$, known as the Hibi ring associated with the distributive lattice $\mathcal{I}(P)$, is generated over K by all the monomials $u_\alpha t \in S$, where $\alpha \in \mathcal{I}(P)$. The toric ring $K[\mathcal{C}(P)]$ is generated by all the monomials $u_A t$ where A is an antichain in P. In addition, as we have already mentioned in the Introduction, both rings are algebras with straightening laws on $\mathcal{I}(P)$ over K.

We recall the definition of an ASL as it was introduced in [7]. For a quick introduction to this topic, we refer to [7] and ([8], Chapter XIII). Algebras with straightening laws turned out to be useful tools in studying determinantal rings. Let K be a field, $R = \bigoplus_{i \geq 0} R_i$ with $R_0 = K$ be a graded K-algebra, H a finite poset, and $\varphi : H \to R$ an injective map which maps each $\alpha \in H$ to a homogeneous element $\varphi(\alpha) \in R$ with $\deg \varphi(\alpha) \geq 1$. A *standard monomial* in R is a product $\varphi(\alpha_1)\varphi(\alpha_2) \cdots \varphi(\alpha_k)$ where $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$ in H.

Definition 1. *The K-algebra R is called an algebra with straightening laws on H over K if the following conditions hold:*

(1) The set of standard monomials is a K–basis of R;
(2) If $\alpha, \beta \in H$ are incomparable and if $\varphi(\alpha)\varphi(\beta) = \sum c_i \varphi(\gamma_{i1}) \ldots \varphi(\gamma_{ik_i})$, where $c_i \in K \setminus \{0\}$ and $\gamma_{i1} \leq \ldots \leq \gamma_{ik_i}$, is the unique expression of $\varphi(\alpha)\varphi(\beta)$ as a linear combination of standard monomials, then $\gamma_{i1} \leq \alpha, \beta$ for all i.

The above relations $\varphi(\alpha)\varphi(\beta) = \sum c_i \varphi(\gamma_{i1}) \ldots \varphi(\gamma_{ik_i})$ are called the straightening relations of R and they generate all the relations of of R.

Let us go back to the toric rings $K[\mathcal{O}(P)]$ and $K[\mathcal{C}(P)]$.

One considers $\varphi : \mathcal{I}(P) \to K[\mathcal{O}(P)]$ defined by $\varphi(\alpha) = u_\alpha t$ for every $\alpha \in \mathcal{I}(P)$. As it was proved by Hibi in [3], $K[\mathcal{O}(P)]$ is an ASL on $\mathcal{I}(P)$ over K with the straightening relations $\varphi(\alpha)\varphi(\beta) = \varphi(\alpha \cap \beta)\varphi(\alpha \cup \beta)$, where α, β are incomparable elements in $\mathcal{I}(P)$.

On the other hand, one defines $\psi : \mathcal{I}(P) \to K[\mathcal{C}(P)]$ by setting $\psi(\alpha) = u_{\max \alpha} t$ for all $\alpha \in \mathcal{I}(P)$ where $\max \alpha$ denotes the set of the maximal elements in α. Note that, for every $\alpha \in \mathcal{I}(P)$, $\max \alpha$ is an antichain in P and each antichain $A \subset P$ determines a unique ideal $\alpha \in \mathcal{I}(P)$, namely, the poset ideal generated by A. Therefore, ψ is an injective well defined map and by ([4], Theorem 3.1), the ring $K[\mathcal{C}(P)]$ is an ASL on $\mathcal{I}(P)$ over K with the straightening relations

$$\psi(\alpha)\psi(\beta) = \psi(\alpha * \beta)\psi(\alpha \cup \beta),$$

where $\alpha * \beta$ is the poset ideal of P generated by $\max(\alpha \cap \beta) \cap (\max \alpha \cup \max \beta)$.

We observe that one may also consider $K[\mathcal{C}(P^*)]$ as an ASL on $\mathcal{I}(P)$, where P^* is the dual poset of P. We may define $\delta : \mathcal{I}(P) \to K[C(P^*)]$ by $\delta(\alpha) = u_{\min \overline{\alpha}} t$ for $\alpha \in \mathcal{I}(P)$, where $\min \overline{\alpha}$ is the set of minimal elements in $\overline{\alpha}$ and $\overline{\alpha}$ is the filter $P \setminus \alpha$ of P. We recall that a *filter* γ in P (or *dual order ideal*) is a subset of P with the property that for every $p \in \gamma$ and every $q \in P$ with $q \geq p$, we have $q \in \gamma$. Thus, a filter in P is simply a poset ideal in the dual poset P^*. The ring $K[C(P^*)]$ is an ASL on $\mathcal{I}(P)$ over K as well with the straightening relations

$$\delta(\alpha)\delta(\beta) = \delta(\alpha \cap \beta)\delta(\alpha \circ \beta)$$

for incomparable elements $\alpha, \beta \in \mathcal{I}(P)$, where $\alpha \circ \beta$ is the poset ideal of P which is the complement in P of the filter generated by $\min(\overline{\alpha} \cap \overline{\beta}) \cap (\min \overline{\alpha} \cup \min \overline{\beta})$. Let us also observe that all the algebras $K[\mathcal{O}(P)], K[\mathcal{C}(P)]$, and $K[\mathcal{C}(P^*)]$ are compatible algebras with straightening laws.

3. Proof of Theorem 1

We clearly have (i) $\Rightarrow$ (ii). Let us now prove (ii) $\Rightarrow$ (iii). By hypothesis, the straightening relations of $K[\mathcal{O}(P)], K[\mathcal{C}(P)]$, and $K[\mathcal{C}(P^*)]$ coincide. Therefore, we must have

$$\alpha \cap \beta = \alpha * \beta \text{ and } \overline{\alpha} \cap \overline{\beta} = \overline{\alpha \circ \beta} \tag{1}$$

for all α, β incomparable elements in $\mathcal{I}(P)$. From the second equality in (1), it follows that $\overline{\alpha} \cap \overline{\beta}$ is the filter of P generated by $\min(\overline{\alpha} \cap \overline{\beta}) \cap (\min \overline{\alpha} \cup \min \overline{\beta})$. Assume that there exists two incomparable elements $p, p' \in P$ such that there exists $q \in P$ with $q > p$ and $q > p'$. Consider $\overline{\alpha}$ the filter generated by p and $\overline{\beta}$ the filter generated by p'. Then, $\min(\overline{\alpha} \cap \overline{\beta}) \cap (\min \overline{\alpha} \cup \min \overline{\beta}) = \emptyset$, but obviously, $\overline{\alpha} \cap \overline{\beta} \neq \emptyset$. This shows that, for any two incomparable elements $p, p' \in P$, there is no upper bound for p and p'.

Similarly, by using the first equality in Equation (1), we derive that, for any two incomparable elements $p, p' \in P$, there is no lower bound for p and p'. This shows that every connected component of the poset P is a chain.

Finally, we prove (iii) $\Rightarrow$ (i). Let P be a poset such that all its connected components are chains and assume that the cardinality of P is equal to n. Let $\{\omega_\alpha\}_{\alpha \in \mathcal{I}(P)}$ be the generators of $K[\Omega] \subset S$ and assume that the straightening relations of $K[\Omega]$ are $\varphi(\alpha)\varphi(\alpha') = \varphi(\beta)\varphi(\beta')$ where $\beta \subseteq \alpha \cap \alpha'$, $\beta' \supseteq \alpha \cup \alpha'$, and α, α' are incomparable elements in $\mathcal{I}(P)$. We have to show that, for all α, α' incomparable elements in $\mathcal{I}(P)$, we have $\beta = \alpha \cap \alpha'$ and $\beta' = \alpha \cup \alpha'$.

We proceed by induction on

$$k = n - (\operatorname{rank}(\alpha \cup \alpha') - \operatorname{rank}(\alpha \cap \alpha')).$$

Let us recall that, if $\gamma \in \mathcal{I}(P)$, then rank γ denotes the rank of the subposet of $\mathcal{I}(P)$ consisting of all elements $\delta \in \mathcal{I}(P)$ with $\delta \subseteq \gamma$.

If $k = 0$, that is, $\operatorname{rank}(\alpha \cup \alpha') - \operatorname{rank}(\alpha \cap \alpha') = n$, then $\alpha \cup \alpha' = P$ and $\alpha \cap \alpha' = \emptyset$, thus $\beta = \alpha \cap \alpha'$ and $\beta' = \alpha \cup \alpha'$. Assume that the desired conclusion is true for $\operatorname{rank}(\alpha \cup \alpha') - \operatorname{rank}(\alpha \cap \alpha') = n - k$ with $k \geq 0$. Let us choose now α, α' incomparable in $\mathcal{I}(P)$ such that $\operatorname{rank}(\alpha \cup \alpha') - \operatorname{rank}(\alpha \cap \alpha') = n - k - 1$ and assume that we have a straightening relation $\varphi(\alpha)\varphi(\alpha') = \varphi(\beta)\varphi(\beta')$ with $\beta \subsetneq \alpha \cap \alpha'$ or $\beta' \supsetneq \alpha \cup \alpha'$. By duality, we may reduce to considering $\beta' \supsetneq \alpha \cup \alpha'$. In other words, in $K[\Omega]$, we have

$$\omega_\alpha \omega_{\alpha'} = \omega_\beta \omega_{\beta'}, \text{ with } \beta \subseteq \alpha \cap \alpha' \text{ and } \beta' \supsetneq \alpha \cup \alpha'.$$

As P is a direct sum of chains, we may find $p \in \max(\alpha \cup \alpha')$ and $q \in \beta' \setminus (\alpha \cup \alpha')$ such that q covers p in P, that is, $q > p$ and there is no other element q' in P with $q > q' > p$. Without loss of generality, we may assume that $p \in \alpha'$. Let α_1 be the poset ideal of P generated by $\alpha' \cup \{q\}$. As all the connected components of P are chains, we have $\alpha_1 = \alpha' \cup \{q\}$ since there are no other elements in P which are smaller than q except those that are on the same chain as p and q, which are in α'. Moreover, by the choice of q, we have

$$\alpha_1 \subseteq \beta' \text{ and } \alpha \cap \alpha_1 = \alpha \cap (\alpha' \cup \{q\}) = \alpha \cap \alpha'.$$

On the other hand,

$$\operatorname{rank}(\alpha \cup \alpha_1) - \operatorname{rank}(\alpha \cap \alpha_1) = \operatorname{rank}(\alpha \cup \alpha' \cup \{q\}) - \operatorname{rank}(\alpha \cap \alpha')$$

$$= \operatorname{rank}(\alpha \cup \alpha') + 1 - \operatorname{rank}(\alpha \cap \alpha') = n - k.$$

By the inductive hypothesis, it follows that $\varphi(\alpha)\varphi(\alpha_1) = \varphi(\alpha \cap \alpha_1)\varphi(\alpha \cup \alpha_1)$, or, equivalently, in $K[\Omega]$ we have the equality $\omega_\alpha \omega_{\alpha_1} = \omega_{\alpha \cap \alpha_1} \omega_{\alpha \cup \alpha_1}$. Thus, we have obtained the following equalities in $K[\Omega]$:

$$\omega_\alpha \omega_{\alpha'} = \omega_\beta \omega_{\beta'} \text{ and } \omega_\alpha \omega_{\alpha_1} = \omega_{\alpha \cap \alpha'} \omega_{\alpha \cup \alpha_1}.$$

This implies that

$$\omega_{\alpha \cap \alpha'} \omega_{\alpha'} \omega_{\alpha \cup \alpha_1} = \omega_\beta \omega_{\alpha_1} \omega_{\beta'}. \tag{2}$$

In addition, we have:

$$\alpha \cap \alpha' \subset \alpha' \subset \alpha \cup \alpha_1 \text{ and } \beta \subset \alpha \cap \alpha' = \alpha \cap \alpha_1 \subset \alpha_1 \subset \beta'.$$

This implies that the monomials in Equation (2) are distinct standard monomials in $K[\Omega]$, which is in contradiction to the condition that the standard monomials form a K-basis in $K[\Omega]$. Therefore, our proof is completed.

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