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A Note on Type 2 Degenerate q-Euler Polynomials

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Abstract: Recently, type 2 degenerate Euler polynomials and type 2 q-Euler polynomials were studied, respectively, as degenerate versions of the type 2 Euler polynomials as well as a q-analog of the type 2 Euler polynomials. In this paper, we consider the type 2 degenerate q-Euler polynomials, which are derived from the fermionic p-adic q-integrals on $\mathbb{Z}_p$, and investigate some properties and identities related to these polynomials and numbers. In detail, we give for these polynomials several expressions, generating function, relations with type 2 q-Euler polynomials and the expression corresponding to the representation of alternating integer power sums in terms of Euler polynomials. One novelty about this paper is that the type 2 degenerate q-Euler polynomials arise naturally by means of the fermionic p-adic q-integrals so that it is possible to easily find some identities of symmetry for those polynomials and numbers, as were done previously.

Keywords: type 2 degenerate q-Euler polynomials; fermionic p-adic q-integral

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1. Introduction

We would like to introduce the type 2 degenerate q-Euler polynomials and numbers by making use of the fermionic p-adic q-integrals, as a degenerate version as well as a q-analog of the type 2 Euler polynomials and derive some basic results for them.

Studying degenerate versions and q-analogs of some known special polynomials and numbers are both very good ways of naturally introducing new special polynomials and numbers. In these two ways of constructing new polynomials and numbers, the Volkenborn integrals (also called p-adic invariant integrals), the fermionic p-adic integrals, the bosonic p-adic q-integrals, and the fermionic p-adic q-integrals have played very important roles and they will continue to do so.

For those polynomials and numbers, we derive several expressions, generating function, relations with type 2 q-Euler polynomials and the expression corresponding to the representation of alternating integer power sum in terms of Euler polynomials.

Motivation for introducing the type 2 degenerate q-Euler polynomials and numbers is to study their number-theoretic and combinatorial properties, and their applications in mathematics, science and engineering. One novelty about this paper is that they arise naturally by means of the fermionic p-adic q-integrals so that it is possible to easily find some identities of symmetry for those polynomials and numbers, as it done, for example, in [1]. We spend the rest of this section in recalling what are needed in the sequel.

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of

$\mathbb{Q}_p$, respectively. The p-adic nom $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. It is known that the ordinary Euler numbers are defined by the recurrence relation (see [2–9])

$$(E^* + 1)^n + E_n^* = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{1}$$

with the usual convention about replacing E^{*i} by E_i^*. The Euler polynomials of degree n are given either by (see [1,10–19])

$$E_n^*(x) = (E^* + x)^n = \sum_{l=0}^{n} \binom{n}{l} E_l^* x^{n-l}, \ (n \geq 0),$$

or by

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n^*(x) \frac{t^n}{n!}. \tag{2}$$

From Equations (1) and (2), we note that

$$\begin{aligned} 2 \sum_{l=0}^{n-1} (-1)^l e^{lt} &= \frac{2}{e^t + 1} (e^{nt} + 1), \ (n \equiv 1 \pmod 2) \\ &= \sum_{k=0}^{\infty} \left(E_k^*(n) + E_k^* \right) \frac{t^k}{k!}. \end{aligned} \tag{3}$$

Thus, by Equation (3), we get (see [12–14])

$$\sum_{l=0}^{n-1} (-1)^l l^k = \frac{1}{2} \left(E_k^*(n) + E_k^* \right), \tag{4}$$

where $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$.

The type 2 Euler numbers are the sequence E_n, $(n \geq 0)$, of integers defined by (see [1])

$$\operatorname{sech} t = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \tag{5}$$

where $\operatorname{sech} t$ is the hyperbolic secant function.

Then, we have (see [1]):

$$E_n = 2^n E_n^* \left(\frac{1}{2} \right), \ (n \geq 0). \tag{6}$$

Let f be a continuous function on $\mathbb{Z}_p$. Then, the fermionic p-adic integral of f on $\mathbb{Z}_p$ is defined by Kim as (see [18])

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) &= \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) \mu_{-1}(x + p^N \mathbb{Z}_p) \\ &= \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) (-1)^x. \end{aligned} \tag{7}$$

Here, the continuity is the usual one. It means that it is continuous at every point of $\mathbb{Z}_p$. The continuity at a point in $\mathbb{Z}_p$ can be given, for example, as the usual $\epsilon - \delta$ definition.

From Equation (7), we have the following lemma.

Lemma 1. *Let f be a continuous function on $\mathbb{Z}_p$. Then, we have*

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2f(0). \tag{8}$$

Proof.

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x+1)(-1)^x \\ &= \lim_{N\to\infty} -\sum_{x=0}^{p^N-1} f(x+1)(-1)^{x+1} \\ &= \lim_{N\to\infty} -\sum_{x=1}^{p^N} f(x)(-1)^x \\ &= \lim_{N\to\infty} \left(f(p^N) - \sum_{x=0}^{p^N-1} f(x)(-1)^x + f(0) \right) \\ &= 2f(0) - \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x).\end{aligned}$$

□

By Equation (8), we easily get (see [18])

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_{-1}(x) &= \frac{2}{e^t + e^{-t}} \\ &= \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.\end{aligned} \tag{9}$$

From Equation (9), we have (see [13])

$$\int_{\mathbb{Z}_p} (2x+1)^n d\mu_{-1}(x) = E_n, \ (n \geq 0). \tag{10}$$

The type 2 Euler polynomials are given by the generating function (see [1]):

$$\frac{2}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{11}$$

Here, we remind the reader of the fact that generating functions are important tools with many applications not only in mathematics but also in physics. For this, we let the reader refer to [20,21].

From Equation (8), we can derive the following integral equation.

$$\int_{\mathbb{Z}_p} e^{(2y+x+1)t} d\mu_{-1}(y) = \frac{2}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{12}$$

Thus, by Equation (12), we get

$$\int_{\mathbb{Z}_p} (2y+x+1)^n d\mu_{-1}(y) = E_n(x), \ (n \geq 0). \tag{13}$$

We need the following generalization of Equation (8):

Lemma 2. *For any positive integer n, we have*

$$\int_{\mathbb{Z}_p} f(x+n)d\mu_{-1}(x) + (-1)^{n-1}\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l). \tag{14}$$

Proof. We show Equation (14) by induction on n. It holds for $n = 1$ by Equation (8). Assume that Equation (14) holds. Then, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x+n+1)d\mu_{-1}(x) &= -\int_{\mathbb{Z}_p} f(x+n)d\mu_{-1}(x) + 2f(n)\\ &= -\left((-1)^n\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l}f(l)\right) + 2f(n)\\ &= -(-1)^n\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) + 2\sum_{l=0}^{n}(-1)^{(n+1)-1-l}f(l).\end{aligned}$$

□

From Equations (10), (13) and (14), we obtain that (see [1]),

$$E_m(2n) + E_m = 2\sum_{l=0}^{n-1}(-1)^l(2l+1)^m,$$

where $n \equiv 1 \pmod 2$, and $m \geq 0$.

The degenerate exponential function is defined by (see [3])

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}},\ e_\lambda(t) = e_\lambda^1(t).$$

Recently, the type 2 degenerate Euler polynomials were defined in [1] by the following:

$$\int_{\mathbb{Z}_p} e_\lambda^{2y+x+1}(t)d\mu_{-1}(y) = \frac{2}{e_\lambda(t) + e_\lambda^{-1}(t)}e_\lambda^x(t) = \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{15}$$

From Equation (15), we have (see [1])

$$\int_{\mathbb{Z}_p} (2y+x+1)_{n,\lambda}d\mu_{-1}(y) = E_{n,\lambda}(x),\ (n \geq 0), \tag{16}$$

where $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $(x)_{0,\lambda} = 1$.

Let q be an indeterminate in $\mathbb{C}_p$ (or $\mathbb{C}$). We assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/p-1}$. The q-analog of the number x is defined as $[x]_q = \frac{1-q^x}{1-q}$, usually called $q-$ *bracket* or $q-$ *number* (see [12,19]).

Let f be a continuous function on $\mathbb{Z}_p$. Then, the fermionic p-adic q-integral of f on $\mathbb{Z}_p$ is defined by Kim as (see [18])

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{-q}(x+p^N\mathbb{Z}_p)\\ &= \lim_{N\to\infty}\frac{1+q}{1+q^{p^N}}\sum_{x=0}^{p^N-1} f(x)(-q)^x.\end{aligned} \tag{17}$$

From Equation (17), we have the next lemma.

Lemma 3. *For any positive integer n, we have*

$$q^n \int_{\mathbb{Z}_p} f(x+n)d\mu_{-q}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l). \tag{18}$$

In particular, we have

$$q \int_{\mathbb{Z}_p} f(x+1)d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q f(0), \tag{19}$$

Proof. First, we prove Equation (19).

$$\begin{aligned}
q \int_{\mathbb{Z}_p} f(x+1)d\mu_{-q}(x) &= q \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x+1)(-q)^x \\
&= -\lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x+1)(-q)^{x+1} \\
&= -\lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=1}^{p^N} f(x)(-q)^x \\
&= -\lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \Big(-f(0) + \sum_{x=0}^{p^N-1} f(x)(-q)^x + f(p^N)(-q)^{p^N} \Big) \\
&= -\frac{1+q}{2} \Big(-f(0) + \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) - f(0) \Big).
\end{aligned}$$

Next, we show Equation (18) by induction on n. Observe that Equation (18) holds true for $n=1$ by Equation (19). Assume that Equation (18) is true. Then, we have

$$\begin{aligned}
&q^{n+1} \int_{\mathbb{Z}_p} f(x+n+1)d\mu_{-q}(x) = q^n \Big(-\int_{\mathbb{Z}_p} f(x+n)d\mu_{-q}(x) + [2]_q f(n) \Big) \\
&= -\Big((-1)^n \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l) \Big) + [2]_q q^n f(n) \\
&= (-1)^{n+1} \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) + [2]_q \sum_{l=0}^{n} (-1)^{(n+1)-1-l} q^l f(l).
\end{aligned}$$

□

The Carlitz's q-Euler polynomials can be represented by the p-adic q-integral on $\mathbb{Z}_p$ as (see [18]).

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(x) = E_{n,q}(x), (n \geq 0). \tag{20}$$

When $x=0$, $E_{n,q} = E_{n,q}(0)$ are called the Carlitz q-Euler numbers.
From Equation (19), we get that (see [12])

$$q(qE_q+1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n=0, \\ 0, & \text{if } n>0, \end{cases} \tag{21}$$

with the usual convention about replacing E_q^n by $E_{n,q}$.

In 2015, Dolgy et al. [14] introduced the degenerate Carlitz's q-Euler polynomialsm which are given by

$$[2]_q \sum_{m=0}^{\infty} (-q)^m e_\lambda^{[x+m]_q}(t) = \sum_{n=0}^{\infty} E_{n,q}(x|\lambda) \frac{t^n}{n!}. \tag{22}$$

From Equation (22), we have that (see [14])

$$E_{n,q}(x|\lambda) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m ([x+m]_q)_{n,\lambda}. \tag{23}$$

Motivated by Equations (12) and (15), we would like to consider the type 2 degenerate q-Euler polynomials and investigate some properties for these polynomials in the following section.

This paper is organized as follows. In Section 1, we review some known results. To be specific, we recall type 2 Euler polynomials and type 2 degenerate Euler polynomials in connection with fermionic p-adic integrals, and Carlitz's q-Euler polynomials and degenerate Carlitz's q-Euler polynomials in relation to fermionic p-adic q-integrals. In Section 2, by virtue of fermionic p-adic q-integrals, we introduce type 2 degenerate q-Euler polynomials. Then, we present for these polynomials several expressions, generating function, relations with type 2 q-Euler polynomials and the expression corresponding to the representation of alternating integer power sums in terms of Euler polynomials. In Section 3, we give the conclusion of this paper.

2. Type 2 Degenerate *q*-Euler Polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/p-1}$ and $\lambda \in \mathbb{C}_p$ with $|\lambda|_p < p^{-1/p-1}$.

We now consider the *type 2 degenerate q-Euler polynomials* given by the fermionic p-adic q-integral on $\mathbb{Z}_p$

$$\int_{\mathbb{Z}_p} e_\lambda^{[x+1+2y]_q}(t) d\mu_{-q}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x|\lambda) \frac{t^n}{n!}. \tag{24}$$

By Equation (24), we get

$$\int_{\mathbb{Z}_p} \left([x+1+2y]_q\right)_{n,\lambda} d\mu_{-q}(y) = \mathcal{E}_{n,q}(x|\lambda), \ (n \geq 0). \tag{25}$$

We observe here that

$$\begin{aligned} \lim_{q\to 1} \lim_{\lambda \to 0} \int_{\mathbb{Z}_p} \left([x+1+2y]_q\right)_{n,\lambda} d\mu_{-q}(y) &= \int_{\mathbb{Z}_p} (x+1+2y)^n d\mu_{-1}(y) \\ &= E_n(x), \ (n \geq 0). \end{aligned}$$

When $x = 0$, $\mathcal{E}_{n,q}(0|\lambda) = \mathcal{E}_{n,q}(\lambda)$ are called the *type 2 degenerate q-Euler numbers*.

In [4], the degenerate Stirling numbers of the first kind, denoted by $S_{1,\lambda}(n,k)$, are defined as

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{1,\lambda}(n,k) x^k, \ (n \geq 0). \tag{26}$$

From Equation (25) and (26), we have

$$\mathcal{E}_{n,q}(x|\lambda) = \sum_{k=0}^{n} S_{1,\lambda}(n,k) \int_{\mathbb{Z}_p} [x+2y+1]_q^k d\mu_{-q}(y). \tag{27}$$

Recall that the type 2 q-Euler polynomials and the type 2 q-Euler numbers are, respectively, given by

$$\int_{\mathbb{Z}_p} [x+2y+1]_q^n d\mu_{-q}(y) = \mathcal{E}_{n,q}(x), (k \geq 0), \ \int_{\mathbb{Z}_p} [2y+1]_q^n d\mu_{-q}(y) = \mathcal{E}_{n,q}, \ (n \geq 0). \tag{28}$$

Therefore, by Equations (27) and (28), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,q}(x|\lambda) = \sum_{l=0}^{n} S_{1,\lambda}(n,l)\mathcal{E}_{l,q}(x).$$

We observe that

$$\begin{aligned}
\int_{\mathbb{Z}_p} ([x+2y+1]_q)_{n,\lambda}\, d\mu_{-q}(y) &= \sum_{k=0}^{n} S_{1,\lambda}(n,k) \int_{\mathbb{Z}_p} [x+2y+1]_q^k d\mu_{-q}(y) \\
&= \sum_{k=0}^{n} \frac{S_{1,\lambda}(n,k)}{(q-1)^k} \sum_{m=0}^{k} \binom{k}{m} q^{m(x+1)}(-1)^{k-m} \int_{\mathbb{Z}_p} q^{2my} d\mu_{-q}(y) \\
&= \frac{[2]_q}{2} \lim_{N\to\infty} \sum_{y=0}^{p^N-1} (-q)^y \sum_{k=0}^{n} \frac{S_{1,\lambda}(n,k)}{(q-1)^k} \sum_{m=0}^{k} \binom{k}{m} q^{m(x+1)}(-1)^{k-m} q^{2my} \\
&= [2]_q \sum_{k=0}^{n} \frac{S_{1,\lambda}(n,k)}{(q-1)^k} \sum_{m=0}^{k} \binom{k}{m} q^{m(x+1)}(-1)^{k-m} \frac{1}{1+q^{2m+1}} \\
&= [2]_q \sum_{k=0}^{n} \frac{S_{1,\lambda}(n,k)}{(1-q)^k} \sum_{m=0}^{k} \binom{k}{m} (-q^{x+1})^m \frac{1}{1+q^{2m+1}}.
\end{aligned} \tag{29}$$

Therefore, by Equation (29), we obtain the following theorem.

Theorem 2. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,q}(x|\lambda) = [2]_q \sum_{k=0}^{n} \frac{S_{1,\lambda}(n,k)}{(1-q)^k} \sum_{m=0}^{k} \binom{k}{m} (-q^{x+1})^m \frac{1}{1+q^{2m+1}}.$$

From Equation (24), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x|\lambda)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+2y+1]_q}{\lambda}} d\mu_{-q}(y) \\
&= \int_{\mathbb{Z}_p} e^{\frac{[x+2y+1]_q}{\lambda}\log(1+\lambda t)} d\mu_{-q}(y) \\
&= \sum_{k=0}^{\infty} \lambda^{-k} \int_{\mathbb{Z}_p} [x+2y+1]_q^k d\mu_{-q}(y) \frac{1}{k!} (\log(1+\lambda t))^k \\
&= \sum_{k=0}^{\infty} \lambda^{-k} \mathcal{E}_{k,q}(x) \sum_{n=k}^{\infty} S_1(n,k)\lambda^n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \lambda^{n-k} \mathcal{E}_{k,q}(x) S_1(n,k) \right) \frac{t^n}{n!}.
\end{aligned} \tag{30}$$

By comparing the coefficients on both sides of Equation (30), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,q}(x|\lambda) = \sum_{k=0}^{n} \lambda^{n-k} S_1(n,k)\mathcal{E}_{k,q}(x),$$

where $S_1(n,k)$ are the Stirling numbers of the first kind.

From Equation (24), we obtain that

$$\int_{\mathbb{Z}_p} e^{[x+2y+1]_q t} d\mu_{-q}(y) = \sum_{k=0}^{\infty} \mathcal{E}_{k,q}(x|\lambda) \frac{1}{k!} \left(\frac{e^{\lambda t} - 1}{\lambda} \right)^k,$$

where

$$\begin{aligned} \sum_{k=0}^{\infty} \mathcal{E}_{k,q}(x|\lambda) \frac{1}{k!} \left(\frac{e^{\lambda t} - 1}{\lambda} \right)^k &= \sum_{k=0}^{\infty} \mathcal{E}_{k,q}(x|\lambda) \sum_{n=k}^{\infty} S_2(n,k) \lambda^{n-k} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} S_2(n,k) \lambda^{n-k} \mathcal{E}_{k,q}(x|\lambda) \right) \frac{t^n}{n!}, \end{aligned} \tag{31}$$

where $S_2(n,k)$ are the Stirling numbers of the second kind.

On the other hand,

$$\int_{\mathbb{Z}_p} e^{[x+2y+1)]_q t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{n!}. \tag{32}$$

Therefore, by Equations (31) and (32), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, we have*

$$\mathcal{E}_{n,q}(x) = \sum_{k=0}^{n} S_2(n,k) \lambda^{n-k} \mathcal{E}_{k,q}(x|\lambda).$$

Using Theorem 2, we can derive Equation (33).

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x|\lambda) \frac{t^n}{n!} &= [2]_q \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{S_{1,\lambda}(n,k)}{(1-q)^k} \sum_{m=0}^{k} \binom{k}{m} (-q^{x+1})^m \frac{1}{1+q^{2m+1}} \right) \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} (-q)^l \sum_{k=0}^{n} \frac{S_{1,\lambda}(n,k)}{(1-q)^k} (1-q^{x+2l+1})^k \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} (-q)^l \sum_{k=0}^{n} S_{1,\lambda}(n,k) [x+2l+1]_q^k \frac{t^n}{n!} \\ &= [2]_q \sum_{l=0}^{\infty} (-q)^l \sum_{n=0}^{\infty} \left([x+2l+1]_q \right)_{n,\lambda} \frac{t^n}{n!} \\ &= [2]_q \sum_{l=0}^{\infty} (-q)^l (1+\lambda t)^{\frac{[x+2l+1]_q}{\lambda}} \\ &= [2]_q \sum_{l=0}^{\infty} (-q)^l e_{\lambda}^{[x+2l+1]_q}(t). \end{aligned} \tag{33}$$

Therefore, by Equation (33), we obtain the generating function for the type 2 degenerate q-Euler polynomials.

Theorem 5. *Let $F(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x|\lambda) \frac{t^n}{n!}$. Then, we have*

$$F(t) = [2]_q \sum_{m=0}^{\infty} (-q)^m e_{\lambda}^{[x+2m+1]_q}(t).$$

From Equation (19), we have that

$$\begin{aligned} & q^n \int_{\mathbb{Z}_p} [2x+2n+1]_q^m d\mu_{-q}(x) \\ & = -\int_{\mathbb{Z}_p} [2x+1]_q^m d\mu_{-q}(x) + [2]_q \sum_{l=0}^{n-1} (-q)^l [2l+1]_q^m, \end{aligned} \tag{34}$$

where $n \in \mathbb{N}$, with $n \equiv 1 \pmod 2$.

Thus, we have shown the following result.

Theorem 6. *For $m \geq 0$, and $n \in \mathbb{N}$, with $n \equiv 1 \pmod 2$, we have*

$$q^n \mathcal{E}_{m,q}(2n) + \mathcal{E}_{m,q} = [2]_q \sum_{l=0}^{n-1} (-q)^l [2l+1]_q^m.$$

Let us take $f(x) = \left([2x+1]_q\right)_{m,\lambda}$, $(m \geq 0)$ in Equation (19). Then, we get

$$\begin{aligned} & q^n \int_{\mathbb{Z}_p} \left([2x+2n+1]_q\right)_{m,\lambda} d\mu_{-q}(x) \\ & = (-1)^n \int_{\mathbb{Z}_p} \left([2x+1]_q\right)_{m,\lambda} d\mu_{-q}(x) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l \left([2l+1]_q\right)_{m,\lambda}, \end{aligned} \tag{35}$$

where $n \in \mathbb{N}$.

Thus, we obtain the following theorem.

Theorem 7. *For $m \geq 0$, and $n \in \mathbb{N}$, we have*

$$q^n \mathcal{E}_{m,q}(2n|\lambda) + (-1)^{n-1} \mathcal{E}_{m,q}(\lambda) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l \left([2l+1]_q\right)_{m,\lambda}.$$

For $d \in \mathbb{N}$, with $d \equiv 1 \pmod 2$, we have

$$\int_{\mathbb{Z}_p} f(y) d\mu_{-q}(y) = \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} f(a+dy) d\mu_{-q^d}(y) \tag{36}$$

Let us take $f(y) = \left([2y+x+1]_q\right)_{n,\lambda}$, $(n \geq 0)$. Then, we have

$$\begin{aligned} \mathcal{E}_{n,q}(x|\lambda) &= \int_{\mathbb{Z}_p} \left([2y+x+1]_q\right)_{n,\lambda} d\mu_{-q}(y) \\ &= \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} \left([2(a+dy)+x+1]_q\right)_{n,\lambda} d\mu_{-qd}(y) \\ &= \sum_{a=0}^{d-1} (-q)^a [d]_q^n \int_{\mathbb{Z}_p} \left(\left[\frac{2a+x+1}{d}+2y\right]_{qd}\right)_{n,\frac{\lambda}{[d]_q}} d\mu_{-qd}(y) \\ &= [d]_q^n \sum_{a=0}^{d-1} (-q)^a \mathcal{E}_{n,qd}\left(\frac{2a+x+1-d}{d} \Big| \frac{\lambda}{[d]_q}\right), \end{aligned}$$

where $d \in \mathbb{N}$, with $d \equiv 1 \pmod 2$. Therefore, we have

$$\mathcal{E}_{n,q}(x|\lambda) = [d]_q^n \sum_{a=0}^{d-1} (-q)^a \mathcal{E}_{n,qd}\left(\frac{2a+x+1-d}{d} \Big| \frac{\lambda}{[d]_q}\right),$$

where $d \in \mathbb{N}$, with $d \equiv 1 \pmod 2$, and $n \geq 0$.

3. Conclusions

There are various ways of introducing new special polynomials and numbers.

One way of introducing new special polynomials and numbers is to study various degenerate versions of some known special polynomials and numbers. This idea traces at least back to Carlitz [11]. It is noteworthy that degenerate versions can be investigated not only for some polynomials but also for some transcendental functions. The reader may refer to the work in [3] for this instance.

Another way of introducing new special polynomials and numbers is to study various q-analogs of some known special polynomials and numbers. It turns out that the fermionic p-adic q-integrals, together with the bosonic p-adic q-integrals, are very powerful and fruitful tools in naturally constructing such q-analogs.

In this paper, the type 2 degenerate q-Euler polynomials and numbers are introduced and investigated as a degenerate version as well as a q-analog of type 2 Euler polynomials by using the fermionic p-adic q-integrals [17–19,22,23]. In this paper, some results about those polynomials and numbers are obtained. In detail, we give for them several expressions, generating function, relations with type 2 q-Euler polynomials and the expression corresponding to the representation of alternating integer power sums in terms of Euler polynomials.

We are planning to study more detailed results relating to those polynomials and numbers in a forthcoming paper. More generally, the fermionic p-adic integrals, the bosonic p-adic q-integrals, the fermionic p-adic q-integrals and the Vokenborn integrals (also called the bosonic p-adic integrals) have been very useful and fruitful in naturally introducing special polynomials and numbers and in studying various properties of them, for example in discovering symmetric identities relating to such polynomials and numbers. Anyone is invited to join this fascinating pursuit of research.

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