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Article

Cohen-Macaulay and (S_2) Properties of the Second Power of Squarefree Monomial Ideals

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Abstract: We show that Cohen-Macaulay and (S_2) properties are equivalent for the second power of an edge ideal. We give an example of a Gorenstein squarefree monomial ideal I such that S/I^2 satisfies the Serre condition (S_2), but is not Cohen-Macaulay.

Keywords: Stanley-Reisner ideal; edge ideal; Cohen-Macaulay; (S_2) condition

1. Introduction

Let K be a fixed field. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring with $\deg x_i = 1$ for all $i \in [n] = \{1, 2, \ldots, n\}$. Let I be a squarefree monomial ideal.

For a Stanley-Reisner ring S/I, the Cohen-Macaulay and (S_2) properties are different in general. For instance, consider the Stanley-Reisner ring of a non-Cohen-Macaulay manifold, e.g., a torus, which satisfies the (S_2) condition. However, for some special classes of such rings, they are known to be equivalent. The quotient ring of the edge ideal of a very well-covered graph (see [1]) and a Stanley-Reisner ring with "large" multiplicity (see [2] for the precise statement) are such examples. What about the powers of squarefree monomial ideals?

As for the third and larger powers, the following is proven in [3]:

Theorem 1. *Let I be a squarefree monomial ideal. Then, the following conditions are equivalent for a fixed integer $m \geq 3$:*

1. *S/I is a complete intersection.*
2. *S/I^m is Cohen-Macaulay.*
3. *S/I^m satisfies the Serre condition (S_2).*

Then, what about the second power of a squarefree monomial ideal? This is the theme of this article. If the second power I^2 is Cohen-Macaulay, I is not necessarily a complete intersection. Gorenstein ideals with height three give such examples.

In Section 3, we prove that the Cohen-Macaulay and (S_2) properties are equivalent for the second power of a squarefree monomial ideal generated in degree two:

Theorem 2. *Let I be a squarefree monomial ideal generated in degree two. Then, the following conditions are equivalent:*

1. *S/I^2 is Cohen-Macaulay.*

2. *S/I^2 satisfies the Serre condition (S_2).*

In Section 4, we first give an upper bound of the number of variables in terms of the dimension of S/I when I is a squarefree monomial ideal generated in degree two and S/I^2 has the Cohen-Macaulay (equivalently (S_2)) property. Using a computer, we classify squarefree monomial ideals I generated in degree two with $\dim S/I \leq 4$ such that S/I^2 have the Cohen-Macaulay (equivalently (S_2)) property. Since not many examples of squarefree monomial ideals I generated in degree two such that S/I^2 are Cohen-Macaulay are known, new examples might be useful. See [4,5] for the two- and three-dimensional cases, respectively, and [6,7] for the higher dimensional case. See also [6,8] for the fact that for a very well-covered graph G, the second power $I(G)^2$ is not Cohen-Macaulay if the edge ideal $I(G)$ of G is not a complete intersection.

In Section 5, we give an example of a Gorenstein squarefree monomial ideal I such that S/I^2 satisfies the Serre condition (S_2), but is not Cohen-Macaulay. Hence, the Cohen-Macaulay and (S_2) properties are different for the second power in general.

2. Preliminaries

2.1. Stanley-Reisner Ideals

We recall some notation on simplicial complexes and their Stanley-Reisner ideals. We refer the reader to [9–11] for the detailed information.

Set $V = [n] = \{1, 2, \ldots, n\}$. A nonempty subset Δ of the power set 2^V of V is called a *simplicial complex* on V if the following two conditions are satisfied: (i) $\{v\} \in \Delta$ for all $v \in V$, and (ii)$F \in \Delta$, $H \subseteq F$ imply $H \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ. The dimension of F, denoted by $\dim F$, is defined by $\dim F = |F| - 1$. The dimension of Δ is defined by $\dim \Delta = \max\{\dim F : F \in \Delta\}$. We call a maximal face of Δ a *facet* of Δ. Let $\mathcal{F}(\Delta)$ denote the set of all facets of Δ. We call Δ *pure* if all its facets have the same dimension. We call Δ *connected* if for any pair (p, q), $p \neq q$, of vertices of Δ, there is a chain $p = p_0, p_1, p_2, \ldots, p_k = q$ of vertices of Δ such that $\{p_{i-1}, p_i\} \in \Delta$ for $i = 1, 2, \ldots, k$.

The *Stanley-Reisner ideal* I_Δ of Δ is defined by:

$$I_\Delta = (x_{i_1} x_{i_2} \cdots x_{i_p} \,:\, 1 \leq i_1 < \cdots < i_p \leq n,\ \{x_{i_1}, \ldots, x_{i_p}\} \notin \Delta).$$

The quotient ring $K[\Delta] = K[x_1, \ldots, x_n]/I_\Delta$ is called the *Stanley-Reisner ring* of Δ.

We say that Δ is a Cohen-Macaulay (resp. Gorenstein) complex if $K[\Delta]$ is a Cohen-Macaulay (resp. Gorenstein) ring. A Gorenstein complex Δ is called *Gorenstein** if x_i divides some minimal monomial generator of I_Δ for each i.

For a face $F \in \Delta$, the *link* and *star* of F are defined by:

$$\begin{aligned} \operatorname{link}_\Delta F &= \{H \in \Delta \,:\, H \cup F \in \Delta,\ H \cap F = \emptyset\}, \\ \operatorname{star}_\Delta F &= \{H \in \Delta \,:\, H \cup F \in \Delta\}. \end{aligned}$$

The Stanley-Reisner ideal I_Δ of Δ has the minimal prime decomposition:

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F,$$

where $P_F = (x \in [n] \setminus F)$ for each $F \in \mathcal{F}(\Delta)$. We call I_Δ *unmixed* if all P_F have the same height for $F \in \mathcal{F}(\Delta)$. Note that Δ is *pure* if and only if I_Δ is unmixed. We define the ℓ^{th} symbolic power of I_Δ by:

$$I_\Delta^{(\ell)} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^\ell.$$

For a Noetherian ring A, the following condition (S_i) for $i = 1, 2, \ldots$ is called *Serre's condition*:

$$(S_i)\ \operatorname{depth} A_P \geq \min\{\operatorname{height} P, i\} \ \text{ for all } P \in \operatorname{Spec}(A).$$

See [12] for more information for Stanley-Reisner rings satisfying Serre's condition (S_i).

To introduce a characterization of the (S_2) property for the second symbolic power of a Stanley-Reisner ideal, we first define the diameter of a simplicial complex. Let Δ be a connected simplicial complex. For p, q being two vertices of Δ, the *distance* between p and q is the minimal length k of chains $p = p_0, p_1, p_2, \ldots, p_k = q$ of vertices of Δ such that $\{p_{i-1}, p_i\} \in \Delta$ for $i = 1, 2, \ldots, k$. The *diameter*, denoted by $\operatorname{diam} \Delta$, is the maximal distance between two vertices in Δ. We set $\operatorname{diam} \Delta = \infty$ if Δ is disconnected. The (S_2) property of the second symbolic power of a Stanley-Reisner ideal is characterized as follows:

Theorem 3. *([7], Corollary 3.3) Let Δ be a pure simplicial complex. Then, the following conditions are equivalent*:

1. $S/I_\Delta^{(2)}$ *satisfies* (S_2).
2. $\operatorname{diam}(\operatorname{link}_\Delta F) \leq 2$ *for any face* $F \in \Delta$ *with* $\dim \operatorname{link}_\Delta F \geq 1$.

2.2. Edge Ideals

Let G be a graph, which means a finite simple graph, which has no loops and multiple edges. We denote by $V(G)$ (resp. $E(G)$) the set of vertices (resp. edges) of G. We call $F \subseteq V(G)$ an *independent set* of G if any $e \in E(G)$ is not contained in F. The independence complex $\Delta(G)$ of G is defined by:

$$\Delta(G) = \{F \subset V(G) \,:\, e \not\subseteq F \text{ for any } e \in E(G)\},$$

which is a simplicial complex on the vertex set $V(G)$. We define $\alpha(G)$ by:

$$\alpha(G) = \dim \Delta(G) + 1.$$

We define the *neighbor set* $N_G(a)$ of a vertex a of G by:

$$N_G(a) = \{b \in V \,:\, ab \in E(G)\}.$$

Set $N_G[a] := \{a\} \cup N_G(a)$, which is called the *closed neighbor set* of a vertex a of G. For $S \subseteq V(G)$, we denote by $G \backslash S$ the induced subgraph on the vertex set $V(G) \backslash S$. Set $G_S := G \backslash N_G[S]$, where $N_G[S] := \cup_{x \in S} N_G[x]$. If $S \in \Delta(G)$, then:

$$\operatorname{link}_{\Delta(G)}(S) = \Delta(G_S).$$

See ([11], Lemma 7.4.3). For $ab \in E(G)$, set $G_{ab} := G \backslash (N_G(a) \cup N_G(b))$.

Set $V(G) = \{1, \ldots, n\}$. Then, the *edge ideal* of G, denoted by $I(G)$, is a squarefree monomial ideal of $S = K[x_1, \ldots, x_n]$ defined by:

$$I(G) = (x_i x_j \,:\, \{x_i, x_j\} \in E(G)).$$

Note that $I(G) = I_{\Delta(G)}$. We call G *well-covered* (or *unmixed*) if $I(G)$ is unmixed.

Theorem 4 ([13,14])**.** *Let G be a graph. Then, the following conditions are equivalent:*

1. *G is triangle-free.*
2. $I(G)^{(2)} = I(G)^2$.

Theorem 5 ([15]). *Let G be a graph. Then, the following conditions are equivalent:*

1. *G is triangle-free, and $I(G)$ is Gorenstein.*
2. *$S/I(G)^2$ is Cohen-Macaulay.*

3. The Second Power of Edge Ideals

In this section, we show that the Cohen-Macaulay and (S_2) properties are equivalent for the second power of an edge ideal.

Lemma 1. *Let G be a graph with $\alpha(G) \geq 2$. The following conditions are equivalent:*

1. *$S/I(G)^{(2)}$ satisfies the (S_2) property,*
2. *G is a well-covered graph and satisfies $\operatorname{diam} \Delta(G_F) \leq 2$ for all the independent sets F of G such that $|F| \leq \alpha(G) - 2$,*
3. *G_{ab} is well-covered and satisfies $\alpha(G_{ab}) = \alpha(G) - 1$ for all $ab \in E(G)$.*

Proof. (1)$\Leftrightarrow$ (2): By [12], Theorem 8.3, $I(G)$ satisfies the (S_2) property if so does $S/I(G)^{(2)}$. Using [12], Corollary 5.4, we obtain that $\Delta(G)$ is pure. This means that G is well-covered, and thus:

$$\dim \operatorname{link}_{\Delta(G)}(F) = \dim \Delta(G) - |F|$$

and $\operatorname{link}_{\Delta(G)}(F) = \Delta(G_F)$. The result is implied by Theorem 3.

(2) $\Rightarrow$ (3): For all $ab \in E(G)$, we have:

$$\alpha(G_{ab}) \leq \alpha(G) - 1.$$

Let F be an independent set of G_{ab}. If $|F| < \alpha(G) - 1$, then $|F| \leq \alpha(G) - 2$. Recall that $G_{ab} = G \backslash (N_G(a) \cup N_G(b))$ and $F \subseteq V(G_{ab})$. This implies that $a, b \notin N_G[F]$. Hence, we obtain that $\{a, b\}$ is an edge of G_F. In other words, $\{a, b\}$ is not an independent set of G_F. By the assumption, $\operatorname{diam} \Delta(G_F) \leq 2$, there is a vertex $c \in V(G_F)$ such that $\{a, c\}, \{c, b\}$ are independent sets of G_F. Thus, $ac, bc \notin E(G_F)$. Hence, $c \in V(G_{ab})$. Therefore, $F \cup \{c\}$ is an independent of G_{ab}. Then, G_{ab} is well-covered, and moreover, $\alpha(G_{ab}) = \alpha(G) - 1$.

(3) $\Rightarrow$ (2): By [15], Lemma 4.1 (2), G is a well-covered graph. We will prove that $\operatorname{diam} \Delta(G_F) \leq 2$ for all independent set F with $|F| \leq \alpha(G) - 2$ by induction on $\alpha(G)$.

If $\alpha(G) = 2$, then we must prove $\operatorname{diam} \Delta(G) \leq 2$. For all $a, b \in V(G)$, we assume $\{a, b\} \notin \Delta(G)$. Then, $ab \in E(G)$. By the assumption, $\alpha(G_{ab}) = \alpha(G) - 1 = 1 > 0$. Therefore, we can take a vertex c in G_{ab}, and thus, $ac, bc \notin E(G)$. Hence, $\{a, c\}, \{b, c\} \in \Delta(G)$. Therefore, we conclude that $\operatorname{diam} \Delta(G) \leq 2$.

Let $\alpha(G) > 2$, and suppose that the assertion is true for all graphs G' with the same structure as G satisfying the condition "*G_{ab} is well-covered and satisfies $\alpha(G_{ab}) = \alpha(G) - 1$ for all $ab \in E(G)$*" with $\alpha(G') < \alpha(G)$. For all independent set F of G such that $|F| \leq \alpha(G) - 2$, we divide the proof into the following two cases:

Case 1: $F = \emptyset$. In this case, we need to prove that $\operatorname{diam} \Delta(G) \leq 2$. In fact, using the same argument as above, we obtain $\operatorname{diam} \Delta(G) \leq 2$.

Case 2: $F \neq \emptyset$. Let $x \in F$. Recall that G is a well-covered graph, and thus, we have $\alpha(G_x) = \alpha(G) - 1$. Hence, $|F \backslash \{x\}| = |F| - 1 \leq \alpha(G) - 3 = \alpha(G_x) - 2$. Note that for all $ab \in E(G_x)$, we have that $(G_x)_{ab}$ and $(G_{ab})_x$ are two induced subgraphs of G on vertex set $V(G) \backslash (N_G[x] \cup N_G(a) \cup N_G(b))$. Thus, $(G_x)_{ab} = (G_{ab})_x$. By the assumption and [15], Lemma 4.1 (1), $(G_{ab})_x$ is a well-covered graph with $\alpha((G_{ab})_x) = \alpha(G_{ab}) - 1$. Therefore, $(G_x)_{ab}$ is also a well-covered graph. Moreover,

$$\alpha((G_x)_{ab}) = \alpha((G_{ab})_x) = \alpha(G_{ab}) - 1 = \alpha(G) - 2 = \alpha(G_x) - 1.$$

Thus, G_x has the same structure as G satisfying the condition "*G_{ab} is well-covered and satisfies $\alpha(G_{ab}) = \alpha(G) - 1$ for all $ab \in E(G)$*" with $\alpha(G_x) < \alpha(G)$. By the induction hypothesis, we obtain $\operatorname{diam} \Delta((G_x)_{F\setminus\{x\}}) \leq 2$. Note that:

$$(G_x)_{F\setminus\{x\}} = G_x \setminus N_G[F\setminus\{x\}] = G\setminus(N_G[x] \cup N_G[F\setminus\{x\}]) = G\setminus(N_G[F]) = G_F.$$

Therefore, $\Delta(G_F) = \Delta((G_x)_{F\setminus\{x\}})$. Therefore, we conclude that $\operatorname{diam} \Delta(G_F) \leq 2$. □

Then, we get the following theorem.

Theorem 6. *Let G be a graph. The following conditions are equivalent:*

1. *$S/I(G)^2$ satisfies the (S_2) property,*
2. *$S/I(G)^2$ is Cohen-Macaulay,*
3. *G is triangle-free, and G_{ab} is a well-covered graph with $\alpha(G_{ab}) = \alpha(G) - 1$ for all $ab \in E(G)$.*

Proof. By the statements of Conditions (1), (2) and (3), without loss of generality, we can assume that G contains no isolated vertices.

(2) $\Leftrightarrow$ (3): By [15], Theorem 4.4, $S/I(G)^2$ is Cohen-Macaulay if and only if G is triangle-free and in W_2, which is a well-covered graph such that the removal of any vertex of G leaves a well-covered graph with the same independence number as G. By [15], Lemma 4.2, this is equivalent to the condition that G is triangle-free and G_{ab} is a well-covered graph with $\alpha(G_{ab}) = \alpha(G) - 1$ for all $ab \in E(G)$.

(2) $\Rightarrow$ (1): It is obvious.

(1) $\Rightarrow$ (3): If $\alpha(G) = 1$, then G is a complete graph. By the assumption, G is one edge. Therefore, the statement holds true. Now, we assume $\alpha(G) \geq 2$. We know that $S/I(G)^2$ satisfies that (S_2) property if and only if $S/I(G)^{(2)}$ satisfies the (S_2) property and $I(G)^2$ has no embedded associated prime, which means $I(G)^2 = I(G)^{(2)}$. By Theorem 4 and Lemma 1, G is triangle-free, and G_{ab} is well-covered with $\alpha(G_{ab}) = \alpha(G) - 1$ for all $ab \in E(G)$. □

Question. *If $S/I(G)^{(2)}$ satisfies the (S_2) property, then is it Cohen-Macaulay?*

The question is affirmative if G is a triangle-free graph by Theorems 4 and 6.

4. Classification

The purpose of the section is to classify all graphs G such that $S/I(G)^2$ is Cohen-Macaulay with dimension less than five. First, we give an upper bound of the number of vertices of a graph G such that $S/I(G)^2$ is Cohen-Macaulay.

4.1. Upper Bound of the Number of Vertices

Theorem 7 (Upper bound). *Let G be a graph with the vertex set $[n]$. Suppose G has no isolate vertex. If $S/I(G)^2$ is d-dimensional Cohen-Macaulay, where $d \geq 3$, then we have $n \leq \frac{d^2+3d-2}{2}$.*

Proof. We prove this by induction on d. For $d = 3$, we have $n \leq 8$ by [5] (see Proposition 3). Set $N(d) = \frac{d^2+3d-2}{2}$. Let n be the number of vertices of G such that $S/I(G)^2$ is d-dimensional and Cohen-Macaulay. Let $i \in [n]$. Then, we have $n = |V(\operatorname{star}_{\Delta(G)}\{i\})| + |([n] \setminus V(\operatorname{star}_{\Delta(G)}\{i\})|$. Since G is triangle-free by Theorem 5, an edge among $\{i,p\}$, $\{i,q\}$ and $\{p,q\}$ belongs to $\Delta(G)$ for any $p,q \in ([n] \setminus V(\operatorname{star}_{\Delta(G)}\{i\})$, where $p \neq q$. By the definition of $\operatorname{star}_{\Delta(G)}\{i\}$, we have $\{i,p\}, \{i,q\} \notin \Delta(G)$. Then, we have $\{p,q\} \in \Delta(G)$. By the fact that $I(G)$ is generated in degree two, all minimal non-faces of $\Delta(G)$ have cardinality two. Now, we know that $\{p,q\} \in \Delta(G)$ for any $p,q \in ([n] \setminus V(\operatorname{star}_{\Delta(G)}\{i\})$; hence, we have $[n] \setminus V(\operatorname{star}_{\Delta(G)}\{i\}) \in \Delta(G)$. By the assumption that $S/I(G)^2$ is d-dimensional, we have $|[n] \setminus V(\operatorname{star}_{\Delta(G)}\{i\})| \leq d$. Since $\Delta(G)$ is Gorenstein*, so is $\operatorname{link}_{\Delta(G)}\{i\}$ by [10], Theorem

5.1. By Theorem 5, $I^2_{\mathrm{link}_{\Delta(G)}\{i\}}$ is Cohen-Macaulay. Hence, $|V(\mathrm{star}_{\Delta(G)}\{i\})| = |V(\mathrm{link}_{\Delta(G)}\{i\})| + 1 \leq N(d-1)+1$ by the induction hypothesis. Therefore, $n \leq N(d-1)+d+1 = \frac{(d-1)^2+3(d-1)-2}{2}+d+1 = \frac{d^2+3d-2}{2} = N(d)$. $\square$

4.2. Classification

In this subsection, we classify all graphs G such that $S/I(G)^2$ is Cohen-Macaulay with dimension less than five.

Proposition 1. (One-dimensional case) *Let G be a graph with the vertex set $[n]$. Suppose G has no isolate vertex. Then, $S/I(G)^2$ is one-dimensional Cohen-Macaulay if and only if $n = 2$ and $I(G) = (x_1x_2)$.*

Proposition 2 ([4])**.** (Two-dimensional case) *Let G be a graph with the vertex set $[n]$. Suppose G has no isolate vertex. Then, $S/I(G)^2$ is two-dimensional Cohen-Macaulay if and only if $I(G)$ is one of the following up to the permutation of variables:*

1. *If $n = 4$, then (x_1x_3, x_2x_4).*
2. *If $n = 5$, then $(x_1x_3, x_1x_4, x_2x_3, x_2x_5, x_4x_5)$.*

Proposition 3 ([5])**.** (Three-dimensional case) *Let G be a graph with the vertex set $[n]$. Suppose G has no isolate vertex. Then, $S/I(G)^2$ is three-dimensional Cohen-Macaulay if and only if $I(G)$ is one of the following up to the permutation of variables:*

1. *If $n = 6$, then (x_1x_4, x_2x_5, x_3x_6).*
2. *If $n = 7$, then $(x_1x_5, x_1x_6, x_2x_5, x_2x_7, x_3x_4, x_6x_7)$.*
3. *If $n = 8$, then $(x_1x_2, x_1x_5, x_1x_8, x_2x_3, x_3x_4, x_4x_5, x_4x_8, x_5x_6, x_6x_7, x_7x_8)$.*

Using a computer with Nauty [16] and CoCoA [17], we classify four-dimensional case: By Theorem 7, it is enough to search for them up to $n = 13$.

Theorem 8. (Four-dimensional case) *Let G be a graph with the vertex set $[n]$. Suppose G has no isolate vertex. Then, $S/I(G)^2$ is four-dimensional Cohen-Macaulay if and only if $I(G)$ is one of the following up to the permutation of variables:*

1. *If $n = 8$, then $(x_1x_5, x_2x_6, x_3x_7, x_4x_8)$.*
2. *If $n = 9$, then $(x_1x_5, x_2x_6, x_3x_7, x_1x_8, x_4x_8, x_4x_9, x_5x_9)$.*
3. *If $n = 10$, then*
   - (a) $(x_1x_5, x_2x_6, x_3x_7, x_1x_8, x_4x_8, x_2x_9, x_4x_9, x_5x_9, x_4x_{10}, x_5x_{10}, x_6x_{10})$.
   - (b) $(x_1x_5, x_2x_6, x_1x_7, x_3x_7, x_3x_8, x_5x_8, x_2x_9, x_4x_9, x_4x_{10}, x_6x_{10})$.
4. *If $n = 11$, then*
   - (a) $(x_1x_5, x_2x_6, x_3x_7, x_1x_8, x_4x_8, x_2x_9, x_4x_9, x_5x_9, x_3x_{10}, x_4x_{10}, x_5x_{10}, x_6x_{10}, x_4x_{11}, x_5x_{11}, x_6x_{11}, x_7x_{11})$.
   - (b) $(x_1x_5, x_2x_6, x_1x_7, x_3x_7, x_3x_8, x_5x_8, x_2x_9, x_4x_9, x_1x_{10}, x_4x_{10}, x_6x_{10}, x_4x_{11}, x_5x_{11}, x_6x_{11}, x_7x_{11})$.
5. *If $n = 12$, then*

   $(x_1x_5, x_2x_6, x_1x_7, x_3x_7, x_2x_8, x_4x_8, x_2x_9, x_3x_9, x_5x_9, x_1x_{10}, x_4x_{10}, x_6x_{10}, x_4x_{11}, x_5x_{11}, x_6x_{11}, x_7x_{11}, x_3x_{12}, x_5x_{12}, x_6x_{12}, x_8x_{12})$.
6. *If $n = 13$, then*

   $(x_1x_5, x_2x_6, x_1x_7, x_3x_7, x_2x_8, x_4x_8, x_2x_9, x_3x_9, x_5x_9, x_1x_{10}, x_3x_{10}, x_4x_{10}, x_6x_{10}, x_3x_{11}, x_5x_{11}, x_6x_{11}, x_8x_{11}, x_2x_{12}, x_4x_{12}, x_5x_{12}, x_7x_{12}, x_4x_{13}, x_6x_{13}, x_7x_{13}, x_9x_{13})$.

See [18] for the concrete algorithm we used. By Theorem 6 in this case, the Cohen-Macaulay property is equivalent to the (S_2) property, which is independent of the base field K.

5. Example

In this section, we give an example of a Gorenstein squarefree monomial ideal I such that S/I^2 satisfies the Serre condition (S_2), but it is not Cohen-Macaulay.

The Cohen-Macaulay property of I_Δ^2 implies the "Gorenstein" property of I_Δ. More precisely:

Theorem 9 ([7])**.** *Let Δ be a simplicial complex on $[n]$. Suppose that S/I_Δ^2 is Cohen-Macaulay over any field K. Then, Δ is Gorenstein for any field K.*

In [7], the authors asked the following question:

Question. *Let Δ be a simplicial complex on $[n]$. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring for a fixed field K. Suppose Δ satisfies the following conditions:*

1. *Δ is Gorenstein.*
2. *S/I_Δ^2 satisfies the Serre condition (S_2).*

Then, is it true that S/I_Δ^2 is Cohen-Macaulay?

Using a list in [19] and CoCoA, we have the following counter-example:

Example 1. *Let K be a field of characteristic zero. Set:*

$$I_\Delta = (x_1x_{10}, x_3x_9, x_2x_9, x_7x_8, x_2x_8, x_4x_7, x_5x_6, x_3x_6, x_4x_5, x_6x_8x_{10}, x_2x_5x_{10}, x_1x_4x_9, x_1x_3x_7).$$

Then, the following conditions hold:

1. *Δ is Gorenstein.*
2. *S/I_Δ^2 satisfies the Serre condition (S_2).*
3. *S/I_Δ^2 is not Cohen-Macaulay.*

We explain how to find the example. The manifold page of Lutz [19] gives a classification of all triangulations Δ of the three-sphere with 10 vertices, which shows that there are 247,882 types. Using Theorem 3, we checked the Serre condition (S_2) for them, and there were only nine types such that S/I_Δ^2 satisfies the Serre condition (S_2). Among the nine types, there was only one simplicial complex Δ such that S/I_Δ^2 is not Cohen-Macaulay, which is the above example. Note that a triangulation Δ of a sphere is always Gorenstein. See [18] for more information.

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