Modified Proximal Algorithms for Finding Solutions of the Split Variational Inclusions

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Abstract: We investigate the split variational inclusion problem in Hilbert spaces. We propose efficient algorithms in which, in each iteration, the stepsize is chosen self-adaptive, and proves weak and strong convergence theorems. We provide numerical experiments to validate the theoretical results for solving the split variational inclusion problem as well as the comparison to algorithms defined by Byrne et al. and Chuang, respectively. It is shown that the proposed algorithms outrun other algorithms via numerical experiments. As applications, we apply our method to compressed sensing in signal recovery. The proposed methods have as a main advantage that the computation of the Lipschitz constants for the gradient of functions is dropped in generating the sequences.

Keywords: split variational inclusion problem; compressed sensing; proximal algorithm; hilbert spaces

1. Introduction

Let $H$ be a real Hilbert space. Then, $B : H \to 2^H$ is called monotone if $\langle u - v, x - y \rangle \geq 0$ for each $u \in Bx, v \in By$. Moreover, $B$ is maximal monotone provided its graph is not properly included in the graph of other monotone mappings. Many problems in optimization can be reduced to finding $x^* \in H$ such that $0 \in Bx^*$, Martinet [1] and Rockafellar [2] suggested the proximal method for solving this problem. They construct the sequence $\{x_n\} \subset H$ by choosing $x_1 \in H$ and putting

$$x_{n+1} = J_{\beta_n}^B x_n, \quad n \in \mathbb{N},$$

where $\{\beta_n\} \subseteq (0, \infty)$, $B$ is a set-valued maximal monotone operator and $J_{\beta}^B$ is defined by $J_{\beta}^B = (I + \beta B)^{-1}$ for each $\beta > 0$. We see that Equation (1) is equivalent to $x_n - x_{n+1} \in \beta_n Bx_{n+1}, \quad n \in \mathbb{N}$.

The split variational inclusion problem (SVIP) was first investigated by Moudafi [3]. The problem consists of finding $x^* \in H_1$ such that

$$0 \in B_1(x^*) \quad \text{and} \quad 0 \in B_2(Ax^*),$$

where $H_1$ and $H_2$ are real Hilbert spaces, $B_1$ and $B_2$ are set-valued mappings on $H_1$ and $H_2$. In addition, $A : H_1 \to H_2$ is a bounded and linear operator and $A^*$ is the adjoint of $A$. We know that the SVIP is a generalization of the split feasibility problem that was investigated by Censor and Elfving [4] in Euclidean spaces. See [4–9]. In this paper, we denote by $\Omega$ the solution set of SVIP. Suppose that $\Omega$ is nonempty.

In 2011, Byrne et al. [6] established a weak convergence theorem for SVIP as follows:
Theorem 1. Let $H_1$ and $H_2$ be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators. Let $\beta > 0$ and $\gamma \in (0, \frac{2}{\|A\|^2})$. Let $\{x_n\}$ be generated by
\[
x_{n+1} = J^\beta_{\gamma}(x_n - \gamma A^*(I - J^\beta_{\alpha_n})Ax_n), \quad n \in \mathbb{N}.
\]

Then, $\{x_n\}$ converges weakly to $x^* \in \Omega$.

In 2015, Chuang [10] introduced the following iteration for SVIP in Hilbert spaces. Chuang [10] established its convergence as follows:

Theorem 2. Let $H_1$ and $H_2$ be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded and linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators. Choose $\delta \in (0, 1)$ and let $\{\beta_n\} \subseteq (0, \infty)$ and $\{\gamma_n\} \subseteq (0, \frac{\delta}{\|A\|^2})$ and assume that
\[
\sum_{n=1}^{\infty} \gamma_n = \infty, \quad \sum_{n=1}^{\infty} \frac{\gamma_n^2}{\beta_n} < \infty, \quad \liminf_{n \to \infty} \beta_n > 0.
\]

If $H_1$ is finite dimensional, then $\lim_{n \to \infty} x_n = x^* \in \Omega$.

Chuang [10] also provided the following result.

Theorem 3. Let $H_1$ and $H_2$ be infinite dimensional Hilbert spaces, $A : H_1 \to H_2$ be a bounded and linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone mappings. Choose $\delta \in (0, 1)$ and let $\{\beta_n\} \subseteq (0, \infty)$, $\liminf_{n \to \infty} \beta_n > 0$ and $\{\gamma_n\} \subseteq (0, \frac{\delta}{\|A\|^2})$ with $\inf_{n \in \mathbb{N}} \gamma_n > 0$. Then, $x_n \to x^* \in \Omega$.

In 2013, Chuang [11] proved strong convergence theorem for SVIP using the following algorithm.

**Algorithm 1:**

[11]

For $n \in \mathbb{N}$, set $y_n$ as
\[
y_n = J^\beta_{\gamma_n}(x_n - \gamma_n A^*(I - J^\beta_{\beta_n})Ax_n),
\]

where $\gamma_n > 0$ is chosen such that
\[
\gamma_n \|A^*(I - J^\beta_{\beta_n})Ax_n - A^*(I - J^\beta_{\beta_n})Ay_n\| \leq \delta \|x_n - y_n\|, \quad 0 < \delta < 1.
\]

The iterative $x_{n+1}$ is generated by
\[
x_{n+1} = J^\beta_{\alpha_n}(x_n - \alpha_n D(x_n, \gamma_n)),
\]

where
\[
D(x_n, \gamma_n) = x_n - y_n + \gamma_n (A^*(I - J^\beta_{\beta_n})Ay_n - A^*(I - J^\beta_{\beta_n})Ax_n)
\]

and
\[
\alpha_n = \frac{\|x_n - y_n, D(x_n, \gamma_n)\|}{\|D(x_n, \gamma_n)\|^2}.
\]

Theorem 4. Let $H_1$ and $H_2$ be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded and linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two set-valued maximal monotone operators. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in $[0, 1]$ with $a_n + b_n + c_n + d_n = 1$ and $0 < a_n < 1$ for each $n \in \mathbb{N}$.
Let \( \{ \beta_n \} \subseteq (0, \infty) \) and let \( \{ \gamma_n \} \subseteq (0, \frac{2}{\|A\|_2^2 + 1}) \). Let \( \{ v_n \} \) be a bounded sequence in \( H_1 \). Fix \( u \in H_1 \) and let the sequence \( \{ x_n \} \subseteq H_1 \) be generated by

\[
x_{n+1} = a_n u + b_n x_n + c_n \beta_n (x_n - \gamma_n (I - \beta_n^{\ast} A)x_n) + d_n v_n
\]

for each \( n \in \mathbb{N} \). Suppose that

(i) \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{d_n}{a_n} = 0; \; \sum_{n=1}^{\infty} a_n = \infty; \; \sum_{n=1}^{\infty} d_n < \infty \);

(ii) \( \liminf_{n \to \infty} c_n \gamma_n > 0, \; \liminf_{n \to \infty} b_n c_n > 0, \; \liminf_{n \to \infty} \beta_n > 0 \).

Then, \( \lim_{n \to \infty} x_n = x^{\ast} \), where \( x^{\ast} = P_{\Omega} u \) and \( P_{\Omega} u \) is nearest to \( u \).

We aim to find the approximate algorithms with a new step size which is self-adaptive (see López et al. [8]) for solving our SVIP and prove its convergence. We present numerical examples and the comparison to algorithms of Byrne et al. [6] and algorithms of Chuang [10,11]. We also obtain the result for split feasibility problem (SFP) and its applications to compressed sensing in signal recovery. It reveals that our methods have a better convergence than those of Byrne et al. [6] and Chuang [10,11].

2. Preliminaries

We next provide some basic concepts for our proof. In what follows, we shall use the following symbols:

- \( \rightharpoonup \) stands for the weak convergence,
- \( \rightarrow \) stands for the strong convergence.

Recall that a mapping \( T : H \to H \) is called

(1) nonexpansive if, for all \( x, y \in H \),

\[
\| Tx - Ty \| \leq \| x - y \|. \tag{11}
\]

(2) firmly-nonexpansive if, for all \( x, y \in H \),

\[
\| Tx - Ty \|^2 \leq \langle Tx - Ty, x - y \rangle. \tag{12}
\]

It is clear that \( I - T \) is also firmly-nonexpansive when \( T \) is firmly-nonexpansive. We know that, for each \( x, y \in H \),

\[
\langle x, y \rangle = \frac{1}{2} \| x \|^2 + \frac{1}{2} \| y \|^2 - \frac{1}{2} \| x - y \|^2 \tag{13}
\]

and

\[
\| tx + (1 - t)y \|^2 = t \| x \|^2 + (1 - t) \| y \|^2 - t(1 - t) \| x - y \|^2 \tag{14}
\]

for all \( x, y \in H \) and for all \( t \in [0, 1] \).

The following lemma can be found in [12].

**Lemma 1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T : C \to C \) be a nonexpansive mapping. If \( x_n \rightharpoonup x \in C \) and \( \lim_{n \to \infty} \| x_n - Tx_n \| = 0 \), then \( x = Tx \).

We use \( \text{Fix}(T) \) by the fixed point set of a mapping \( T \), that is, \( \text{Fix}(T) = \{ x \in H : x = Tx \} \) and \( D(T) \) by the domain of a mapping \( T \), i.e., \( D(T) = \{ x \in H : T(x) \neq \emptyset \} \).

The following lemma can be found in [11,13].

**Lemma 2.** Let \( H \) be a real Hilbert space and let \( B : H \to 2^H \) be a maximal monotone operator. Then,
Let $B$ be a nonempty subset of a Hilbert space $H$. Let $\gamma > 0, B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ be maximal monotone operators. Let $x^* \in H_1$.

(i) $J_{\beta}^B$ is single-valued and firmly nonexpansive for each $\beta > 0$;
(ii) $\mathcal{D}(J_{\beta}^B) = H$ and $\text{Fix}(J_{\beta}^B) = \{x \in \mathcal{D}(B) : 0 \in Bx\}$;
(iii) $\|x - J_{\beta}^Bx\| \leq \|x - J_{\beta'}^Bx\|$ for all $0 < \beta \leq \gamma$ and for all $x \in H$;
(iv) If $B^{-1}(0) \neq \emptyset$, then we have $\|x - J_{\beta}^Bx\|^2 + \|J_{\beta}^Bx - x^*\|^2 \leq \|x - x^*\|^2$ for all $x \in H$, each $x^* \in B^{-1}(0)$, and each $\beta > 0$;
(v) If $B^{-1}(0) \neq \emptyset$, then we have $(x - J_{\beta}^Bx, J_{\beta}^Bx - w) \geq 0$ for all $x \in H$, each $w \in B^{-1}(0)$, and each $\beta > 0$.

**Lemma 3.** Let $H_1$ and $H_2$ be real Hilbert spaces. Let $A : H_1 \rightarrow H_2$ be a bounded and linear operator. Let $\gamma > 0, B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ be maximal monotone operators. Let $x^* \in H_1$.

(i) If $x^*$ is a solution of (SVIP), then $J_{\beta}^B(x^* - \gamma A^*(I - J_{\beta}^B)A)x^*) = x^*$.
(ii) Suppose that $J_{\beta}^B(x^* - \gamma A^*(I - J_{\beta}^B)A)x^*) = x^*$ and the solution set of (SVIP) is nonempty. Then, $x^*$ is a solution of (SVIP).

**Lemma 4.** Let $H_1$ and $H_2$ be real Hilbert spaces. Let $A : H_1 \rightarrow H_2$ be a bounded and linear operator and $\beta > 0$. Let $B : H_2 \rightarrow 2^{H_2}$ be a maximal monotone operator. Define a mapping $T : H_1 \rightarrow H_1$ by $Tx := A^*(I - J_{\beta}^B)Ax$ for each $x \in H_1$. Then,

(i) $\|A(I - J_{\beta}^B)Ax - A(I - J_{\beta}^B)Ay\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in H_1$;
(ii) $\|A^*(I - J_{\beta}^B)Ax - A^*(I - J_{\beta}^B)Ay\|^2 \leq \|A\|^2 : \langle Tx - Ty, x - y\rangle$ for all $x, y \in H_1$.

The following lemma can be found in [14].

**Lemma 5.** Let $C$ be a nonempty subset of a Hilbert space $H$. Let $\{x_n\}$ be a sequence in $H$ that satisfies the following assumptions:

(i) $\lim_{n \to \infty} \|x_n - x\|$ exists for each $x \in C$;
(ii) every sequential weak limit point of $\{x_n\}$ is in $C$.

Then, $\{x_n\}$ weakly converges to a point in $C$.

The following lemma can be found in [15].

**Lemma 6.** Assume $\{s_n\} \subseteq (0, \infty)$ such that

\[
s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \delta_n, \quad n \geq 1, \quad (15)
\]

\[
s_{n+1} \leq s_n - \lambda_n + \varphi_n, \quad n \geq 1, \quad (16)
\]

where $\{\alpha_n\} \subseteq (0, 1), \{\lambda_n\} \subseteq (0, 1)$ and $\{\delta_n\}$ and $\{\varphi_n\}$ are real sequences such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n \to \infty} \varphi_n = 0$;
(iii) $\lim_{k \to \infty} \lambda_n = 0$ implies $\lim_{k \to \infty} \sup_{\delta_n} \delta_n \leq 0$ for any subsequence $\{n_k\}$ of $\{n\}$.

Then, $\lim_{n \to \infty} s_n = 0$.

3. Weak Convergence Result

Let $H_1$ and $H_2$ be real Hilbert spaces, $A : H_1 \rightarrow H_2$ be a bounded and linear operator. Let $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ be set-valued maximal monotone operators.

Let $\Omega$ be a solution set of problem (SVIP) and assume that $\Omega \neq \emptyset$. We remark that the stepsize sequence $\{\gamma_n\}$ does not depend on the norm of an operator $A$ as introduced by Byrne et al. [6] and Chuang [10, 11].
**Theorem 5.** Suppose that \( \lim \inf \beta_n > 0 \), \( \inf \rho_n (4 - \rho_n) > 0 \) and \( \lim \theta_n = 0 \). Then, \( \{x_n\} \) defined by Algorithm 2 converges weakly to a solution in \( \Omega \).

**Algorithm 2:**

Choose \( x_1 \in H_1 \) and define

\[
x_{n+1} = \frac{B_{\rho_n}}{B_{\rho_n}} (x_n - \gamma_n g(x_n)),
\]

where

\[
\gamma_n = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2 + \theta_n}, \quad 0 < \rho_n < 4, \quad 0 < \theta_n < 1, \quad \beta_n > 0,
\]

and

\[
f(x_n) = \frac{1}{2} \| (I - B_{\beta_n}^2) A x_n \|^2, \quad g(x_n) = A^\ast (I - B_{\beta_n}^2) A x_n.
\]

**Proof.** Let \( z \in \Omega \). Then, \( z \in B_{\rho_n}^{-1}(0) \) and \( Az \in B_{\beta_n}^{-1}(0) \). Thus, we have \( B_{\beta_n}^2 Az = Az \). Using Lemma 4 (i), we have

\[
\langle x_n - z, g(x_n) \rangle = \langle x_n - z, g(x_n) - g(z) \rangle
= \langle x_n - z, A^\ast (I - B_{\beta_n}^2) A x_n - A^\ast (I - B_{\beta_n}^2) A z \rangle
= \langle A x_n - A z, (I - B_{\beta_n}^2) A x_n - (I - B_{\beta_n}^2) A z \rangle
\geq \| (I - B_{\beta_n}^2) A x_n \|^2
= 2 f(x_n).
\]

From Equation (20), Lemma 2 (iv) and the defining formulas for Algorithm 2

\[
\|x_{n+1} - z\|^2 = \| B_{\rho_n}^1 (x_n - \gamma_n g(x_n)) - z \|^2
\leq \| x_n - z \|^2 - \| x_{n+1} - x_n + \gamma_n g(x_n) \|^2
= \| x_n - z \|^2 + \| \gamma_n g(x_n) \|^2 - 2 \gamma_n \langle x_n - z, g(x_n) \rangle - \| x_{n+1} - x_n + \gamma_n g(x_n) \|^2
\leq \| x_n - z \|^2 + \| \gamma_n g(x_n) \|^2 - 4 \gamma_n f(x_n) - \| x_{n+1} - x_n + \gamma_n g(x_n) \|^2
= \| x_n - z \|^2 + \frac{\rho_n f^2(x_n)}{\| g(x_n) \|^2 + \theta_n} \| g(x_n) \|^2 - \| x_{n+1} - x_n + \gamma_n g(x_n) \|^2
\leq \| x_n - z \|^2 - \frac{\rho_n f^2(x_n)}{\| g(x_n) \|^2 + \theta_n} \| g(x_n) \|^2 - \| x_{n+1} - x_n + \gamma_n g(x_n) \|^2
= \| x_n - z \|^2 - \rho_n (4 - \rho_n) \| g(x_n) \|^2 + \theta_n - \| x_{n+1} - x_n + \gamma_n g(x_n) \|^2.
\]

This implies that, since \( 0 < \rho_n < 4 \),

\[
\| x_{n+1} - z \| \leq \| x_n - z \|.
\]

Thus, \( \lim_{n \to \infty} \| x_n - z \| \) exists. It follows that \( \{x_n\} \) is bounded. Again, by Equation (21), we get

\[
\rho_n (4 - \rho_n) \frac{f^2(x_n)}{\| g(x_n) \|^2 + \theta_n} \leq \| x_n - z \|^2 - \| x_{n+1} - z \|^2.
\]
which yields by our assumptions that

$$\lim_{n \to \infty} \frac{f^2(x_n)}{\|g(x_n)\|^2} = 0. \quad (24)$$

By Lemma 3 (ii), it can be checked that $g$ is a Lipschitzian mapping and thus $\{\|g(x_n)\|\}$ is bounded. Hence, we get $\lim_{n \to \infty} f(x_n) = 0$. This means

$$\lim_{n \to \infty} \| (I - \beta_n^2) Ax_n \| = 0. \quad (25)$$

Furthermore, by Equation (21), we also have

$$\lim_{n \to \infty} \| x_{n+1} - x_n + \gamma_n g(x_n) \| = 0. \quad (26)$$

We note that

$$\gamma_n \| g(x_n) \| = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2 + \theta_n} \|g(x_n)\| \to 0, \text{ as } n \to \infty. \quad (27)$$

Hence, by Equations (26) and (27), we obtain

$$\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0. \quad (28)$$

From Equation (25) and Lemma 2 (iii), we get

$$\lim_{n \to \infty} \| Ax_n - \beta_n^2 Ax_n \| \leq \lim_{n \to \infty} \| Ax_n - \beta_n^2 Ax_n \| = 0, \quad (29)$$

for some $\beta > 0$ such that $\beta_n \geq \beta > 0$ for all $n \in \mathbb{N}$. From Equation (27), we see that

$$\| x_{n+1} - \beta_n^2 x_n \| = \| \beta_n (x_n - \gamma_n g(x_n)) - \beta_n x_n \|$$

$$\leq \| x_n - \gamma_n g(x_n) - x_n \|$$

$$= \gamma_n \| g(x_n) \|$$

$$\to 0 \text{ as } n \to \infty. \quad (30)$$

From Equations (28) and (30), we have

$$\| x_n - \beta_n^2 x_n \| = \| x_n - x_{n+1} + x_{n+1} - \beta_n x_n \|$$

$$\leq \| x_n - x_{n+1} \| + \| x_{n+1} - \beta_n x_n \|$$

$$\to 0 \text{ as } n \to \infty. \quad (31)$$

Lemma 2 (iii) gives

$$\lim_{n \to \infty} \| x_n - \beta_n^2 x_n \| \leq \lim_{n \to \infty} \| x_n - \beta_n^2 x_n \| = 0. \quad (32)$$

Since $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x^* \in H$ with $x_{n_k} \to x^*$. We also have $Ax_{n_k} \to Ax^*$. By Equations (29) and (32), Lemmas 1 and 2 (ii), we obtain $x^* \in \Omega$. Using Lemma 5, we obtain that $\{x_n\}$ converges weakly to a solution in $\Omega$. \(\Box\)

4. Strong Convergence Result

**Theorem 6.** Assume that $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\theta_n\}$ satisfy the assumptions:

(a1) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
Proof. Let \( z = P_{\Omega}u \in \Omega \). Using the line of proof as for Theorem 5, we have

\[
\| f_{\beta_n}^1(x_n - \gamma_n g(x_n)) - z \|^2 \leq \| x_n - z \|^2 - \rho_n (4 - \rho_n) \frac{f^2(x_n)}{\| g(x_n) \|^2 + \theta_n} - \| f_{\beta_n}^1(x_n - \gamma_n g(x_n)) - x_n + \gamma_n g(x_n) \|^2. \tag{36}
\]

Then,

\[
\| x_{n+1} - z \|^2 = \| \alpha_n (u - z) + (1 - \alpha_n) f_{\beta_n}^1(x_n - \gamma_n g(x_n)) - z \|^2 \leq \| (1 - \alpha_n) \| f_{\beta_n}^1(x_n - \gamma_n g(x_n)) - x_n + \gamma_n g(x_n) \|^2 + 2 \alpha_n \langle u - z, x_{n+1} - z \rangle. \tag{37}
\]

Combining Equations (36) and (37), we get

\[
\| x_{n+1} - z \|^2 \leq (1 - \alpha_n) \| x_n - z \|^2 - (1 - \alpha_n) \rho_n (4 - \rho_n) \frac{f^2(x_n)}{\| g(x_n) \|^2 + \theta_n} + (1 - \alpha_n) \| f_{\beta_n}^1(x_n - \gamma_n g(x_n)) - x_n + \gamma_n g(x_n) \|^2 + 2 \alpha_n \langle u - z, x_{n+1} - z \rangle. \tag{38}
\]

Next, we will show that \( \{ x_n \} \) is bounded. Again, using Equation (36),

\[
\| x_{n+1} - z \| = \| \alpha_n u + (1 - \alpha_n) f_{\beta_n}^1(x_n - \gamma_n g(x_n)) - z \| \leq \alpha_n \| u - z \| + (1 - \alpha_n) \| x_n - z \|. \tag{39}
\]

Thus, \( \{ x_n \} \) is bounded. Employing Lemma 6, from Equation (38), we set

\[
\begin{align*}
\mathcal{S}_n &= \| x_n - z \|^2; \\
\varphi_n &= 2 \alpha_n \langle u - z, x_{n+1} - z \rangle; \\
\delta_n &= 2 \langle u - z, x_{n+1} - z \rangle; \\
\lambda_n &= (1 - \alpha_n) \rho_n (4 - \rho_n) \frac{f^2(x_n)}{\| g(x_n) \|^2 + \theta_n} + (1 - \alpha_n) \| f_{\beta_n}^1(x_n - \gamma_n g(x_n)) - x_n + \gamma_n g(x_n) \|^2.
\end{align*}
\]
Thus, Equation (38) reduces to the inequalities
\[
\begin{align*}
    s_{n+1} &\leq (1 - \alpha_n)s_n + \alpha_n\delta_n, \quad n \geq 1, \\
    s_{n+1} &\leq s_n - \lambda_n + \varphi_n.
\end{align*}
\]

Let \( \{n_k\} \subseteq \{n\} \) be such that
\[
\lim_{k \to \infty} \lambda_{n_k} = 0.
\]

Then, we have
\[
\lim_{k \to \infty} \left( (1 - \alpha_{n_k})\rho_{n_k} (4 - \rho_{n_k}) \frac{f^2(x_{n_k})}{\|g(x_{n_k})\|^2 + \theta_{n_k}} + (1 - \alpha_{n_k})\left\| f_{\beta_{n_k}}^1 (x_{n_k} - \gamma_{n_k} g(x_{n_k})) - x_{n_k} + \gamma_{n_k} g(x_{n_k}) \right\|^2 \right) = 0,
\]
which, by using our assumptions, implies
\[
\frac{f^2_{n_k}(x_{n_k})}{\|g_{n_k}(x_{n_k})\|^2} \to 0 \text{ as } k \to \infty,
\]
and
\[
\left\| f_{\beta_{n_k}}^1 (x_{n_k} - \gamma_{n_k} g(x_{n_k})) - x_{n_k} + \gamma_{n_k} g(x_{n_k}) \right\| \to 0 \text{ as } k \to \infty.
\]

Since \( \{\|g_{n_k}(x_{n_k})\|\} \) is bounded, it follows that \( f_{n_k}(x_{n_k}) \to 0 \) as \( k \to \infty \). Thus, we get
\[
\lim_{k \to \infty} \left\| (I - f_{\beta_{n_k}}^1)Ax_{n_k} \right\| = 0.
\]

As the same proof in Theorem 5, we can show that there is \( \{x_{n_j}\} \) of \( \{x_{n_k}\} \) such that \( x_{n_j} \to x^* \in \Omega \). From Lemma 2 (v), we obtain
\[
\limsup_{k \to \infty} (u - z, x_{n_k} - z) = \lim_{k \to \infty} (u - z, x_{n_j} - z) = (u - z, x^* - z) \leq 0.
\]

We see that
\[
\|x_{n_{k+1}} - x_{n_k}\| = \|\alpha_{n_k}u + (1 - \alpha_{n_k})f_{\beta_{n_k}}^1 (x_{n_k} - \gamma_{n_k} g(x_{n_k})) - x_{n_k}\| \\
\leq \alpha_{n_k} \|u - x_{n_k}\| + (1 - \alpha_{n_k})\|f_{\beta_{n_k}}^1 (x_{n_k} - \gamma_{n_k} g(x_{n_k})) - x_{n_k}\| \\
\leq \alpha_{n_k} \|u - x_{n_k}\| + (1 - \alpha_{n_k})\left\| f_{\beta_{n_k}}^1 (x_{n_k} - \gamma_{n_k} g(x_{n_k})) - x_{n_k} + \gamma_{n_k} g(x_{n_k}) \right\| \\
+ (1 - \alpha_{n_k})\gamma_{n_k}\|g(x_{n_k})\| \\
\to 0 \text{ as } k \to \infty.
\]

From Equations (48) and (49), it follows that
\[
\limsup_{k \to \infty} (u - z, x_{n_{k+1}} - z) \leq 0.
\]

Hence, we get
\[
\limsup_{k \to \infty} \delta_{n_k} \leq 0.
\]

Thus, \( \{x_n\} \) converges strongly to \( z = P_\Omega u \) by Lemma 6. \( \square \)
5. Numerical Experiments

We present numerical experiments for our main results.

First, we give a comparison among Theorems 1–3 and 5 for a weak convergence theorem.

The following example is introduced in [10].

Let $B_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $B_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix}, B_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix}, B_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & 2 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \quad (52)$$

We aim to find $x^* = (x_1^*, x_2^*)^T \in \mathbb{R}^2$ such that $B_1(x^*) = (0, 0)^T$ and $B_2(Ax^*) = (0, 0, 0)^T$. In this case, we know that $x_1^* = 1.5$ and $x_2^* = -0.5$.

We set $\gamma_n = 0.001$ in Theorem 1, $\gamma_n = \frac{1}{2n||A||^2}$ in Theorem 2, $\gamma_n = \frac{\delta}{2||A||^2}$ in Theorem 3 and $\gamma_n = \frac{\rho_n f(x_n)}{\xi(x_n) \theta_n}$, $\theta_n = \frac{1}{n^2}$ in Theorem 5. The stopping criterion is given by $\|x_n - x^*\|_2 < \epsilon$.

We test by the following cases:

Case 1: $x_1 = [1, 1]$, $\beta_n = 1, \rho_n = \frac{1}{n^2}$, and $\delta = \frac{1}{4}$,
Case 2: $x_1 = [4, -2]$, $\beta_n = 2, \rho_n = \frac{1}{n^2}$, and $\delta = \frac{1}{2}$,
Case 3: $x_1 = [-5, -3]$, $\beta_n = 3, \rho_n = 2.8$, and $\delta = \frac{3}{1}$,
Case 4: $x_1 = [-2, -7]$, $\beta_n = 4, \rho_n = 3.9$, and $\delta = \frac{5}{1}$.

From Table 1, we see that Theorem 5 using Algorithm 2 has a better convergence rate than other algorithms.

**Table 1. Comparison for Theorems 1–3 and 5 for each case.**

<table>
<thead>
<tr>
<th>Method</th>
<th>$\epsilon = 10^{-4}$</th>
<th>$\epsilon = 10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CPU</td>
<td>Iter</td>
</tr>
<tr>
<td>Case 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theorem 1</td>
<td>0.1091</td>
<td>3657</td>
</tr>
<tr>
<td>Theorem 2</td>
<td>0.0078</td>
<td>131</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>0.0452</td>
<td>777</td>
</tr>
<tr>
<td>Theorem 5</td>
<td>0.0017</td>
<td>66</td>
</tr>
<tr>
<td>Case 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theorem 1</td>
<td>0.1565</td>
<td>4645</td>
</tr>
<tr>
<td>Theorem 2</td>
<td>0.0276</td>
<td>454</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>0.0368</td>
<td>609</td>
</tr>
<tr>
<td>Theorem 5</td>
<td>0.0011</td>
<td>39</td>
</tr>
<tr>
<td>Case 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theorem 1</td>
<td>0.1390</td>
<td>4572</td>
</tr>
<tr>
<td>Theorem 2</td>
<td>0.0280</td>
<td>471</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>0.0635</td>
<td>1048</td>
</tr>
<tr>
<td>Theorem 5</td>
<td>0.0014</td>
<td>45</td>
</tr>
<tr>
<td>Case 4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theorem 1</td>
<td>0.1189</td>
<td>4069</td>
</tr>
<tr>
<td>Theorem 2</td>
<td>0.0213</td>
<td>345</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>0.0686</td>
<td>1159</td>
</tr>
<tr>
<td>Theorem 5</td>
<td>0.0011</td>
<td>34</td>
</tr>
</tbody>
</table>

Second, we give a comparison between Theorems 4 and 6 for a strong convergence theorem by using Example 1.

Choose $a_n = \frac{1}{n^2}, b_n = \frac{1}{5}, c_n = 1 - a_n - b_n, d_n = 0$ and $\gamma_n = \frac{1}{||A||^2}$ in Theorem 4 and set $\theta_n = \frac{1}{n^2}, \alpha_n = \frac{1}{n^3}$ and $\gamma_n = \frac{\rho_n f(x_n)}{\xi(x_n) \theta_n}$ in Theorem 6. In this case, we let $\nu = [2, 2]$.

We test by the following cases:

Case 1: $x_1 = [1, 1], \beta_n = 1$ and $\rho_n = \frac{1.5}{n^2}$.
Case 2: \( x_1 = [4, -2], \beta_n = 2 \) and \( \rho_n = \frac{3.5n}{n+1} \),

Case 3: \( x_1 = [-5, -3], \beta_n = 3 \) and \( \rho_n = 2.8 \),

Case 4: \( x_1 = [-2, -7], \beta_n = 4 \) and \( \rho_n = 3.9 \).

From Table 2, we observe that, in each case, the convergence behavior of Theorem 4 is worse than that of Theorem 6.

<table>
<thead>
<tr>
<th>Case</th>
<th>Method</th>
<th>( \varepsilon = 10^{-4} )</th>
<th>( \varepsilon = 10^{-5} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CPU Iter</td>
<td>CPU Iter</td>
<td></td>
</tr>
<tr>
<td>Case 1</td>
<td>Theorem 4</td>
<td>0.0310</td>
<td>0.0871</td>
</tr>
<tr>
<td></td>
<td>Theorem 6</td>
<td>0.0068</td>
<td>0.0240</td>
</tr>
<tr>
<td>Case 2</td>
<td>Theorem 4</td>
<td>0.0225</td>
<td>0.0790</td>
</tr>
<tr>
<td></td>
<td>Theorem 6</td>
<td>0.0038</td>
<td>0.0117</td>
</tr>
<tr>
<td>Case 3</td>
<td>Theorem 4</td>
<td>0.0204</td>
<td>0.0727</td>
</tr>
<tr>
<td></td>
<td>Theorem 6</td>
<td>0.0043</td>
<td>0.0132</td>
</tr>
<tr>
<td>Case 4</td>
<td>Theorem 4</td>
<td>0.0241</td>
<td>0.0677</td>
</tr>
<tr>
<td></td>
<td>Theorem 6</td>
<td>0.0038</td>
<td>0.0156</td>
</tr>
</tbody>
</table>

6. Split Feasibility Problem

Let \( H_1 \) and \( H_2 \) be real Hilbert spaces. We next study the split feasibility problem (SFP) that is to seek \( x^* \in H_1 \) such that

\[
x^* \in C \text{ and } Ax^* \in Q,
\]

where \( C \) and \( Q \) are nonempty closed convex subsets of \( H_1 \) and \( H_2 \), respectively, and \( A : H_1 \to H_2 \) is a bounded linear operator with the adjoint operator \( A^* \). Many authors introduced various algorithms for solving the SFP [16–19].

Let \( H \) be a Hilbert space and let \( g : H \to (-\infty, \infty] \) be a proper, lower semicontinuous and convex function. The subdifferential \( \partial g \) of \( g \) is defined by

\[
\partial g(x) = \{ z \in H : g(x) + \langle z, y-x \rangle \leq g(y), \forall y \in H \} \tag{54}
\]

for all \( x \in H \). Let \( C \) be a nonempty closed convex subset of \( H \), and \( \iota_C \) be the indicator function of \( C \) defined by

\[
\iota_C(x) = \begin{cases} 
0 & x \in C, \\
\infty & x \notin C.
\end{cases} \tag{55}
\]

The normal cone \( N_Cu \) of \( C \) at \( u \) is defined by

\[
N_Cu = \{ z \in H : \langle z, v-u \rangle \leq 0, \forall v \in C \}. \tag{56}
\]

Then, \( \iota_C \) is a proper, lower semicontinuous and convex function on \( H \). See [20,21]. Moreover, the subdifferential \( \partial \iota_C \) of \( \iota_C \) is a maximal monotone mapping. In this connection, we can define the resolvent \( J_{\partial \iota_C}^\lambda \) of \( \partial \iota_C \) for \( \lambda > 0 \) by

\[
J_{\partial \iota_C}^\lambda x = (I + \lambda \partial \iota_C)^{-1}x \tag{57}
\]

for all \( x \in H \). Hence, we see that

\[
\partial \iota_C x = \{ z \in H : \iota_C(x) + \langle z, y-x \rangle \leq \iota_C(y), \forall y \in H \} = \{ z \in H : \langle z, y-x \rangle \leq 0, \forall y \in C \} = N_Cx \tag{58}
\]
for all $x \in C$. Hence, for each $\beta > 0$, we obtain the following relation:

$$ u = \int_{\beta}^{C} x \iff x \in u + \beta \partial_{C} u $$

$$ \iff x - u \in \beta N C u $$

$$ \iff \langle x - u, y - u \rangle \leq 0, \forall y \in C $$

$$ u = P_{C} x. \quad (59) $$

Consequently, we obtain the following results which are deduced from Algorithm 2.

**Theorem 7.** Assume that $\inf \rho_{n}(4 - \rho_{n}) > 0$ and $\lim_{n \to \infty} \theta_{n} = 0$. Choose $x_{1} \in H_{1}$ and let $\{x_{n}\}$ be defined by

$$ x_{n+1} = P_{C}(x_{n} - \gamma g(x_{n})) $$

where $\gamma_{n} = \frac{\rho_{n}f(x_{n})}{\|g(x_{n})\|^{2} + \theta_{n}}, \quad 0 < \rho_{n} < 4, \quad 0 < \theta_{n} < 1 \quad (60) $$

and

$$ f(x_{n}) = \frac{1}{2}\| (I - P_{Q})Ax_{n} \|^{2}, \quad g(x_{n}) = A^{*}(I - P_{Q})Ax_{n}. \quad (61) $$

Then, $\{x_{n}\}$ converges weakly to a solution in $\Omega$.

By Theorem 1, we obtain the result of Byrne et al. [6].

**Theorem 8.** Let $\{x_{n}\}$ be generated by

$$ x_{n+1} = P_{C}(x_{n} - \gamma A^{*}(I - P_{Q})Ax_{n}), \quad n \in \mathbb{N}, \quad (62) $$

where $H_{1}$ and $H_{2}$ are Hilbert spaces, $A : H_{1} \to H_{2}$ be a bounded and linear operator and $\gamma \in (0, \frac{\beta}{\|A\|})$. Then, $\{x_{n}\}$ converges weakly to $x^{*} \in \Omega$.

Using Chuang’s results in Algorithm 1, we have

**Theorem 9.** Let $H_{1}$ and $H_{2}$ be infinite dimensional Hilbert spaces, $A : H_{1} \to H_{2}$ be a bounded and linear operator. Choose $\delta \in (0,1)$ and $\{\gamma_{n}\} \subseteq (0, \frac{\beta}{\|A\|})$ with $\inf_{n \in \mathbb{N}} \gamma_{n} > 0$. Choose $x_{1} \in H_{1}$. For $n \in \mathbb{N}$, set $y_{n}$ as

$$ y_{n} = P_{C}(x_{n} - \gamma A^{*}(I - P_{Q})Ax_{n}), \quad (63) $$

where $\gamma_{n} > 0$ satisfies

$$ \gamma_{n}\| A^{*}(I - P_{Q})Ax_{n} - A^{*}(I - P_{Q})Ay_{n} \| \leq \delta \| x_{n} - y_{n} \|, \quad 0 < \delta < 1 \quad (64) $$

Construct $x_{n+1}$ by

$$ x_{n+1} = P_{C}(x_{n} - \alpha_{n} D(x_{n}, \gamma_{n})), \quad (65) $$

where

$$ D(x_{n}, \gamma_{n}) = x_{n} - y_{n} + \gamma_{n}(A^{*}(I - P_{Q})Ay_{n} - A^{*}(I - P_{Q})Ax_{n}) \quad (66) $$

and

$$ \alpha_{n} = \frac{\langle x_{n} - y_{n}, D(x_{n}, \gamma_{n}) \rangle}{\| D(x_{n}, \gamma_{n}) \|^{2}}. \quad (67) $$

Then, the sequence $\{x_{n}\}$ converges weakly to $x^{*} \in \Omega$.

From Algorithm 3 and Theorem 6, we have
**Theorem 10.** Assume that \( \{a_n\}, \{\rho_n\} \) and \( \{\theta_n\} \) satisfy the assumptions:

(a1) \( \lim_{n \to \infty} a_n = 0 \) and \( \sum_{n=1}^{\infty} a_n = \infty; \)

(a2) \( \inf_{n} \rho_n (4 - \rho_n) > 0; \)

(a3) \( \lim_{n \to \infty} \theta_n = 0. \)

Choose \( x_1 \in H_1 \) and define \( \{x_n\} \) by

\[
x_{n+1} = a_n u + (1 - a_n) PC(x_n - \gamma_n g(x_n)),
\]

where

\[
\gamma_n = \frac{\rho_n f(x_n)}{\|g(x_n)\|^2 + \theta_n}, \quad 0 < \rho_n < 4, \quad 0 < \theta_n < 1, \quad 0 < a_n < 1,
\]

and

\[
f(x_n) = \frac{1}{2} \|(I - P_Q)Ax_n\|^2, \quad g(x_n) = A^* (I - P_Q)Ax_n.
\]

Then, \( \{x_n\} \) converges strongly to \( z = P_{\Omega} u. \)

We also have the following result.

**Theorem 11.** Let \( \{a_n\}, \{b_n\}, \{c_n\}, \) and \( \{d_n\} \) be sequences of real numbers in \([0,1]\) with \( a_n + b_n + c_n + d_n = 1 \) and \( 0 < a_n < 1 \) for each \( n \in \mathbb{N}. \) Let \( \{v_n\} \) be a bounded sequence in \( H_1. \) Let \( u \in H_1 \) be fixed and \( \{\gamma_n\} \subseteq (0, \frac{2}{\|A\|^2 + 1}). \) Let \( \{x_n\} \) be defined by

\[
x_{n+1} = a_n u + b_n x_n + c_n PC(x_n - \gamma_n A^* (I - P_Q)Ax_n) + d_n v_n
\]

for each \( n \in \mathbb{N}. \) Suppose that

(i) \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0; \) \( \sum_{n=1}^{\infty} a_n = \infty; \) \( \sum_{n=1}^{\infty} d_n < \infty; \)

(ii) \( \liminf_{n \to \infty} c_n \gamma_n > 0 \) and \( \liminf_{n \to \infty} b_n c_n > 0. \)

Then, \( \lim_{n \to \infty} x_n = x^* \), where \( x^* = P_{\Omega} u, A : H_1 \to H_2 \) be a bounded and linear operator. Then, \( \{x_n\} \) converges strongly to a point in \( \Omega. \)

**7. Applications to Compressed Sensing**

In signal processing, we consider the following linear equation:

\[
y = Ax + \varepsilon,
\]

where \( x \in \mathbb{R}^N \) is a sparse vector that has \( m \) nonzero components, \( y \in \mathbb{R}^M \) is the observed data with noisy \( \varepsilon, \) and \( A : \mathbb{R}^N \to \mathbb{R}^M \) (\( M < N \)). It can be seen that Equation (73) relates to the LASSO problem [22]

\[
\min_{x \in \mathbb{R}^N} \frac{1}{2} \|y - Ax\|_2^2 \text{ subject to } \|x\|_1 \leq t,
\]

where \( t > 0. \) In particular, if \( C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\} \) and \( Q = \{y\}, \) then the LASSO problem can be considered as the SFP Equation (53).

The vector \( x \in \mathbb{R}^N \) is generated by the uniform distribution in \([-2,2]\) with \( m \) nonzero components. Let \( A \) be an \( M \times N \) matrix that is generated by the normal distribution with mean zero and the variance one. The observed data \( y \) is generated by white Gaussian noise with signal-to-noise ratio (SNR)40. The process is started with \( t = m \) and initial point \( x_1 = 0. \)
The stopping error is defined by
\[ E_n = \frac{1}{N} \| x_n - x \|_2^2 < \kappa, \tag{75} \]
where \( x_n \) is an estimated signal of \( x \).

We give some numerical results of Theorems 7–9. Choose \( \gamma_n = \rho_n f(x_n) \| g(x_n) \|_2 + \theta_n \), \( \rho_n = 3 \), \( \theta_n = \frac{1}{n^5} \) in Theorem 7 and \( \gamma_n = \frac{\delta}{\| A \|_2} \) in Theorem 8 and \( \delta = 0.8 \), \( \gamma_n = \frac{\delta}{\| A \|_2} \) in Theorem 9.

Tables 3 and 4 show that both the number of iterations and the CPU time in our algorithm in Theorem 7 are less than algorithms in Theorems 8 and 9 have in their computations. Next, we test numerical experiments in signal recovery in the case \( N = 512, M = 256 \) and \( N = 2048, M = 1024 \), respectively.

| Table 3. Numerical results for the LASSO problem in case \( M = 256, N = 512 \). |
|-----------------|-----------------|-----------------|-----------------|
| **m-Sparse**    | **Method**      | **\( \kappa = 10^{-3} \)** | **\( \kappa = 10^{-4} \)** |
|                 | **CPU** | **Iter** | **CPU** | **Iter** |
| \( m = 10 \)    | Theorem 8 | 0.9662 | 44 | 3.6208 | 132 |
|                 | Theorem 9 | 1.3204 | 58 | 4.2151 | 170 |
|                 | Theorem 7 | 0.0054 | 26 | 0.0111 | 63 |
| \( m = 15 \)    | Theorem 8 | 1.3082 | 57 | 2.8470 | 124 |
|                 | Theorem 9 | 1.8984 | 84 | 3.7938 | 170 |
|                 | Theorem 7 | 0.0058 | 36 | 0.0099 | 72 |
| \( m = 20 \)    | Theorem 8 | 1.4928 | 65 | 3.5994 | 161 |
|                 | Theorem 9 | 2.7294 | 122 | 5.7801 | 251 |
|                 | Theorem 7 | 0.0070 | 42 | 0.0143 | 99 |
| \( m = 25 \)    | Theorem 8 | 2.2008 | 98 | 6.0600 | 275 |
|                 | Theorem 9 | 4.1730 | 183 | 18.6269 | 824 |
|                 | Theorem 7 | 0.0107 | 67 | 0.0323 | 227 |

| Table 4. Numerical results for the LASSO problem in case \( M = 2048, N = 1024 \). |
|-----------------|-----------------|-----------------|-----------------|
| **m-Sparse**    | **Method**      | **\( \kappa = 10^{-3} \)** | **\( \kappa = 10^{-4} \)** |
|                 | **CPU** | **Iter** | **CPU** | **Iter** |
| \( m = 30 \)    | Theorem 8 | 47.6530 | 41 | 119.9776 | 101 |
|                 | Theorem 9 | 67.6869 | 57 | 157.1087 | 134 |
|                 | Theorem 7 | 0.0807 | 25 | 0.1899 | 58 |
| \( m = 40 \)    | Theorem 8 | 47.7347 | 41 | 151.0891 | 117 |
|                 | Theorem 9 | 93.1898 | 79 | 306.8623 | 240 |
|                 | Theorem 7 | 0.1007 | 31 | 0.2880 | 82 |
| \( m = 50 \)    | Theorem 8 | 65.1771 | 55 | 136.1508 | 115 |
|                 | Theorem 9 | 99.0021 | 83 | 188.9366 | 158 |
|                 | Theorem 7 | 0.1227 | 35 | 0.2203 | 67 |
| \( m = 60 \)    | Theorem 8 | 76.7457 | 64 | 163.8805 | 138 |
|                 | Theorem 9 | 127.5520 | 106 | 209.5990 | 177 |
|                 | Theorem 7 | 0.1401 | 43 | 0.2449 | 75 |

Finally, we discuss the strong convergence of Theorems 10 and 11. We set \( a_n = \frac{1}{n^{1+1}}, b_n = \frac{1}{n}, c_n = 1 - a_n - b_n, d_n = 0 \) and \( \gamma_n = \frac{1}{\| A \|_2^{2+1}} \) in Theorem 11 and set \( \rho_n = 2, \theta_n = \frac{1}{n^5}, a_n = \frac{1}{n^{1+1}} \) and \( u = [1, 1, \ldots, 1] \) in Theorem 10.

Tables 5 and 6 show that our proposed algorithm in Theorem 10 has a better convergence behavior than the algorithm defined in Theorem 11 in iterations and CPU time.
Table 5. Numerical results for the LASSO problem in case $M = 512, N = 256$.

<table>
<thead>
<tr>
<th>$m$-Sparse</th>
<th>Method</th>
<th>$\kappa = 10^{-3}$</th>
<th>CPU</th>
<th>Iter</th>
<th>$\kappa = 10^{-4}$</th>
<th>CPU</th>
<th>Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 10$</td>
<td>Theorem 11</td>
<td>5.8869</td>
<td>237</td>
<td>28.2850</td>
<td>863</td>
<td>0.0296</td>
<td>157</td>
</tr>
<tr>
<td></td>
<td>Theorem 10</td>
<td>0.0260</td>
<td>155</td>
<td>0.1550</td>
<td>950</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 15$</td>
<td>Theorem 11</td>
<td>6.1204</td>
<td>245</td>
<td>38.9049</td>
<td>1561</td>
<td>0.0377</td>
<td>233</td>
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<tr>
<td></td>
<td>Theorem 10</td>
<td>0.0260</td>
<td>155</td>
<td>0.1550</td>
<td>950</td>
<td></td>
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</tr>
<tr>
<td>$m = 20$</td>
<td>Theorem 11</td>
<td>9.3238</td>
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<td>0.0377</td>
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<td></td>
<td>Theorem 10</td>
<td>0.0420</td>
<td>252</td>
<td>0.1578</td>
<td>858</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6. Numerical results for the LASSO problem in case $M = 2048, N = 1024$.

<table>
<thead>
<tr>
<th>$m$-Sparse</th>
<th>Method</th>
<th>$\kappa = 10^{-3}$</th>
<th>CPU</th>
<th>Iter</th>
<th>$\kappa = 10^{-4}$</th>
<th>CPU</th>
<th>Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 10$</td>
<td>Theorem 11</td>
<td>131.3365</td>
<td>111</td>
<td>578.6894</td>
<td>490</td>
<td>0.2419</td>
<td>74</td>
</tr>
<tr>
<td></td>
<td>Theorem 10</td>
<td>0.3274</td>
<td>101</td>
<td>1.0929</td>
<td>339</td>
<td>0.4633</td>
<td>141</td>
</tr>
<tr>
<td>$m = 20$</td>
<td>Theorem 11</td>
<td>184.8031</td>
<td>157</td>
<td>616.7051</td>
<td>526</td>
<td>0.3274</td>
<td>101</td>
</tr>
<tr>
<td></td>
<td>Theorem 10</td>
<td>0.4633</td>
<td>141</td>
<td>1.4516</td>
<td>339</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m = 30$</td>
<td>Theorem 11</td>
<td>262.3976</td>
<td>224</td>
<td>$1.3220 \times 10^3$</td>
<td>503</td>
<td>0.4633</td>
<td>141</td>
</tr>
<tr>
<td></td>
<td>Theorem 10</td>
<td>0.5393</td>
<td>158</td>
<td>$1.6136 \times 10^3$</td>
<td>1326</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We next provide some experiments in recovering the signal.

From Figures 1–4, we observe that our algorithms can be applied to solve the LASSO problem. Moreover, the proposed algorithms have a better convergence behavior than other methods.
Figure 2. From top to bottom: original signal, measured values, recovered signal by Theorem 8, Theorem 9 and Theorem 7 with $N = 2048$, $M = 1024$ and $m = 40$.

Figure 3. From top to bottom: original signal, measured values, recovered signal by Theorem 11 and Theorem 10 with $N = 512$, $M = 256$ and $m = 10$. 
8. Conclusions

In the present work, we introduce a new approximation algorithm with a new stepsize that involves the self adaptive method for SVIP. The stepsize does not use the Lipschitz constant and the norm of operators in computing. We show its convergence analysis, which was proved under some suitable assumptions. The numerical results showed the efficiency of our algorithms. It is reported that the performance of our algorithms outruns those of Byrne et al. [6] and Chuang [10,11] through experiments.

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