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Chen's Biharmonic Conjecture and Submanifolds with Parallel Normalized Mean Curvature Vector

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Abstract: The well known Chen's conjecture on biharmonic submanifolds in Euclidean spaces states that every biharmonic submanifold in a Euclidean space is a minimal one. For hypersurfaces, we know from Chen and Jiang that the conjecture is true for biharmonic surfaces in \mathbb{E}^3 . Also, Hasanis and Vlachos proved that biharmonic hypersurfaces in \mathbb{E}^4 ; and Dimitric proved that biharmonic hypersurfaces in \mathbb{E}^m with at most two distinct principal curvatures. Chen and Munteanu showed that the conjecture is true for $\delta(2)$ -ideal and $\delta(3)$ -ideal hypersurfaces in \mathbb{E}^m . Further, Fu proved that the conjecture is true for hypersurfaces with three distinct principal curvatures in \mathbb{E}^m with arbitrary m . In this article, we provide another solution to the conjecture, namely, we prove that biharmonic surfaces do not exist in any Euclidean space with parallel normalized mean curvature vectors.

Keywords: biharmonic submanifold; B.-Y. Chen's conjecture; δ -invariant; submanifolds with parallel normalized mean curvature vector

MSC: 53C40; 53D12; 53D40

1. Introduction

Let M be a Riemannian submanifold of a Euclidean m -space \mathbb{E}^m . Denote by Δ the Laplacian of M . Then, M is called *biharmonic* if its position vector field x satisfies (cf. [1])

$$\Delta^2 x = 0. \quad (1)$$

Clearly, minimal submanifolds of \mathbb{E}^m are biharmonic.

The study of biharmonic submanifolds was initiated around the middle of 1980s in the author's program of understanding finite type submanifolds in Euclidean spaces, also independently by Jiang in [2] in his study of Euler–Lagrange's equation of bienergy functional. The author and Jiang proved independently that there are no biharmonic surfaces in \mathbb{E}^3 except the minimal ones. This non-existence result was later generalized by Dimitric in his doctoral thesis [3] at Michigan State University and his paper [4]. More precisely, Dimitric proved in [3,4] that biharmonic curves in Euclidean spaces \mathbb{E}^m are parts of straight lines (i.e., minimal); biharmonic submanifolds of finite type in \mathbb{E}^m are minimal; pseudo umbilical submanifolds M that \mathbb{E}^m with $\dim M \neq 4$ are minimal, as well as biharmonic hypersurfaces in \mathbb{E}^m with at most two distinct principal curvatures are also minimal. In addition, it was known that there exist no biharmonic submanifolds of \mathbb{E}^m which lie in a hypersphere of \mathbb{E}^m (see [1], page 181 or [5], Corollary 7.2).

Based on these results mentioned above, the author made in 1991 the following conjecture (see, e.g., [1,6,7]):

Chen's Conjecture. *Biharmonic submanifolds of Euclidean spaces are minimal.*

This conjecture was proved by Hasanis and Vlachos for hypersurfaces in \mathbb{E}^4 [8] (see also [9] with a different proof). Fu proved in [10,11] that this conjecture is true for hypersurfaces with three distinct principal curvatures in \mathbb{E}^m with arbitrary m . Furthermore, Montaldo, Oniciuc and Ratto [12] proved this conjecture for G -invariant hypersurfaces in \mathbb{E}^m . However, Chen’s conjecture remains open until now. The main difficulty is that the conjecture is a local problem and it is not easy to understand the local structure of submanifolds satisfying $\Delta^2x = 0$ (or, equivalently, $\Delta\eta = 0$, where η is the mean curvature vector). The study of Chen’s conjecture is a very active research subject nowadays.

The theory of δ -invariants was initiated by the author in the early 1990s (see, e.g., [13–16]). In particular, the author introduced the notion of $\delta(k)$ -ideal submanifolds. In [17], the author and Munteanu proved that Chen’s conjecture is true for all $\delta(2)$ -ideal and $\delta(3)$ -ideal hypersurfaces of \mathbb{E}^m with arbitrary m . Under the assumption of completeness, Akutagawa and Maeta [18] proved that biharmonic properly immersed submanifolds in Euclidean spaces are minimal.

In contrast to Euclidean submanifolds, Chen’s conjecture is not always true for submanifolds in pseudo-Euclidean spaces. This fact was achieved in [19,20] by the author and Ishikawa who constructed examples of proper biharmonic surfaces in four-dimensional pseudo-Euclidean spaces \mathbb{E}_s^4 (with index $s = 1, 2, 3$). For hypersurfaces in pseudo-Euclidean spaces, it was also proved in [19,20] that biharmonic surfaces in pseudo-Euclidean 3-spaces are minimal. Furthermore, Arvanitoyeorgos et al. [21] proved that biharmonic Lorentzian hypersurfaces in Minkowski 4-spaces are also minimal.

From the view point of k -harmonic maps, one can define a biharmonic map φ between two Riemannian manifolds as a critical point of the bienergy functional. Jiang showed in [22] that a smooth map φ is biharmonic if and only if its bitension field $\tau_2(\varphi)$ vanishes identically, i.e., $\tau_2(\varphi) = 0$. In 2002, Caddeo and Montaldo showed in [23] that $\tau_2(\varphi) = 0$ holds identically if and only if $\Delta\eta = 0$ holds identically for any isometric immersion $\varphi : M \rightarrow \mathbb{E}^m$. Consequently, both definitions of biharmonicity of Chen and of Jiang coincide for the class of Euclidean submanifolds.

During the past two decades, there has been a lot of research work done on biharmonic submanifolds in spheres or even in generic Riemannian manifolds (see, e.g., [7,23–31]).

The study of submanifolds with parallel normalized mean curvature vector in Euclidean spaces was initiated at the beginning of the 1980s (see [32]). Let M be a submanifold in a Euclidean space whose mean curvature vector η is nowhere zero. If the unit normal vector field $\zeta = \eta/|\eta|$ in the direction of η is parallel in the normal bundle $T^\perp M$, i.e., $D\eta = 0$, then M is said to have *parallel normalized mean curvature vector field*.

Clearly, every non-minimal hypersurface has parallel normalized mean curvature vector. In addition, submanifolds with parallel normalized mean curvature vector generalize submanifolds with parallel mean curvature vector $\eta \neq 0$, since a submanifold has parallel mean curvature vector $\eta \neq 0$ if and only if it has a parallel normalized mean curvature vector with constant mean curvature.

In this article, we provide another solution to Chen’s conjecture. More precisely, we prove the following.

Theorem 1. *A biharmonic surface in \mathbb{E}^m with a parallel normalized mean curvature vector does not exist.*

2. Preliminaries

Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian n -manifold M into a Euclidean m -space \mathbb{E}^m . Denote the Levi–Civita connections of M and \mathbb{E}^m by ∇ and $\tilde{\nabla}$, respectively. Then, the Gauss and Weingarten formulas are given, respectively, by (cf. [5,33])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2}$$

$$\tilde{\nabla}_X \zeta = -A_\zeta X + D_X \zeta, \tag{3}$$

for vector fields X, Y tangent to M and ζ normal to M , where h is the second fundamental form, A is the shape operator and D the normal connection.

It is well known that the second fundamental form h and the shape operator A are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle. \tag{4}$$

The mean curvature vector field η is given by

$$\eta = \left(\frac{1}{n} \right) \text{trace } h. \tag{5}$$

A submanifold is called *totally geodesic* (respectively, *totally umbilical*) if its second fundamental form h satisfies $h = 0$ (respectively, $h = g \otimes \eta$). A submanifold is called *pseudo-umbilical* if the shape operator A_η in the direction of η is proportional to the identity map.

The equations of Gauss and Codazzi are given respectively by

$$\langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle, \tag{6}$$

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{7}$$

for X, Y, Z, W tangent to M , where R is the Riemann curvature tensor of M and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{8}$$

The equation of Ricci is given by

$$\langle R^D(X, Y)\xi, \zeta \rangle = \langle [A_\xi, A_\zeta](X), Y \rangle, \tag{9}$$

for X, Y tangent to M and ξ, ζ normal to M , where R^D is defined by

$$R^D(X, Y)\xi = D_X D_Y \xi - D_Y D_X \xi - D_{[X, Y]}\xi. \tag{10}$$

In terms of a local coordinate system $\{x_1, \dots, x_n\}$ of M , the Laplacian Δ of M is defined by

$$\Delta f = \text{div}(\nabla f) = \Delta f = \sum_{i,j=1}^n g^{ij} \left\{ \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right\},$$

where ∇f denotes the gradient of f , $\text{div}(X)$ the divergence of a vector field X , and $\Gamma_{ij}^k, i, j, k = 1, \dots, n$ are the Christoffel symbols.

It is well known that the mean curvature vector field η of M in \mathbb{E}^m satisfies the Beltrami formula (see, for instance, page 44 of [16])

$$\Delta x = n\eta. \tag{11}$$

Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian manifold M into a Euclidean m -space \mathbb{E}^m . Then, M is called a *biharmonic submanifold* if and only if $\Delta\eta = 0$ holds identically, or equivalently, $\Delta^2 x = 0$ holds identically.

The following characterization of biharmonic submanifolds in \mathbb{E}^m is an immediate consequence of the expression of $\Delta\eta$ given in [34,35].

Theorem 2. Let $\phi : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian n -manifold M into \mathbb{E}^m . Then, M is biharmonic if and only if it satisfies

$$\Delta^D \eta = \sum_{i=1}^n h(A_\eta e_i, e_i), \tag{12}$$

$$n \nabla \langle \eta, \eta \rangle + 4 \operatorname{Tr}(A_{D\eta}) = 0, \tag{13}$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal tangent frame on M .

Condition (12) implies immediately that every biharmonic submanifold of \mathbb{E}^m is minimal if it has parallel mean curvature vector, i.e., $D\eta = 0$.

3. Proof of the Theorem 1

First, we prove the following.

Claim. If M is a biharmonic surface in \mathbb{E}^m with a parallel normalized mean curvature vector, then it has a flat normal connection in \mathbb{E}^m , i.e., $R^D = 0$.

This can be proved as follows. Assume that M is a biharmonic surface in \mathbb{E}^m with parallel normalized mean curvature vector. Let $\{e_3, \dots, e_m\}$ be an orthonormal frame of the normal bundle $T^\perp M$ of M such that $e_3 = \eta / |\eta|$. Then, we get $De_3 = 0$. Thus, it follows from the equation of Ricci that

$$[A_3, A_r] = 0, \quad r = 4, \dots, m, \tag{14}$$

where $A_r = A_{e_r}$. Note that, although A_r is globally defined, A_r 's are not.

On the other hand, since $e_3 = \eta / |\eta|$, we also have

$$\operatorname{Tr}(A_r) = 0, \quad r = 4, \dots, m. \tag{15}$$

By combining (14) and (15), we know that if A_3 is not proportional to the identity map I at a point $p \in M$, then we have $[A_r, A_s] = 0$ at p for $r, s = 3, \dots, m$. Hence, by putting

$$U = \{p \in M : A_3 = \lambda I \text{ for some } \lambda \in \mathbb{R} \text{ at } p\},$$

we have $R^D = 0$ on $M - U$. Clearly, if the interior $\operatorname{int}(U)$ of U is empty, then, by continuity, $R^D = 0$ holds identically on M . Therefore, we may assume $\operatorname{int}(U) \neq \emptyset$. Clearly, each connected component of $\operatorname{int}(U)$ is a pseudo-umbilical surface in \mathbb{E}^m .

Since $De_3 = 0$ and $A_3 = \lambda I$ on U , we obtain from (8) that

$$\langle (\bar{\nabla}_X h)(Y, Z), e_3 \rangle = (X\lambda)(Y, Z). \tag{16}$$

It follows from (16) and the equation of Codazzi that the function λ is constant on each connected component of $\operatorname{int}(U)$. Therefore, $\operatorname{int}(U)$ has a parallel nonzero mean curvature vector on each connected component of $\operatorname{int}(U)$.

On the other hand, we know from [36] that every pseudo-umbilical submanifold of \mathbb{E}^m with a parallel nonzero mean curvature vector is a minimal submanifold of a hypersphere of \mathbb{E}^m . Therefore, each connected component of $\operatorname{int}(U)$ is a minimal submanifold of a hypersphere of \mathbb{E}^m , which is a contradiction since a spherical submanifold cannot be biharmonic in \mathbb{E}^m . Consequently, the normal connection of M must be flat, i.e., $R^D = 0$. This proves the claim.

Now, we return to the proof of Theorem 1. Let M be a biharmonic surface in \mathbb{E}^m with a parallel normalized mean curvature vector and let $\{e_3, \dots, e_m\}$ be an orthonormal frame of $T^\perp M$ such that

$e_3 = \eta/|\eta|$. Then, we get $De_3 = 0$ as before. From the Claim, we have $R^D = 0$. Hence, the equation of Ricci implies

$$[A_r, A_s] = 0, \quad r, s = 3, \dots, m. \tag{17}$$

From the proof of the Claim, we may assume that A_3 is not proportional to the identity map I at any point of M . Thus, we may put it with

$$A_3 = \begin{pmatrix} \kappa & 0 \\ 0 & \mu \end{pmatrix}, \quad \kappa \neq \mu. \tag{18}$$

For each $p \in M$, put $N_p = \{\zeta \in T_p^\perp M : \langle \zeta, \eta \rangle = 0\}$. Let us define a linear map $\gamma_p : N_p \rightarrow \text{End}(T_p M)$ by $\gamma(\zeta) = A_\zeta$. Define $0_p = \gamma^{-1}(0)$ and let N'_p be the subspaces of N_p satisfying

$$N_p = N'_p \oplus 0_p, \quad N'_p \perp 0_p.$$

Put $V = \{p \in M : \dim N'_p = 0\}$ and $W = \{p \in M : \dim N'_p = 1\}$. Clearly, W is an open subset of M . From (15), any A_ζ with $\zeta \in N_p$ has the form

$$A_\zeta = \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix}, \quad \delta \in \mathbb{R},$$

so $\dim N'_p = 0$ if $A_\zeta = 0$ for any $\zeta \in N_p$, or $\dim N'_p = 1$ if $A_\zeta \neq 0$ for some $\zeta \in N_p$. Therefore, $M = V \cup W$.

If $\text{int}(V) \neq \emptyset$ holds, then the first normal subbundle, $\text{Im } h$, is spanned by e_3 on $\text{int}(V)$. Because $De_3 = 0$, the reduction theorem of Erbacher implies that each connected component of $\text{int}(V)$ lies in an affine 3-subspace of \mathbb{E}^m . However, this is impossible, since every biharmonic surface in \mathbb{E}^3 is always minimal. Therefore, we obtain $\text{int}(V) = \emptyset$. Thus, the subset W must be dense in M . Without loss of generality, we may put $M = W$. Thus, we have $\dim N'_p = 1$ for each $p \in M$. Now, if we choose $e_4 \in N'$, then we find

$$A_5 = \dots = A_m = 0. \tag{19}$$

Since e_4 is perpendicular to η , we may put

$$A_4 = \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix}, \quad \delta \neq 0. \tag{20}$$

On the other hand, since M is a biharmonic surface in \mathbb{E}^m , it follows from (12), (18), (20), $\eta = He_3$ and $De_3 = 0$ at a point p that

$$(\Delta H)e_3 = \sum_{i=1}^2 h(A_\eta e_i, e_i) = H\{(\kappa^2 + \mu^2)e_3 + \delta(\kappa - \mu)e_4\}.$$

Thus, we must have $\kappa = \mu$, $A_\eta = \lambda I$, and $\lambda = H\kappa$, which is a contradiction. Consequently, M cannot be biharmonic.

The remaining part of the theorem is easy to verify. \square

4. Some Remarks

1. For biharmonic surfaces in \mathbb{E}^4 , Theorem 1 is due to Sen and Turgay [37].

2. Note that the surface in \mathbb{E}^m given in Theorem 1 of this article is not required to be analytic, in contrast to the results given in [32]. Hence, the theorems and lemmas given in [32] cannot apply to this article.

3. It is interesting to know whether Theorem 1 holds for biharmonic submanifolds in \mathbb{E}^m with a parallel normalized mean curvature vector. In this respect, we make the following conjecture as a special case of Chen's conjecture.

Conjecture. *There do not exist biharmonic submanifolds in Euclidean spaces with a parallel normalized mean curvature vector.*

4. Let M be a biharmonic submanifold of an m -sphere S^n with a parallel normalized mean curvature vector. It was proved in ([38], Theorem 4.3) that, if the shape operator of M in the direction of mean curvature vector field η has at most two distinct principal curvatures, then M has parallel mean curvature vector. One can generalize the method in [38] to any space form to conclude that there exist no biharmonic submanifolds of \mathbb{E}^m with a parallel normalized mean curvature vector field and with at most two distinct principal curvatures in the direction of η .

5. Let M be a compact biconservative (in particular, biharmonic) submanifold of a space form $N^m(c)$ with $Riem^M \geq 0$ and $\dim M \leq 10$. It was proved in [39] that, if M has parallel normalized mean curvature vector, then M has a parallel mean curvature vector field η and with $\nabla A_\eta = 0$.

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