Split Variational Inclusion Problem and Fixed Point Problem for a Class of Multivalued Mappings in $\text{CAT}(0)$ Spaces

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Abstract: The aim of this paper is to introduce a modified viscosity iterative method to approximate a solution of the split variational inclusion problem and fixed point problem for a uniformly continuous multivalued total asymptotically strictly pseudocontractive mapping in $\text{CAT}(0)$ spaces. A strong convergence theorem for the above problem is established and several important known results are deduced as corollaries to it. Furthermore, we solve a split Hammerstein integral inclusion problem and fixed point problem as an application to validate our result. It seems that our main result in the split variational inclusion problem is new in the setting of $\text{CAT}(0)$ spaces.

Keywords: split variational inclusion problem; fixed point problem; $\text{CAT}(0)$ space; total asymptotically strictly pseudocontractive mapping

1. Introduction

1.1. $\text{Cat}(0)$ Space

Let $(X, d)$ be a metric space. A geodesic path joining $x$ and $y$ is a map $c : [0, l] \subset \mathbb{R} \to X$ such that

i. $c(0) = x$, $c(l) = y$ and $d(x, y) = l$.

ii. $c$ is an isometry: $d(c(t), c(s)) = |t - s|$ for all $t, s \in [0, l]$.

In this case, $c([0, l])$ is called a geodesic segment joining $x$ and $y$ which when unique is denoted by $[x, y]$.

The space $(X, d)$ is said to be a geodesic space if any two points of $X$ are joined by a geodesic segment.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space $(X, d)$ consists of three points in $X$ (the vertices of $\Delta$) and a geodesic segment between each pair of vertices (the edges of $\Delta$).

A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\hat{\Delta}(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ in $(\mathbb{R}^2, d)$ such that $d_{\mathbb{R}^2}(\hat{x}_i, \hat{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (Bridson and Haefliger [2]).

A metric space $X$ is said to be a $\text{CAT}(0)$ space if it is geodesically connected and every geodesic triangle in $X$ is at least as ‘thin’ as its comparison triangle in the Euclidean plane.
Let $\Delta$ be a geodesic triangle in $X$, and let $\hat{\Delta}$ be its comparison triangle in $\mathbb{R}^2$. Then, $X$ is said to satisfy $\text{CAT}(0)$ inequality, if, for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \hat{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(x, y).$$

If $x, y_1, y_2 \in X$, and $y_0$ is the midpoint of the segment $[y_1, y_2]$, then, the $\text{CAT}(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \tag{1}$$

It is well known that the following spaces are $\text{CAT}(0)$ spaces: a complete, simply connected Riemannian manifold with non-positive sectional curvature, Pre-Hilbert spaces $\mathbb{R}$, Euclidean buildings $[3]$, R-trees $[18]$, and Hilbert ball with a hyperbolic metric $[10,16]$.

1.2. Some Basic Concepts in Hilbert Space

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $C$ a nonempty closed and convex subset of $H$.

The inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ generates norm via

$$\langle x, x \rangle = \|x\|^2$$

for all $x \in H$.

A mapping $T : C \to C$ is said to be total asymptotically strictly pseudocontractive (see $[4]$), if there exists a constant $\gamma \in [0, 1]$ such that

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + \gamma\|x - T^n x\|^2 + \kappa_n \phi(\|x - y\|) + \mu_n$$

holds for all $x, y \in C$, the sequences $\kappa_n, \mu_n \in [0, \infty)$ satisfy $\lim_{n \to \infty} \kappa_n = \lim_{n \to \infty} \mu_n = 0$, and $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing and continuous mapping with $\phi(0) = 0$.

For concepts such as bounded linear operator and its adjoint operator, maximal monotone operator and metric projection, we refer to Chidume $[5]$.

The metric projection is parity and scale invariant (cf. Proposition 1.26(e) in $[30]$) in the sense that

$$\lambda P_C x = P_C \lambda x, \text{ for every } \lambda \geq 0, \ x \in H,$$

consequently,

$$\lambda (P_C x - x) = P_{C - \lambda x} (\lambda x) - \lambda x, \text{ for every } \lambda \geq 0, \ x \in H.$$

1.3. Counterpart of the above Concepts in the Setting of a $\text{Cat}(0)$ Space

A mapping $T : C \to C$ is said to be total asymptotically strictly pseudocontractive if there exists $\gamma \in [0, 1]$ such that

$$d(T^n x, T^n y)^2 \leq d(x, y)^2 + \gamma d(x, T^n x)^2 + \kappa_n \phi(d(x, y)) + \mu_n$$

holds for all $x, y \in C$, the sequences $\kappa_n, \mu_n \in [0, \infty)$ satisfy $\lim_{n \to \infty} \kappa_n = \lim_{n \to \infty} \mu_n = 0$, and $\phi : [0, \infty) \to [0, \infty)$ is strictly increasing and continuous mapping with $\phi(0) = 0$.

Define an addition $(x, y) \mapsto x \oplus y$ and a scalar multiplication $(\alpha, x) \mapsto \alpha \cdot x$ in the space $X$ as follows: for any $z \in X$ and $\alpha, \beta \in \mathbb{R}$, we denote the point $z = ax \oplus \beta y$ such that $d(x, z) = d((1 - \alpha)x, \beta y)$.

A mapping $A : X \to X$ is said to be linear if for $x, y \in X$, we have

$$A(ax \oplus \beta y) = aA(x) \oplus \beta A(y).$$
A mapping $A : X \rightarrow X$ is said to be bounded if for all $x, y \in X$, there exists $M \geq 0$ such that
\[d(Ax, Ay)^2 \leq Md(x, y)^2,\]

Let $C$ be a nonempty subset of a CAT(0) space $X$.

In [1], a mapping $\langle ., . \rangle : (X \times X) \times (X \times X) \rightarrow R$ is said to be quasi-linearization in $X$ if
\[
\langle \vec{pq}, \vec{rs} \rangle = \frac{1}{2}(d(p, s)^2 + d(q, r)^2 - d(p, r)^2 - d(q, s)^2), \tag{3}
\]
holds for all $p, q, r, s \in X$; here a pair $(p, q) \in X \times X$ is denoted by a vector $\vec{pq}$. Consequently, we have

1. $\langle \vec{pq}, \vec{pq} \rangle = d(p, q)^2$, 
2. for all $p, q, r, s, t, u, v, w \in X$,
\[
\langle \vec{pq}, \vec{rs}, \vec{tu}, \vec{vw} \rangle = \langle \vec{pq}, \vec{tu} \rangle - \langle \vec{pq}, \vec{vw} \rangle - \langle \vec{rs}, \vec{tu} \rangle + \langle \vec{rs}, \vec{vw} \rangle. \tag{4}
\]

A mapping $A^* : X \rightarrow X$ is said to be adjoint operator of $A$ if for all $x, y, w, z \in X$, we have
\[
\langle AxA\vec{w}, \vec{yz} \rangle = \langle A\vec{w}, A^*\vec{yz} \rangle = \langle \vec{w}, A^*yA^z \rangle. \tag{5}
\]

Clearly, $A^*$ is a linear operator when so is $A$. As in a Hilbert space, we have
\[d(A^*y, A^*z)^2 = d(Ax, Aw)^2 \leq Md(x, w)^2,\]
and hence, $A^*$ is bounded in $X$.

For any $x \in X$, there exists a unique point $x_0 \in C$ such that
\[d(x, x_0) \leq d(x, y) \ \forall \ y \in C,\]
and the mapping $P_C : X \rightarrow C$ defined by $P_Cx = x_0$ is called the metric projection of $X$ onto $C$ (cf. Proposition 2.4 in [2]). Equivalently, in view of the characterization of Hossein and Jamal [12], we have
\[
\langle x_0\overline{x}, yx_0\overline{x} \rangle \geq 0,
\]
consequently,
\[P_{Cx} : X \rightarrow \overline{C}_x \text{ is defined by } P_{Cx}(x)\overline{x} = x_0\overline{x}, \]
equivalently,
\[
\langle x_0\overline{x}x_0\overline{x}, y\overline{x}x_0\overline{x} \rangle \geq 0,
\]
\[
\Leftrightarrow \langle x_0\overline{x}, y\overline{x}x_0\overline{x} \rangle \geq 0 \text{ (because } x_0\overline{x} \text{ is an additive identity element),}
\]
\[
\Leftrightarrow \langle x_0\overline{x}, y\overline{x} \rangle - \langle x_0\overline{x}, x_0\overline{x} \rangle \geq 0 \text{ (by (4))}, \tag{6}
\]
where $x_0\overline{x}, y\overline{x} \in \overline{C}_x$. The metric projection is parity and scale invariant in the sense that
\[\lambda P_Cx = P_{C}\lambda x, \text{ for every } \lambda \geq 0, \ x \in X,\]
consequently,

$$\lambda P_{\mathcal{C}x}^x(x) = P_{\lambda \mathcal{C}x}(\lambda x) x, \text{ for every } \lambda \geq 0, x \in X.$$ \hspace{1cm} (7)

1.4. Fixed Point Theory in a CAT(0) Space

Fixed point theory in a CAT(0) space has been introduced by Kirk (see for example [19]). He established that a nonexpansive mapping defined on a bounded, closed and convex subset of a complete CAT(0) space has a fixed point. Consequently, fixed point theorems in CAT(0) spaces have been developed by many mathematicians; see for example [8,29]. More so, some of these theorems in CAT(0) spaces are applicable in many fields of studies such as, graph theory, biology and computer science (see for example [9,18,20,31]).

Let $T : X \rightarrow 2^X$ be a multivalued mapping. A point $x \in X$ is called a fixed point of $T$ if $x \in Tx$ and $F(T) = \{ x \in X : x \in Tx \}$ is called the fixed point set of $T$.

1.5. Our Motivation

As a generalized version of the well known split common fixed point problem, Moudafi [25] introduced the following split monotone variational inclusion (SMVI) by using maximal monotone mappings;

$$\text{find } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*),$$
$$y^* = Ax^* \in H_2 \text{ solves } 0 \in g(y^*) + B_2(y^*),$$

where $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$, $A : H_1 \rightarrow H_2$ is a bounded linear operator, $f : H_1 \rightarrow H_1$ and $g : H_2 \rightarrow H_2$ are given single-valued operators.

In 2000, Moudafi [26] proposed the viscosity approximation method by considering the approximate well-posed problem of a nonexpansive mapping $S$ with a contraction mapping $f$ over a nonempty closed and convex subset; in particular he showed that given an arbitrary $x_1$ in a nonempty closed and convex subset, the sequence $\{ x_n \}$ defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S x_n,$$

where $\{ \alpha_n \} \subset (0, 1)$ with $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, converges strongly to the fixed point set of $S$, $F(S)$.

In [28], viscosity approximation method for split variational inclusion and fixed point problems in Hilbert Spaces was presented as follows.

$$\begin{cases}
  u_n & = j_{\lambda_1}^{B_1}(x_n + \gamma_n A^*(j_{\lambda_2}^{B_2} - I) A x_n); \\
  x_{n+1} & = \alpha_n f(x_n) + (1 - \alpha_n) T^\eta(u_n), \forall n \geq 1,
\end{cases} \hspace{1cm} (8)$$

where $B_1$ and $B_2$ are maximal monotone operators, $j_{\lambda_1}^{B_1}$ and $j_{\lambda_2}^{B_2}$ are resolvent mappings of $B_1$ and $B_2$ respectively, $f$ is a Meir-Keeler mapping, $T$ a nonexpansive mapping, $A^*$ is an adjoint of $A$, $\gamma_n, \alpha_n \in (0, 1)$ and $\lambda > 0$.

In this paper, motivated by (8), we present a modified viscosity algorithm sequence and prove strong convergence theorem for split variational inclusion problem and fixed point problem of a total asymptotically strictly pseudocontractive mapping in the setting of two different CAT(0) spaces. It seems that our main result is new in the setting of CAT(0) spaces.
2. Preliminaries

Denote by $CB(X)$, the collection of all nonempty closed and bounded subsets of $X$ and let $H$ be the Hausdorff metric with respect to the metric $d$; that is,

$$H(A,B) = \max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\} \tag{9}$$

for all $A, B \in CB(X)$, where $d(a,B) = \inf_{b \in B} d(a,b)$ is the distance from the point $a$ to the subset $B$.

Let $X$ be a complete $\text{CAT}(0)$ space with its dual $X^*$ (for details, see [17]). A mapping $G : D(G) \subset X \to 2^X$ is said to be monotone if

$$\langle xy, x'y' \rangle \geq 0 \forall x, y \in D(G), x^* \in Gx, y^* \in Gy.$$

A mapping $G : D(G) \subset X \to 2^X$ is said to be maximal monotone if it is monotone and also has no monotone extension, that is, its graph $gr(G) = \{ (x, x^*) \in X \times X^* : x^* \in G(x) \}$ is not properly contained in the graph of any other monotone operator on $X$.

For $\gamma > 0$, a mapping $B_G = (I + \gamma G)^{-1} : X \to 2^X$ defined by $B_G(x) = \{ z \in X : \frac{1}{\gamma} z \in G(z) \}$ is said to be a resolvent of $G$.

The operator $G$ is said to satisfy the range condition if for every $\gamma > 0, D(B_G) = X$.

Let $X$ be a complete $\text{CAT}(0)$ space and $\{x_n\}$ be a bounded sequence in $X$. Then the asymptotic center of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{ x \in X : \limsup_{n \to \infty} d(x,x_n) \leq \limsup_{n \to \infty} d(z,x_n), \forall z \in X \}.$$

The asymptotic center $A(\{x_n\})$, consists of exactly one point ([6]).

**Definition 1.** A sequence $\{x_n\}$ in a $\text{CAT}(0)$ space $X$ is said to be $\Delta$-convergent to $x \in X$ if $x$ is the unique asymptotic center of any subsequence $\{x_{n_k}\} \subset \{x_n\}$. Symbolically, we write it as $\Delta - \lim x_n = x$ [21, 22].

**Lemma 1.** Let $\{x_n\}$ be a bounded sequence in a complete $\text{CAT}(0)$ space $X$ [21]. Then

i. $\{x_n\}$ has a $\Delta$-convergent subsequence.

ii. The asymptotic center of $\{x_n\} \subset C \subset X$ is in $C$, where $C$ is nonempty, closed and convex.

**Lemma 2.** Let $\{x_n\}$ be a bounded sequence in a complete $\text{CAT}(0)$ space and $A(\{x_n\}) = \{ x \}$. Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$ and $A(\{x_{n_k}\}) = \{ y \}$. If $\lim d(x_n, y)$ exists, then $x = y$ [7].

Let $C$ be closed and convex subset of a $\text{CAT}(0)$ space $X$ and $\{x_n\}$ a bounded sequence in $C$. Then the relation $x_n \rightharpoonup x$ is described by

$$\limsup_{n \to \infty} d(x_n, x) = \inf_{y \in C} \limsup_{n \to \infty} d(x_n, y).$$

**Lemma 3.** [27] Let $C$ be closed and convex subset of a $\text{CAT}(0)$ space $X$ and $\{x_n\}$ a bounded sequence in $C$. Then $\Delta - \lim x_n = x$ implies that $x_n \rightharpoonup x$.

**Lemma 4.** [7] Let $X$ be a $\text{CAT}(0)$ space and $x, y, z \in X$. Then

i. $d((1 - t)x \oplus ty, z) \leq (1 - t)d(x,z) + td(y,z)$, $t \in [0,1]$,

ii. $d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x,z)^2 + td(y,z)^2 - t(1 - t)d(x,y)^2$, $t \in [0,1]$.

**Lemma 5.** [13] Let $X$ be a complete $\text{CAT}(0)$ space, $\{x_n\}$ a sequence in $X$ and $x \in X$. Then $\{x_n\}$, $\Delta$-converges to $x$ if and only if $\limsup_{n \to \infty} \langle x_n, y \rangle = 0$ for all $y \in X$. 
Lemma 6. [34] Let $X$ be a complete CAT(0) space. Then for all $x, y, z \in X$, the following inequality holds
\[ d(x, z)^2 \leq d(y, z)^2 + 2 \langle \bar{x}y, xz \rangle. \]

3. Main Results

Let $X_1$ and $X_2$ be two CAT(0) spaces, $C \subset X_1$ be a closed and convex subset, $A : X_1 \to X_2$ bounded linear and unitary operator, $U : X_1 \to 2^{X_1}$ and $S : X_2 \to 2^{X_2}$ be uniformly continuous and maximal monotone operators, $f : X_1 \to X_1$ contraction mapping and $T : C \to CB(C)$ be uniformly continuous multivalued total asymptotically strictly pseudocontractive mapping defined as
\[ H(T^n x, T^n y)^2 \leq d(x, y)^2 + \gamma d(x, T^n x)^2 + \kappa_n \varphi(d(x, y)) + \mu_n \]
where $x, y \in C$ and the sequences $\kappa_n, \mu_n \in [0, \infty)$ satisfy $\sum_{n=1}^{\infty} \kappa_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Suppose that $\gamma \in [0, 1]$ and $\varphi : [0, \infty) \to [0, \infty)$ is strictly increasing and continuous mapping such that $\varphi(0) = 0$, and $P_{AC} : X_2 \to AC$ and $P_{\gamma AC} : X_2 \to \gamma AC$ are the metric projections onto, respectively, nonempty closed and convex subset $AC$ and $\gamma AC$ of $X_2$, where $AC = \{Ax, \forall x \in C\}$ and $\gamma AC = \{\gamma y Ax, \forall y \in C$ and $Ax$ fixed\}. Let, for $\gamma > 0$, $B^{\gamma}_I : 2^{X_1} \to X_1$ and $B^{\gamma}_I : 2^{X_2} \to X_2$ be resolvent operators for $U$ and $S$, respectively. Denoted by $VIP(U, \gamma)$ and $VIP(S, \gamma)$, and $F(T)$ the solution set of variational inequality problems with respect to $U$ and $S$ and fixed point problem with respect to $T$.

As in [25], we define the split variational inclusion (SVI) as follows:
\[ \text{find } x \in X_1 \text{ such that } \bar{x}x \in U(x) \text{ and } Ax \in X_2 \text{ solves } AxAx \in S(Ax), \]
where $\bar{x}x$ and $AxAx$ are the additive identity elements in $X_1$ and $X_2$, respectively.

Denoted by $F(T)$ is the fixed point set of a map $T$, let $F(T) \neq \emptyset$ and $p \in F(T)$. Then $T$ is multivalued total quasi-asymptotically strictly pseudocontractive mapping if
\[ H(T^n x, T^n p)^2 \leq d(x, p)^2 + \gamma d(x, T^n x)^2 + \kappa_n \varphi(d(x, p)) + \mu_n. \]

Remark 1. Please note that a multivalued total asymptotically strictly pseudocontractive mapping is multivalued total quasi-asymptotically strictly pseudocontractive provided, its fixed point set is nonempty.

Throughout this paper we shall strictly employ the above terminology.

For a bounded sequence $\{x_n\}$ in $C$, we employ the notion:
\[ \limsup_{n \to \infty} d(x_n, x) = \inf_{y \in C} \limsup_{n \to \infty} d(x_n, y), \tag{10} \]
equivalently $x$ is the asymptotic center of each subsequence of $\{x_n\}$.

Following Karapinar et al [14], we first establish a demiclosedness principle based on (10).

Lemma 7. (Demiclosedness of $T$) Let $T$ be a multivalued total asymptotically strictly pseudocontractive mapping on a closed and convex subset $C$ of a CAT(0) space $X$. Let $\{x_n\}$ be a bounded sequence in $C$ such that $\Delta - \lim x_n = x$ and $\lim d(x_n, Tx_n) = 0$. Then $x \in Tx$.

Proof. By the hypothesis $\Delta - \lim x_n = x$ and so by Lemma 3, we get $\{x_n\} \to x$. Then by Lemma 1 (ii), we arrive at $A(\{x_n\}) = \{x\}$. Let $\lim d(x_n, Tx_n) = 0$. So we obtain
\[ \limsup_{n \to \infty} d(x_n, y) = \limsup_{n \to \infty} d(y, Tx_n), \tag{11} \]
for all \( y \in C \). From the hypothesis that \( T \) is multivalued total asymptotically strictly pseudocontractive mapping and by (11), choosing \( y \in Tx \), we have

\[
\limsup_{n \to \infty} d(x_n, y)^2 = \limsup_{n \to \infty} d(y, Tx_n)^2 \leq \limsup_{n \to \infty} H(Tx_n, Tx)^2 \leq \limsup_{n \to \infty} \left\{ d(x_n, x)^2 + \gamma d(x_n, Tx_n)^2 + \kappa_n \varphi(d(x_n, x)) + \mu_n \right\} \]

\[
= \limsup_{n \to \infty} d(x_n, x)^2. \quad (12)
\]

\( \square \)

By (1), we get

\[
d \left( x_n, \frac{x \oplus y}{2} \right)^2 \leq \frac{1}{2} d(x_n, x)^2 + \frac{1}{2} d(x_n, y)^2 - \frac{1}{4} d(x, y)^2.
\]

Let \( n \to \infty \) and take superior limit on the both sides of the above inequality and get

\[
\limsup_{n \to \infty} d \left( x_n, \frac{x \oplus y}{2} \right)^2 \leq \frac{1}{2} \limsup_{n \to \infty} d(x_n, x)^2 + \frac{1}{2} \limsup_{n \to \infty} d(x_n, y)^2 - \frac{1}{4} \limsup_{n \to \infty} d(x, y)^2.
\]

Since \( A(\{ x \}) = \{ x \} \), therefore we have

\[
\limsup_{n \to \infty} d(x_n, x)^2 \leq \limsup_{n \to \infty} \left( x_n, \frac{x \oplus y}{2} \right)^2 \leq \frac{1}{2} \limsup_{n \to \infty} d(x_n, x)^2 + \frac{1}{2} \limsup_{n \to \infty} d(x_n, y)^2 - \frac{1}{4} \limsup_{n \to \infty} d(x, y)^2,
\]

which implies that

\[
\limsup_{n \to \infty} d(x_n, x)^2 \leq \limsup_{n \to \infty} d(x_n, y)^2. \quad (13)
\]

By (12) and (13), we conclude that \( x = y \) and therefore \( x \in Tx \), as desired.

Next, we prove our main result as follows.

**Theorem 1.** Let \( x_1 \in X_1 \) be chosen arbitrarily and the sequence \( \{ x_n \} \) be defined as follows;

\[
\begin{align*}
    y_n &= {B}_{\gamma_n}^U\left( a_n x_n \oplus (1 - a_n) \lambda_n A^* P_{A^* A x_n} B_{\gamma_n}^S(A x_n) A x_n \right) \\
    x_{n+1} &= \beta_n f(x_n) \oplus (1 - \beta_n) z_n, \quad z_n \in \{ T_n y_n \}, \quad n \geq 1,
\end{align*}
\]

(14)

where \( A^* \) is the adjoint operator of \( A \), and \( M, \lambda_n, a_n, \beta_n \in [0, 1] \). Suppose that \( AC \) is closed and convex, \( P_{AC} B_{\gamma_n}^S \) is demiclosed and \( \Gamma = \{ x \in VIP(U, \gamma) : Ax \in VIP(S, \gamma) \} \cap \{ x \in F(T) \} \neq \emptyset \), and the following conditions are satisfied;

1. there exists constant \( N > 0 \) such that \( \varphi(r) \leq Nr, r \geq 0 \);
2. \( \lim \beta_n = \lim a_n = 0 \);
3. \( T \) satisfies the asymptotically regular condition \( \lim d(y_n, T_n y_n) = 0 \).

Then \( \{ x_n \} \) converges strongly to a point \( x \in \Gamma \), where \( P_{AC} B_{\gamma_n}^S(Ax) = B_{\gamma_n}^S(Ax) \).

**Proof.** We will divide the proof into three steps.
Substituting (16) into (15), we get

\[ d(y_n, p)^2 = d \left( B_{\infty}^1 \left( a_n x_n \oplus (1 - a_n) \lambda_n A^* P_{AC_n, Ax_n} B_{\infty}^1 (Ax_n) A x_n, p \right) \right)^2 \]

\[ \leq a_n d(x_n, p)^2 + (1 - a_n) d \left( \lambda_n A^* P_{AC_n, Ax_n} B_{\infty}^1 (Ax_n) A x_n, p \right)^2 \]  

(15)

whereas, by (6), (4), (5), and boundedness, linearity and unitary property of \( A \), we have,

\[ d \left( \lambda_n A^* P_{AC_n, Ax_n} B_{\infty}^1 (Ax_n) \right) A x_n, p \right)^2 \]

\[ = \left( \lambda_n A^* P_{AC_n, Ax_n} B_{\infty}^1 (Ax_n) \right) A x_n p \left( \lambda_n A^* P_{AC_n, Ax_n} B_{\infty}^1 (Ax_n) \right) A x_n p \]

\[ \leq \frac{\lambda_n A^* P_{AC_n, Ax_n} B_{\infty}^1 (Ax_n)}{A x_n A x_n} \]

\[ \leq - \left( \frac{\lambda_n A^* P_{AC_n, Ax_n} B_{\infty}^1 (Ax_n)}{A x_n A x_n} \right) A x_n A x_n p \]

(16)

Substituting (16) into (15), we get

\[ d(y_n, p)^2 \leq -(1 - a_n) \left\langle \frac{\lambda_n A^* P_{AC_n, Ax_n} B_{\infty}^1 (Ax_n)}{A x_n A x_n} \right\rangle A x_n A x_n p \]

\[ + (M(1 - a_n) + a_n) d(x_n, p)^2 \]

(17)

\[ \leq d(x_n, p)^2. \]  

(18)

By Remark 1, (14), (2), Lemma 4(ii) and (9), we get

\[ d(x_{n+1}, p)^2 = d(\beta_n f(x_n) \oplus (1 - \beta_n) z_n, p)^2 \]

\[ \leq \beta_n d(f(x_n), p)^2 + (1 - \beta_n) d(z_n, p)^2 - \beta_n (1 - \beta_n) d(f(y_n), z_n)^2 \]

\[ \leq \beta_n d(f(x_n), f(p))^2 + d(f(p), p)^2 + (1 - \beta_n) H(T_n y_n, T_n p)^2 \]

\[ \leq \beta_n d(f(p), p)^2 + (1 - (1 - \xi) \beta_n) (1 + \kappa_n N) d(x_n, p)^2 \]

\[ + (1 - \beta_n) \gamma d(y_n, T_n y_n)^2 + (1 - \beta_n) \mu_n. \]

(19)

Since \( \Sigma_{n=1}^{\infty} \kappa_n < \infty \), \( \Sigma_{n=1}^{\infty} \mu_n < \infty \) and \( \gamma \) is arbitrary in \([0, 1]\), therefore by (19), we get

\[ d(x_{n+1}, p)^2 \leq \beta_n d(f(p), p)^2 + (1 - (1 - \xi) \beta_n) d(x_n, p)^2 \]

\[ \leq \max \left\{ d(x_n, p)^2, \frac{1}{1 - \xi} d(f(p), p)^2 \right\} \]

(20)
Thus, \( \Delta \) and therefore by (21), (22) and Lemma 6, we get also bounded.

**Step two.** We will show that \( \lim_{n \to \infty} d(P_{AC_n}B_{T_n}^S(Ax_n), Ax_n) = 0. \)

By Lemmas 1 and 2, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \Delta - \lim_{n \to \infty} x_{n_k} = x \in C. \)

Thus, \( \Delta - \lim_{n \to \infty} x_n = x. \) By Lemmas 5 and 6, we get

\[
d(x_{n+1}, x_n)^2 \leq 2d(x_{n+1}, x)^2 + 2d(x, x_n)^2
\]

\[
= 2\left(\langle \hat{x}_{n+1}, \hat{x}_{n+1} \rangle - \langle \hat{x}, \hat{x} \rangle \right) + 2\langle \hat{x}, \hat{x} \rangle
\]

\[
\to 0 \text{ as } n \to \infty.
\]

(21)

This implies that \( x_n \to x \) as \( n \to \infty. \)

In addition, by Lemma 4(ii) we have

\[
d(y_n, x_{n+1})^2 = d(y_n, \beta_n f(x_n) \oplus (1 - \beta_n)z_n)^2
\]

\[
\leq \beta_n d(y_n, f(x_n))^2 + (1 - \beta_n)d(y_n, z_n)^2
\]

\[
\leq \beta_n d(y_n, f(x_n))^2 + (1 - \beta_n)d(y_n, T_ny_n)^2
\]

\[
\to 0 \text{ as } n \to \infty.
\]

(22)

and therefore by (21), (22) and Lemma 6, we get

\[
d(y_n, x_n)^2 \leq d(y_n, x_{n+1})^2 + d(x_{n+1}, x_n)^2
\]

\[
\to 0 \text{ as } n \to \infty.
\]

(23)

This implies that \( y_n \to x \) as \( n \to \infty. \)

As \( \lambda_n \) is arbitrary in \([0, 1]\), so by (17), (3) and (7) we arrive at

\[
(1 - \alpha_n) \left( \lambda_n P_{AC_nAx_n}B_{T_n}^S(Ax_n)Ax_nAx_nApAx_n \right) \leq d(x_n, p)^2 - d(y_n, p)^2
\]

\[
\Rightarrow 2(1 - \alpha_n) d \left( \lambda_n P_{AC_nAx_n}B_{T_n}^S(Ax_n)Ax_n \right)^2 \leq d(x_n, p)^2 - d(y_n, p)^2
\]

\[
\Rightarrow 2(1 - \alpha_n) d \left( P_{AC_nAx_n}B_{T_n}^S(Ax_n)Ax_n \right)^2 \leq d(x_n, p)^2 - d(y_n, p)^2
\]

\[
\Rightarrow 2(1 - \alpha_n) d \left( P_{AC_nAx_n}B_{T_n}^S(Ax_n)Ax_n \right)^2 \leq d(x_n, p)^2 - d(y_n, p)^2
\]

\[
\to 0 \text{ as } n \to \infty.
\]

(24)

It follows from (24) that

\[
d \left( P_{AC_n}B_{T_n}^S(Ax_n), Ax_n \right) \to 0 \text{ as } n \to \infty.
\]

(25)
Step three. We show that $x_n \to x \in \Gamma$.

By (14), we obtain

$$
\alpha_n x_n \oplus (1 - \alpha_n)\lambda_n A^* \overset{P_{AC,Ax_n}}{\to} B_{\gamma_n}^S (Ax_n) Ax_n \in y_n \oplus \gamma_n U_n (y_n)
$$

$$
\implies x_n \in y_n \oplus \gamma_n U_n (y_n) \quad \text{(since $\alpha_n$ is arbitrary in $[0, 1]$)}.
$$

$$
\implies \overline{x_n y_n} \in \gamma_n U_n (y_n)
$$

(26)

Since $U$ and $S$ are uniformly continuous, therefore it follows by (26), as $n \to \infty$, that $\overline{x_n y_n} \in U(x)$.

In addition, it is clear that $\Delta - \lim_{n \to \infty} Ax_n = Ax$. So by using (25) and applying the demiclosedness of $P_{AC} B_{\gamma_n}^S$, we have that $Ax Ax \in SAx$, as $P_{AC} B_{\gamma_n}^S Ax = B_{\gamma_n}^S Ax$. On the other hand, by Lemma 7 and $\Delta - \lim_{n \to \infty} y_n = x$ (by (23)), we have by the hypothesis $\lim_{n \to \infty} d(Ty_n, y_n) = 0$ that $x \in Tx$, as $T$ is uniformly continuous. Hence, $x \in \Gamma$. \(\square\)

The proof is completed.

If $U : X_1 \to X_1$ and $S : X_2 \to X_2$ are total asymptotically strictly pseudocontractive in Theorem 1 and their fixed point sets $F(U)$ and $F(S)$ are nonempty, then we get:

**Corollary 1.** Let $x_1 \in X_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;

$$
y_n = U_n \left( \alpha_n x_n \oplus (1 - \alpha_n) \lambda_n A^* \overset{P_{AC,Ax_n}}{\to} S_n (Ax_n) Ax_n \right)
$$

$$
x_{n+1} = \beta_n f(x_n) \oplus (1 - \beta_n) z_n, \quad z_n \in \{T_n y_n\}, \quad n \geq 1,
$$

where $A^*$ is the adjoint operator of $A$, and $M, \lambda_n, \alpha_n, \beta_n \in [0, 1]$. Suppose that $AC$ is closed and convex, $\Gamma = \{x \in F(U) : Ax \in F(S)\} \cap \{x \in F(T)\} \neq \emptyset$, and the following conditions are satisfied;

1. there exists constant $N > 0$ such that $\varphi(r) \leq Nr, \quad r \geq 0$;
2. $\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \alpha_n = 0$;
3. $T$ satisfies the asymptotically regular condition $\lim_{n \to \infty} d(y_n, T_n y_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $x \in \Gamma$, where $P_{AC} S(Ax) = S(Ax)$.

**Remark 2.** Corollary 1 is about split common fixed point problem and fixed point problem. Hence, this result is new in the literature; in particular, it generalizes similar results in [24,33] from Banach space setting to CAT(0) spaces.

In Theorem 1, let $\overset{P_{AC,Ax_n}}{\to} (Ax_n) Ax_n = P_{AC,Ax_n} B_{\gamma_n}^S (Ax_n) Ax_n$ and $P_C = B_{\gamma_n}^U$, where $P_C : X_1 \to C$ is the metric projection of $X_1$ onto $C$. Then we get the following result.

**Corollary 2.** Let $x_1 \in X_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;

$$
y_n = P_{C_n} \left( \alpha_n x_n \oplus (1 - \alpha_n) \lambda_n A^* \overset{P_{AC,Ax_n}}{\to} (Ax_n) Ax_n \right)
$$

$$
x_{n+1} = \beta_n f(x_n) \oplus (1 - \beta_n) z_n, \quad z_n \in \{T_n y_n\}, \quad n \geq 1,
$$

where $A^*$ is the adjoint operator of $A$, and $M, \lambda_n, \alpha_n, \beta_n \in [0, 1]$. Suppose that $AC$ is closed and convex, $\Gamma = \{x \in C : Ax \in AC\} \cap \{x \in F(T)\} \neq \emptyset$, and the following conditions are satisfied;

1. there exists a constant $N > 0$ such that $\varphi(r) \leq Nr, \quad r \geq 0$;
2. $\lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \alpha_n = 0$;
3. \( T \) satisfies the asymptotically regular condition \( \lim_{n \to \infty} d(y_n, T_n y_n) = 0. \)

Then \( \{x_n\} \) converges strongly to a point \( x \in \Gamma. \)

**Remark 3.** As Corollary 2 deals with split feasibility problem and fixed point problem so it is a new result in the literature. It also extends similar results in Banach spaces \([15,32]\) to the case of \( \text{CAT}(0) \) spaces.

4. **Application to Split Hammerstein Integral Inclusion and Fixed Point Problem**

   An integral equation of Hammerstien-type is of the form
   \[
   u(x) + \int_C k(x,y)f(y,u(y))dy = g(x)
   \]
   (see \([11]\)).

   By writing the above equation in the following form
   \[
   u + KFu = g,
   \]
   without loss of generality, we have
   \[
   u + KFu = 0. \tag{27}
   \]

   If instead of the singlevalued maps \( f \) and \( k \), we have the multivalued functions \( f \) and \( k \), then we obtain Hammerstein integral inclusion in the form \( 0 \in u \oplus KFu \), where \( F : X_1 \to CB(X_1) \)
   defined by \( Fu(y) : = \{v(y) : v \text{ is some selection of } f(\cdot,u(\cdot))\} \) and \( K : X_1 \to CB(X_1) \) defined by \( Kv(x) : = \{w(x) : w \text{ is some selection of } k(\cdot,y)\} \), are bounded and maximal monotone operators (see for example \([23]\)).

   So the split Hammerstein integral inclusion problem is formulated as: find \( x^*, y^* \in X_1 \times X_1 \) such that, for \( v(\cdot) \in Fu(\cdot) \) and \( w(\cdot) \in Kv(\cdot) \)
   \[
   x^* \oplus w(v(x^*)) = 0 \text{ with } v(x^*) = y^* \text{ and } w(y^*) \oplus x^* = 0
   \]
   and \( Ax^*, Ay^* \in X_2 \times X_2 \) such that, for \( v'(\cdot) \in F'u(\cdot) \) and \( w'(\cdot) \in K'v'(\cdot), \)
   \[
   Ax^* \oplus w'(v'(Ax^*)) = 0 \text{ with } v'(Ax^*) = Ay^* \text{ and } w'(Ay^*) \oplus Ax^* = 0
   \]
   where \( F' : X_2 \to CB(X_2) \) and \( K' : X_2 \to CB(X_2) \), defined as \( F \) and \( K \), respectively, are also bounded and maximal monotone.

**Lemma 8.** Let \( X \) be a \( \text{CAT}(0) \) space, \( E : = X \times X \) and let \( F : \text{dom}(F) \subseteq X \to CB(X), K : \text{dom}(K) \subseteq X \to CB(X) \) be two multivalued maps. Define \( D : \text{dom}(F) \times \text{dom}(K) \to CB(E) \) by \( D(x,y) : = \left( \begin{array}{c}
F(x) \\
K(x)
\end{array} \right) \times (Ky \oplus x) \forall x, y \in \text{dom}(F) \times \text{dom}(K) = \left\{ (v(y)y, w(x) \ominus x) : v(y) \in Fu(y), w(x) \in Kv(x) \right\}. \)

Suppose that \( F \) and \( K \) are monotone. Then \( D \) is monotone.
Proof. Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in E$ and let $\bar{w}_1 \in D(z_1), \bar{w}_2 \in D(z_2)$. Then $z_1 = (v_1(y_1), w_1(x_1) \oplus x_1), z_2 = (v_2(y_2), w_2(x_2) \oplus x_2)$, for some $v_1(y_1) \in Fu_1$, $v_2(y_2) \in Fu_2$, $w_1(x_1) \in Ky_1$ and $w_2(x_2) \in Ky_2$. Therefore, by monotonicity of $F$ and $K$, we get

$$\langle z_1 z_2, \bar{z}_1 \bar{z}_2 \rangle = \langle (x_1, y_1), v_1(y_1) \oplus y_2, w_1(x_1) \oplus x_1, \bar{z}_1 \bar{z}_2 \rangle$$

$$= \langle x_1, v_1(y_1) \oplus y_2, x_1, \bar{z}_1 \rangle + \langle y_1, w_1(x_1) \oplus x_1, \bar{z}_2 \rangle$$

$$= \langle x_1, v_1(y_1) \oplus y_2, x_1, \bar{z}_1 \rangle - \langle x_1, y_1, \bar{z}_1 \rangle + \langle y_1, w_1(x_1) \oplus x_1, \bar{z}_2 \rangle$$

$$+ \langle y_1, y_2, x_1, \bar{z}_2 \rangle = \langle x_1, v_1(y_1) \oplus y_2, \bar{z}_1 \rangle + \langle y_1, y_2, \bar{z}_2 \rangle \geq 0.$$

This completes the proof of the lemma.

By Lemma 8, we have two resolvent mappings,

$$B_D^D = (I + \gamma D)^{-1}$$

and

$$B_D^{D'} = (I + \gamma D')^{-1},$$

where $D' : dom(F') \times dom(K') \rightarrow CB(E)$ is defined by

$$D'(Ax, Ay) = (F'AxAy) \times (K'Ax \oplus Ax)$$

$\forall Ax, Ay \in dom(F') \times dom(K') \subseteq \left\{ (v'(Ay)Ax, \lambda(x) \odot Ax) : v'(Ay) \in F'u(Ay), \lambda(x) \in K'\lambda(x) \right\}.$

Now $D$ and $D'$ are maximal monotone by Lemma 8. When $U = D$ and $S = D'$ in Theorem 1, the algorithm (1) becomes

$$y_n = B_D^{\beta_n}(x_n \odot (1 - \alpha_n)A^* \odot A^{\beta_n} \odot B_D^{D'}(Ax_n)Ax_n)$$

and its strong convergence is guaranteed, which solves the split Hammerstein integral inclusion problem and fixed point problem for the mappings involved in this scheme.

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