*mathematics*

Article

# Multiple Solutions for Nonlocal Elliptic Systems Involving p(x)-Biharmonic Operator

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**Abstract:** This paper analyzes the nonlocal elliptic system involving the p(x)-biharmonic operator. We give the corresponding variational structure of the problem, and then by means of Ricceri's Variational theorem and the definition of general Lebesgue-Sobolev space, we obtain sufficient conditions for the infinite solutions to this problem.

**Keywords:** nonlocal elliptic system; p(x)-biharmonic operator; variable exponent space; variational theorem

## 1. Introduction

This article analyzes the system

$$\begin{cases} \Delta^2_{p(x)}u(x) - M\Big(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\Big)\Delta_{p(x)}u(x) + \rho(x)|u|^{p(x)-2}u(x) = \lambda f(x,u)\ in\, \Omega, \\ u = \Delta u = 0 \quad on\, \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ with a smooth boundary. $p(x) \in C(\overline{\Omega})$, $\lambda > 0, \rho(x) \in L^\infty(\Omega), \Delta^2_{p(x)}(u)$ is the operator defined as $\Delta(|\Delta u|^{p(x)-2}\Delta u)$. $\frac{N}{2} < p^- := essinf_{x\in\Omega}p(x) \leq p^+ := esssup_{x\in\Omega}p(x) < \infty$. The continuous function $M : [0,\infty) \to \mathbb{R}$ satisfies $0 < m_0 \leq M(t) \leq m_1$ for every $t \geq 0$. $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ has certain conditions.

The biharmonic equation have many applications, such as describing the theorem of beam vibration, image processing, and so on. In [1], under appropriate conditions and Ricceri's three critical point theory, Li &Tang researched a class of p-biharmonic problems, and three solutions were obtained under the navier boundary value. In [2], Wang & An studied the problem:

$$\begin{cases} \Delta^2 u(x) - M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u(x) = \lambda f(x,u) \quad in\, \Omega, \\ u = \Delta u = 0 \quad on\, \partial\Omega. \end{cases} \tag{2}$$

In [3], when the nonlinear term $f(x,u)$ satisfying the (AR) condition, using the mountain pass theorem and local minimum theorem, two non-trivial solutions of the p-biharmonic system have been obtained. The authors in [4] researched the same problem in [3] and obtained multiple solutions according to Ricceri's variational Principle.

The p(x)-biharmonic problem is the general form of the p-biharmonic problem. The operator is no longer a satisfied homogeneous and pointwise identity. The p(x)-biharmonic problem

$$\begin{cases} \Delta\Big(|\Delta u|^{p(x)-2}\Delta u\Big) = f(x,u)\ in\, \Omega, \\ u = \Delta u = 0 \quad \partial\Omega, \end{cases} \tag{3}$$

has also been studied a lot, see [5–10]. When $f(x,u) = \lambda|u|^{p(x)-2}u$ in problem (3), Ayoujil & EI Amross [5] used Ljusternik-Schnirelmann critical point theorem and found that there are multiple eigenvalues to this problem. When $f(x,u) = \lambda V(x)|u|^{q(x)-2}u$ in problem (3), $1 < q(x) < p(x) < \frac{N}{2} < s(x)$, $V(x) \in L^{s(x)}$, there are multiple eigenvalues to this problem in the neighborhood of the origin in [7]. In [9], Kong studied the p(x)-Biharmonic equation with the Mountain pass theorem. In [11], Miao obtained the many solutions to the $(p_1(x), \cdots, p_n(x))$-biharmonic problem.

The nonlocal problems that arise in elasticity and population models and have attracted much attention in recent years, see [12–19]. In [14], Dai & Hao considered the nonlocal system

$$-M\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right)\Delta_{p(x)}u(x) = f(x,u), \tag{4}$$

with a Dirichlet boundary condition. When the nonlinear term $f(x,u)$ satisfying the (Ambrosetti -Rabinowitz condition, there are multiple solutions to the problem (4) using the Fountain theorem. In [20], multiple solutions for the $(p(x), q(x))$ problems with the Kirchhoff type were obtained using Ricceri's critical point theorem. Based on the Ricceri's variational principle, Miao [18] studied the $(p_1(x), \cdots, p_n(x))$ problems with a Kirchhoff operator.

Although there have been many important results for biharmonic and nonlocal equations in recent years, the corresponding results have also been applied in practice. However, there is little research about the nonlocal elliptic systems involving p(x)-Biharmonic operators. At present, the problem of the variable index has important applications in many disciplines and fields. The chief aim of this article is to research the system (1) under appropriate conditions using Ricceri's variational principle.

## 2. Preliminaries

This section we introduce important theorems on $L^{p(x)}(\Omega), W^{2,p(x)}(\Omega)$ which we will use in this paper.

$$L^{p(x)}(\Omega) = \{u\,|\text{u is measurable}, \int_\Omega |u(x)|^{p(x)}dx < \infty\}$$

has the norm

$$|u|_{p(x)} = \inf\left\{\tau > 0\,|\int_\Omega \left|\frac{u(x)}{\tau}\right|^{p(x)} dx \le 1\right\}.$$

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega)\,|\,D^\gamma u \in L^{p(x)}(\Omega),\ |\gamma| \le m, m \in \mathbb{Z}_+\}$$

has the norm

$$\|u\|_{m,p(x)} = \Sigma_{|\gamma|\le m}|D^\gamma u|_{p(x)}$$

$\gamma$ is the multi-index and $|\gamma|$ is the order.

The closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$ is the $W_0^{m,p(x)}(\Omega)$. From [21], $L^{p(x)}(\Omega), W^{m,p(x)}(\Omega)$ are Banach space and have reflexivity, uniform convexity and separability.

**Proposition 1.** *([21]) Suppose* $\frac{1}{p(x)} + \frac{1}{p^0(x)} = 1$, *then* $L^{p^0(x)}(\Omega)$ *and* $L^{p(x)}(\Omega)$ *are conjugate space, and satisfy the Holder inequality:*

$$\left|\int_\Omega uvdx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p^0)^-}\right)|u|_{p(x)}|v|_{p^0(x)}, \quad u \in L^{p(x)}(\Omega), v \in L^{p^0(x)}(\Omega).$$

*We denote* $X := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$ *and has the norm*

$$\|u\| = \inf\left\{\lambda > 0\,|\int_\Omega\left(\left|\frac{\Delta u(x)}{\lambda}\right|^{p(x)} + \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} + \left|\frac{u(x)}{\lambda}\right|^{p(x)}\right)dx \le 1\right\},$$

$X$ is separable and reflexive Banach spaces. By [22], $\|\cdot\|, \|\cdot\|_{2,p(\cdot)}$ and $|\triangle u|_{p(\cdot)}$ are equivalent norms of $X$.

**Proposition 2.** *([21,23]) For every $\forall u \in L^{p(x)}(\Omega)$, let $J(u) = \int_\Omega |u|^{p(x)}dx$, we have:*

*(1)* $|u|_{p(x)} < 1(=1; > 1) \Leftrightarrow J(u) < 1(=1; > 1)$;

*(2)* $|u|_{p(x)} \geq 1 \Rightarrow |u|_{p(x)}^{p^-} \leq J(u) \leq |u|_{p(x)}^{p^+}$;

*(3)* $|u|_{p(x)} \leq 1 \Rightarrow |u|_{p(x)}^{p^+} \leq J(u) \leq |u|_{p(x)}^{p^-}$;

*(4)* $|u|_{p(x)} \to 0 \Leftrightarrow J \to 0$.

According to the Proposition 2, for $u \in X$, we can deduce the following conclusions:

$$\|u\|^{p^-} \leq \int_\Omega \left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho(x)|u|^{p(x)}\right)dx \leq \|u\|^{p^+} \quad \text{if } \|u\| \geq 1, \tag{5}$$

$$\|u\|^{p^+} \leq \int_\Omega \left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)} + \rho(x)|u|^{p(x)}\right)dx \leq \|u\|^{p^-} \quad \text{if } \|u\| \leq 1. \tag{6}$$

The inequality of (5) and (6) play an important role in the deduction of main conclusions.

**Proposition 3.** *([10]) When $p^- > \frac{N}{2}$, $\Omega \subset \mathbb{R}^N$ is a bounded region, the $X \mapsto C(\overline{\Omega})$ is a compact embedding.*

According to the Proposition 3, $\forall u \in X$, there exists a constant $K > 0$ that depends on $p(\cdot), N, \Omega$:

$$\|u\|_\infty = \sup_{x\in\Omega}|u(x)| \leq K\|u\|. \tag{7}$$

Define the function $I_\lambda : X \to \mathbb{R}$

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

$$\Phi(u) = \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx + \hat{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right) + \int_\Omega \frac{\rho(x)|u|^{p(x)}}{p(x)}dx, \tag{8}$$

$$\Psi(u) = \int_\Omega F(x,u)dx. \tag{9}$$

where $\hat{M}(t) = \int_0^t M(s)ds$, $F(x,u) = \int_0^u f(x,t)dt$.

The functional $\Phi, \Psi : X \to \mathbb{R}$ are Gateaux differentiable functions, $\forall v \in X$, we have:

$$\langle \Phi'(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v dx + M\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega \rho(x)|u|^{p(x)-2}uv dx,$$

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u)v dx,$$

Furthermore, we can deduce that in order to find the weak solution of Problem (1), we can turn to find the critical point of $I_\lambda$. $\forall u \in X$, $\|u\| > 1$, according to (5), we can get

$$\begin{aligned}\Phi(u) &\geq \tfrac{1}{p^+}\int_\Omega |\Delta u|^{p(x)}dx + m_0\int_\Omega \tfrac{|\nabla u|^{p(x)}}{p(x)}dx + \tfrac{1}{p^+}\int_\Omega \rho(x)|u(x)|^{p(x)}dx \\ &\geq \tfrac{M^-}{p^+}\|u\|^{p^-},\end{aligned}$$

where $M^- = \min\{1, m_0\}$. Clearly $\Phi$ is coercive.

## 3. Main Results

The following results are due to Ricceri's [24].

**Theorem 1.** *([24]) Suppose Banach space $X$ is reflexive; $\Psi, \Phi : X \to \mathbb{R}$ are Gateaux differential equations, $\Psi$ satisfies sequentially weakly upper semicontinuity, $\Phi$ satisfies coercive and sequentially weakly lower semicontinuity. When $r > \inf\limits_X \Phi$, denote*

$$\varphi(r) := inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup\limits_{v\in\Phi^{-1}((-\infty,r))} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

*and*

$$\gamma := \operatorname{liminf}_{r\to+\infty} \varphi(r),\ \delta := \operatorname{liminf}_{r\to(\inf\limits_X \Phi)^+} \varphi(r).$$

*Hence, one has*

*(a) $\forall r > \inf\limits_X \Phi$ and $\lambda \in (0, \frac{1}{\varphi(r)})$, the functional $I_\lambda = \Phi - \lambda\Psi$ has a global minimum in $\Phi^{-1}((-\infty, r))$, which is a critical point (local minimum) of $I_\lambda$ in $X$.*
*(b) If $\gamma < +\infty$, then, $\forall\lambda \in (0, \frac{1}{\gamma})$, one of the following two conclusions holds: either*
*(b1) $I_\lambda$ has a global minimum, or*
*(b2) $I_\lambda$ has a sequence local minimum (critical points) denoted by $\{u_n\}$, $\lim\limits_{n\to+\infty} \Phi(u_n) = +\infty$.*
*(c) If $\delta < +\infty$, then, $\forall\lambda \in (0, \frac{1}{\delta})$, one of the following two conclusions holds: either*
*(c1) $\Phi$ has a global minimum which is a local minimum of $I_\lambda$, or*
*(c2) $I_\lambda$* has a sequence of pairwise distinct local minimum (critical points) which weakly converges to a global minimum of $\Phi$.

According Theorem 1, we can receive the following conclusions

**Theorem 2.** *Suppose*
*(A1) $\forall (x, t) \in \Omega \times [0, +\infty), F(x, t) \geq 0$.*
*(A2) Denote $S(\widetilde{x}, l_2)$ as a ball with center at $\widetilde{x}$ and radius of $l_2, \widetilde{x} \in \Omega$, $0 < l_1 < l_2 . S(\widetilde{x}, l_2) \subset \Omega$ If we put*

$$\alpha := \operatorname{liminf}_{\sigma\to+\infty} \frac{\int_\Omega \sup\limits_{|t|<\sigma} F(x,t)dx}{\sigma^{p^-}},$$

$$\beta := \operatorname{limsup}_{t\to+\infty} \frac{\int_{S(\widetilde{x},l_1)} F(x,t)dx}{t^{p^+}}.$$

*One has*

$$\alpha < L\beta,$$

*where*

$$L = \frac{p^- M^-}{p^+ K^{p^-}(\theta_1 + m_1\theta_2 + |\rho(x)|_\infty \theta_3)},$$

$$\theta_1 = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} \left(\frac{2N}{l_2^2 - l_1^2}\right)^{p^+} (l_2^N - l_1^N),$$

$$\theta_2 = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \left(\frac{2}{l_2^2 - l_1^2}\right)^{p^+} \frac{l_2^{p^+ +N} - l_1^{p^+ +N}}{p^+ + N},$$

$$\theta_3 = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left[\frac{l_1^N}{N} + \left(\frac{1}{l_2^2 - l_1^2}\right)^{p^+}\int_{l_1}^{l_2}(l_2^2 - r^2)^{p^+} r^{N-1}dr\right],$$

*Then, for every*

$$\lambda \in \Lambda := \frac{M^-}{p^+ K^{p^-}}\left(\frac{1}{L\beta}, \frac{1}{\alpha}\right),$$

*there exists a series of unbounded weak solutions to the problem (1).*

**Proof.** Based on previous results, $\Phi, \Psi$ meet Theorem 1. Since $\Phi(0) = 0, \Psi(0) \geq 0$, then, when $r > 0$, we have:

$$\varphi(r) \leq \frac{\sup_{\Phi(v)<r} \int_\Omega F(x,v)dx}{r}. \tag{10}$$

From (A2), there exists a sequence $\{\xi_n\}$ and $\xi_n \to +\infty$ as $n \to +\infty$, the following conclusion is valid:

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|<\xi_n} F(x,t)dx}{\xi_n^{p^-}} = \alpha < +\infty. \tag{11}$$

From Proposition 2, $\forall u \in X$, we have:

$$\Phi(u) \geq M^- \min\left\{\frac{\|u\|^{p^-}}{p^+}, \frac{\|u\|^{p^+}}{p^+}\right\}.$$

Put $r_n = \frac{M^-}{p^+}\left(\frac{\xi_n}{K}\right)^{p^-}$ for all $n \in \mathbb{N}$. When $u \in \Phi^{-1}(-\infty, r_n)$, we have $\Phi(u) < r_n$.

If $\|u\| \leq 1$, we can deduce $\frac{M^-}{p^+}\|u\|^{p^+} < r_n$, then $\|u\| < \left(\frac{r_n p^+}{M^-}\right)^{\frac{1}{p^+}}$.

If $\|u\| \geq 1$, we can deduce $\frac{M^-}{p^+}\|u\|^{p^-} < r_n$, then $\|u\| < \left(\frac{r_n p^+}{M^-}\right)^{\frac{1}{p^-}}$.

Hence for $n$ large enough ($r_n > M^-$),

$$\|u\| < \left(\frac{r_n p^+}{M^-}\right)^{\frac{1}{p^-}}.$$

From (7),

$$|u(x)| < K\left(\frac{r_n p^+}{M^-}\right)^{\frac{1}{p^-}} = \xi_n$$

Then the inclusion of sets is valid

$$\Phi^{-1}(-\infty, r_n) \subseteq \{|u(x)| < \xi_n; u \in X\}. \tag{12}$$

From (10)–(12), we have

$$\varphi(r_n) \leq \frac{\sup_{\Phi(v)<r_n} \int_\Omega F(x,v)dx}{r_n} \leq \frac{\int_\Omega \sup_{|t|<\xi_n} p^+ K^{p^-} F(x,t)dx}{M^- \xi_n^{p^-}}. \tag{13}$$

Combination condition (A2), We can get the following conclusion

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \frac{p^+ K^{p^-}\alpha}{M^-} < \infty. \tag{14}$$

Hence, $\Lambda \subset (0, \frac{1}{\gamma})$.

Next step, we want to check that $I_\lambda$ does not have global minimum for $\lambda \in \Lambda$. Indeed, since

$$\frac{1}{\lambda} < \frac{p^+ K^{p^-} L\beta}{M^-},$$

we consider a positive real sequence $\{\eta_n\} \to +\infty$ as $n \to +\infty$ and $\theta > 0$ such that

$$\frac{1}{\lambda} < \theta < \frac{p^+ K^{p^-} L}{M^-} \cdot \frac{\int_{S(\widetilde{x}, l_1)} F(x, \eta_n) dx}{\eta_n^{p^+}}. \tag{15}$$

For $n$ large enough, denote $u_n \in X$ by

$$u_n(x) = \begin{cases} 0, & x \in \Omega \backslash S(\widetilde{x}, l_2), \\ \eta_n, & x \in S(\widetilde{x}, l_1), \\ \frac{\eta_n}{l_2^2 - l_1^2}(l_2^2 - |x - \widetilde{x}|^2), & x \in S(\widetilde{x}, l_2) \backslash S(\widetilde{x}, l_1). \end{cases} \tag{16}$$

Then

$$\frac{\partial u_n(x)}{\partial x_i} = \begin{cases} 0, & x \in \Omega \backslash S(\widetilde{x}, l_2) \cup S(\widetilde{x}, l_1), \\ \frac{2\eta_n}{l_2^2 - l_1^2}(\widetilde{x}_i - x_i), & x \in S(\widetilde{x}, l_2) \backslash S(\widetilde{x}, l_1), \end{cases}$$

$$\frac{\partial^2 u_n(x)}{\partial x_i^2} = \begin{cases} 0, & x \in \Omega \backslash S(\widetilde{x}, l_2) \cup S(\widetilde{x}, l_1), \\ \frac{-2\eta_n}{l_2^2 - l_1^2}, & x \in S(\widetilde{x}, l_2) \backslash S(\widetilde{x}, l_1). \end{cases}$$

We see that

$$\int_\Omega |\Delta u_n|^{p(x)} dx = \int_{S(\widetilde{x}, l_2) \backslash S(\widetilde{x}, l_1)} |\Delta u_n|^{p(x)} dx \le \theta_1 \eta_n^{p^+}, \tag{17}$$

$$\int_\Omega |\nabla u_n|^{p(x)} dx = \int_{S(\widetilde{x}, l_2) \backslash S(\widetilde{x}, l_1)} |\nabla u_n|^{p(x)} dx \le \theta_2 \eta_n^{p^+}, \tag{18}$$

$$\int_\Omega |u_n|^{p(x)} dx = \int_{S(\widetilde{x}, l_1)} |\eta_n|^{p(x)} dx + \int_{S(\widetilde{x}, l_2) \backslash S(\widetilde{x}, l_1)} |u_n|^{p(x)} dx \le \theta_3 \eta_n^{p^+}, \tag{19}$$

therefore,

$$\Phi(u_n) \le \frac{1}{p^-}(\theta_1 + m_1\theta_2 + |\rho(x)|_\infty \theta_3)\eta_n^{p^+}. \tag{20}$$

At the same time, from (A1), we can get

$$\Psi(u_n) = \int_\Omega F(x, u_n) dx \ge \int_{S(\widetilde{x}, l_1)} F(x, \eta_n) dx. \tag{21}$$

from (15), (20) and (21). When $n$ is sufficiently large, we can deduce that

$$\begin{aligned} I_\lambda(u_n) &= \Phi(u_n) - \lambda\Psi(u_n) \\ &\le \tfrac{1}{p^-}(\theta_1 + m_1\theta_2 + |\rho(x)|_\infty\theta_3)\eta_n^{p^+} - \lambda \textstyle\int_{S(\widetilde{x}, l_1)} F(x, \eta_n) dx \\ &\le \tfrac{M^-}{p^+ K^{p^-} L}\eta_n^{p^+} - \lambda\theta \tfrac{M^-}{p^+ K^{p^-} L}\eta_n^{p^+} \\ &< \tfrac{M^-}{p^+ K^{p^-} L}(1 - \lambda\theta)\eta_n^{p^+}, \end{aligned} \tag{22}$$

so

$$I_\lambda(u_n) = -\infty, \text{ as } n \to +\infty.$$

According to Theorem 1 (b), Theorem 2 is proved. □

**Theorem 3.** *Suppose (A1) holds and:*

*(A3)* $\forall x \in \Omega, F(x,0) = 0$;

*(A4) Let* $\widetilde{x} \in \Omega, 0 < l_1 < l_2$,

$$\alpha^0 := \liminf_{\sigma \to 0^+} \frac{\int_\Omega \sup_{|t|<\sigma} F(x,t)dx}{\sigma^{p^+}},$$

$$\beta^0 := \limsup_{t \to 0^+} \frac{\int_{S(\widetilde{x},l_1)} F(x,t)dx}{t^{p^-}},$$

*such that*

$$\alpha^0 < L_1 \beta^0,$$

*where*

$$L_1 = \frac{p^- M^-}{p^+ K^{p^+}(\theta_1' + m_1\theta_2' + |\rho(x)|_\infty \theta_3')},$$

$$\theta_1' = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\left(\frac{2N}{l_2^2 - l_1^2}\right)^{p^-}(l_2^N - l_1^N),$$

$$\theta_2' = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left(\frac{2}{l_2^2 - l_1^2}\right)^{p^-}\frac{l_2^{p^-+N} - l_1^{p^-+N}}{p^- + N},$$

$$\theta_3' = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left[\frac{l_1^N}{N} + \left(\frac{1}{l_2^2 - l_1^2}\right)^{p^-}\int_{l_1}^{l_2}(l_2^2 - r^2)^{p^-} r^{N-1} dr\right],$$

*Hence, for every*

$$\lambda \in \Lambda_1 := \frac{M^-}{p^+ K^{p^+}}\left(\frac{1}{L_1\beta^0}, \frac{1}{\alpha^0}\right),$$

*the problem (1) has infinity solutions which converges to 0.*

**Proof.** According to (A3), we have $\min_X \Phi = 0$.

$\{\xi_n\}$ is a real sequence, $\xi_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n \to +\infty} \frac{\int_\Omega \sup_{|t|<\xi_n} F(x,t)dx}{\xi_n^{p^+}} = \alpha^0 < \infty. \tag{23}$$

Put $r_n = \frac{M^- \xi_n^{p^+}}{p^+ K^{p^+}}$ for all $n \in \mathbb{N}$. Hence, by assumption (A4), we can get

$$\delta \le \liminf_{n \to +\infty} \varphi(r_n) \le \frac{p^+ K^{p^+}}{M^-} \cdot \frac{\int_\Omega \sup_{|t|<\xi_n} F(x,t)dx}{\xi_n^{p^+}} = \frac{p^+ K^{p^+} \alpha_0}{M^-} < +\infty. \tag{24}$$

It is clear that $\Lambda_1 \subseteq (0, \frac{1}{\delta})$.

We now show that $I_\lambda$ does not take the local minimum at 0. For $\lambda \in \Lambda_1$, we have

$$\frac{1}{\lambda} < \frac{p^+ K^{p^+} L_1 \beta_0}{M^-}$$

$\{\eta_n\}$ is a positive sequences $\eta_n \to 0^+$ as $n \to +\infty$. $\theta > 0$ satisfies

$$\frac{1}{\lambda} < \theta < \frac{p^+ K^{p^+} L_1}{M^-} \cdot \frac{\int_{S(x_0,R_1)} F(x,\eta_n)dx}{\eta^{p^-}}. \tag{25}$$

Let $\{u_n(x)\}$ defined by (16), we have:

$$\Phi(u_n) \le \frac{1}{p^-}(\theta_1' + m_1\theta_2' + |\rho(x)|_\infty \theta_3')\eta_n^{p^-}. \tag{26}$$

Combining (25) and (26), we have:

$$I_\lambda(u_n) = \Phi(u_n) - \lambda\Psi(u_n) < \frac{(1-\lambda\theta)M^-\eta_n^{p^-}}{p^+K^{p^+}L_1} < 0 = I_\lambda(0). \tag{27}$$

According to Theorem 1 (c), Theorem 3 is proved. $\{u_n\}$ is the solution satisfying the condition and $u_n \to 0$. □

**Example 1.** *Let $\Omega = ((-1,1))^2$, $M(t) = a + bt$ $(a,b > 0)$ for all $t \ge 0$, then $m_0 = a$. $p(x)$ defined on $\Omega$ by $p(x_1,x_2) = {x_1}^2 + {x_2}^2 + 3$. $\{a_n\}$ is an increasing sequence given by:*

$$a_1 = 2, a_{n+1} = n(a_n)^2 + 2 (n \ge 1).$$

Define the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$f(x_1,x_2,t) = \begin{cases} (a_{n+1})^6 e^{1-\frac{1}{1-(t-a_{n+1})^2}+{x_1}^2+{x_2}^2} \frac{2(a_{n+1}-t)}{\left[1-(t-a_{n+1})^2\right]^2}, & if\ (x_1,x_2,t) \in \Omega \times \cup_{n\ge1} S(a_{n+1},1), \\ 0, & otherwise, \end{cases}$$

where $S(a_{n+1},1)$ denotes a unit ball with center at $a_{n+1}$. Then we can calculate

$$F(x_1,x_2,t) = \begin{cases} (a_{n+1})^6 e^{1-\frac{1}{1-(t-a_{n+1})^2}+{x_1}^2+{x_2}^2}, & if\ (x_1,x_2,t) \in \Omega \times \cup_{n\ge1} S(a_{n+1},1), \\ 0, & otherwise, \end{cases}$$

$F$ is nonnegative and $F(x_1,x_2,0) = 0$ for $(x_1,x_2) \in \Omega$. The maximum of $F$ on $\Omega \times S(a_{n+1},1)$ is:

$$F(x_1,x_2,a_{n+1}) = (a_{n+1})^6 e^{{x_1}^2+x_2^2}.$$

Therefore:

$$\text{limsup}_{n\to\infty} \frac{F(x_1,x_2,a_{n+1})}{{a_{n+1}}^5} = +\infty.$$

Then we can then get

$$\beta = \text{limsup}_{t\to+\infty} \frac{\int_{S(\widetilde{x},l_1)} F(x_1,x_2,t)dx_1dx_2}{t^5} = |S(\widetilde{x},l_1)| \text{limsup}_{t\to+\infty} \frac{F(x_1,x_2,t)}{t^5} = +\infty,$$

where $|S(\widetilde{x},l_1)|$ is the measure of $S(\widetilde{x},l_1)$. On the other hand, by choosing $\sigma_n = a_{n+1} - 1$, then we have $\sup_{|t|<\sigma_n} F(x_1,x_2,t) = (a_n)^6 e^{x_1^2+x_2^2}$.

Hence

$$\lim_{n\to\infty} \frac{\sup_{|t|<\sigma_n} F(x_1,x_2,t)}{(a_{n+1}-1)^3} = 0$$

and:

$$\liminf_{\sigma\to+\infty}\frac{\sup_{|t|<\sigma}F(x_1,x_2,t)}{\sigma^3}=0.$$

Therefore:

$$\alpha:=\liminf_{\sigma\to+\infty}\frac{\int_\Omega \sup_{|t|<\sigma}F(x_1,x_2,t)dx_1dx_2}{\sigma^3}=|\Omega|\liminf_{\sigma\to+\infty}\frac{\sup_{|t|<\sigma}F(x_1,x_2,t)}{\sigma^3}=0<L\beta=+\infty,$$

Then from Theorem 2, for every $\lambda>0$, the problem

$$\begin{cases}\Delta\left(|\Delta u|^{x_1^2+x_2^2+1}\Delta u\right)-\left(a+b\int_\Omega\frac{|\nabla u(x)|^{x_1^2+x_2^2+3}}{x_1^2+x_2^2+3}dx\right)div\left(|\nabla u|^{x_1^2+x_2^2+1}\nabla u\right)+\rho(x)|u|^{x_1^2+x_2^2+1}u(x)=\lambda f(x,u)\ in\ \Omega,\\ u=\Delta u=0\quad on\ \partial\Omega,\end{cases}$$

exists as a series of unbounded weak solutions.

## 4. Discussion

When $p(x)=p$ in problem (1), the corresponding conclusions were given in [4]. The study of a nonlocal type problem involving p-biharmonic operator has been extended to the p(x)-biharmonic operator and reached more general conclusions. The results obtained in this paper can provide a theoretical basis for future research on such problems.

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