mathematics

MDPI

Article

Cohen Macaulay Bipartite Graphs and Regular Element on the Powers of Bipartite Edge Ideals

Arindam Banerjee [1,*] and Vivek Mukundan [2]

1 Ramakrishna Mission Vivekananda Educational and Research Institute, Belur, West Bengal 711202, India
2 Department of Mathematics, University of Virginia, Charlottesville, VA 22902, USA
* Correspondence: 123.arindam@gmail.com

Received: 8 August 2019; Accepted: 13 August 2019; Published: 20 August 2019

Abstract: In this article, we discuss new characterizations of Cohen-Macaulay bipartite edge ideals. For arbitrary bipartite edge ideals $I(G)$, we also discuss methods to recognize regular elements on $I(G)^s$ for all $s \geq 1$ in terms of the combinatorics of the graph G.

Keywords: Cohen Macaulay; Bipartite graphs; regular elements on powers of bipartite graphs; colon ideals; depth of powers of bipartite graphs; dstab; associated graded rings

1. Introduction

The interplay between the combinatorics of finite simple graphs G and the algebra of the underlying edge ideals $I(G)$ has been studied by various researches during the last few decades. The algebraic invariants that have been particularly prone to combinatorial interpretation are regularity, projective dimension, depth, and Betti numbers. In this article, we study the depth of powers of edge ideals of bipartite graphs. Combinatorics of bipartite graphs have been particularly ripe with interesting algebraic counterparts in the edge ideals and their powers. Interested readers are referred to [1–3], etc. In this paper, we continue the study pursued by the same authors in [3]. We study the closely related topics of combinatorial characterization of regular elements and Cohen-Macaulayness of various powers of bipartite edge ideals.

In section two of this paper, we offer a new characterization of Cohen-Macaulay bipartite edge ideals. We characterize it using colon ideals of the form $(I(G)^2 : e)$, where e is an edge/generator of $I(G)$, somehow in the same way as it is done in [3,4], etc., in the study of regularity. An often quoted and important characterization of Cohen-Macaulay bipartite edge ideals is due to Herzog-Hibi in [2]. In this article, we also give a new proof of this characterization ([2]). One important feature of our proof is that it does not use Hall's marriage theorem or any variant of it as it was done in [2]. Throughout this article, we refer to S as the polynomial ring $\mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Our main results in this section are as follows:

Theorem 1. *Let G be a bipartite graph with partition $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_{n'}\}$. Then the following are equivalent*

1. *$S/I(G)$ is Cohen-Macaulay*
2. *$n = n'$ and there exists a re-ordering of the vertex sets V_1, V_2 such that*
   - *(a) $x_i y_i \in I(G)$ for all i*
   - *(b) If $x_i y_j \in I(G)$ then $i \leq j$.*
   - *(c) If $x_i y_j, x_j y_k \in I(G)$ then $x_i y_k \in E$.*
3. *$I(G)$ is unmixed and $S/I(G)$ is connected in codimension one.*
4. *$n = n'$ and there exists exactly n edges $e_1, \ldots, e_n$ such that $(I(G)^2 : e_i) = I(G)$ and for $i \neq j$, e_i and e_j are disjoint.*

5. $n = n'$ *and there exists exactly n edges* $e_1, \ldots, e_n$ *such that* $(I(G)^2 : e_i)$ *is Cohen-Macaulay and for* $i \neq j$, e_i *and* e_j *are disjoint.*

For arbitrary bipartite edge ideals, it is often hard to compute the depth of powers of its edge ideals $I(G)^s$ for all $s \geq 1$. Even if G is Cohen-Macaulay, it is not so easy to compute the depth $S/I(G)^s$ for $s \geq 2$. It is well known that depth $S/I(G)^s$ is asymptotically equal to the number of connected components of G ([5]). An important invariant related to the study of depth $S/I(G)^s$ is the dstab $I(G)$ which measures the minimal t for which depth $S/I(G)^t$ equals the number of connected components of G. To study such invariants the same authors in [3] characterized regular elements on $I(G)^s$ for any unmixed bipartite graphs G. In the third section of this paper we characterize elements of the form $x_\nu - y_\mu$ that are regular on the powers $I(G)^s$ of a bipartite edge ideal G. This is a generalization of the similar result proved in [3]. Our characterization turns out to be the exactly same as the $\star$-condition proved there. To signify its usefulness we call it the neighborhood properties (we refer to the definition in Definition 12) Our main result proved here is as follows:

Theorem 2. *Let G be a bipartite graph and suppose that* $x_\mu \in V_1$ *and* $y_\nu \in V_2$ *satisfies the neighborhood properties. Then* $x_\mu - y_\nu$ *is an regular element on* $S/I(G)^s$ *for all s.*

2. Structure of Cohen-Macaulay and Unmixed Bipartite Graphs

A characterization theorem for Cohen-Macaulay bipartite graphs was given by Herzog-Hibi in [2].

Theorem 3. (Herzog-Hibi, [2]) *Let G be a bipartite graph with partition* $V_1 = \{x_1, \ldots, x_n\}$ *and* $V_2 = \{y_1, \ldots, y_{n'}\}$. *Then the following are equivalent*

1. $S/I(G)$ *is Cohen-Macaulay*
2. $n = n'$ *and there exists a re-ordering of the vertex sets* V_1, V_2 *such that*
   (a) $x_i y_i \in I(G)$ *for all i*
   (b) *If* $x_i y_j \in I(G)$ *then* $i \leq j$.
   (c) *If* $x_i y_j, x_j y_k \in I(G)$ *then* $x_i y_k \in E$.

The following theorem is an improvement of the Herzog-Hibi characterization (Theorem 3). We are grateful to Prof. Huneke for the ideas presented in this proof. It is important to notice here that the following theorem does not make use of the Halls marriage theorem which is an important element of any proofs known to us of Theorem 3.

Definition 1. *(Definition, p. 498, [6]) Let I be an ideal in a polynomial ring S such that* $I = P_1 \cap \cdots \cap P_k$, $P_i \in \mathrm{Spec}(S), 1 \leq i \leq k$. *We say that the ring* S/I *is connected in codimension one if for any two primes* $Q', Q'' \in \mathrm{Min}(S/I)$, *there is a sequence of minimal primes* $Q' = Q_1, \ldots, Q_r = Q'' \in \mathrm{Min}(S/I)$ *such that for each* $i = 1, 2, \ldots, r-1$, $ht(Q_i + Q_{i+1}) = 1$ *in* S/I.

Theorem 4. *Let G be a bipartite graph with partition* $V_1 = \{x_1, \ldots, x_n\}$ *and* $V_2 = \{y_1, \ldots, y_{n'}\}$. *Then the following are equivalent*

1. $S/I(G)$ *is Cohen-Macaulay*
2. $n = n'$ *and there exists a re-ordering of the vertex sets* V_1, V_2 *such that*
   (a) $x_i y_i \in I(G)$ *for all i*
   (b) *If* $x_i y_j \in I(G)$ *then* $i \leq j$.
   (c) *If* $x_i y_j, x_j y_k \in I(G)$ *then* $x_i y_k \in E$.
3. $I(G)$ *is unmixed and* $S/I(G)$ *is connected in codimension one.*

Proof. First we show (2) $\Rightarrow$ (1). We prove by induction on n. If $n = 1$, then $I(G) = (x_1 y_1)$ and hence clearly $S/I(G)$ is Cohen-Macaulay. Now assume that the result is true for $n - 1$ and let G be a graph which satisfies the conditions $(a)-(c)$ of (2) on $2n$ vertices (with partition $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_n\}$). Consider

$$0 \to \frac{S}{(I(G):x_1)} \to \frac{S}{I(G)} \to \frac{S}{((I(G),x_1)} \to 0 \tag{1}$$

Notice that $(I(G), x_1) = (I(G'), x_1)$, where G' is the graph obtained by deleting x_1 and y_1 from G. Clearly G' satisfies the conditions $(a)-(c)$ of (2) and hence $S/I(G')$ is Cohen-Macaulay (on $2n-2$ vertices) by induction. So $S/(I(G), x_1)$ is Cohen-Macaulay of dimension n. Let $\{y_1, y_{i_1}, ..., y_{i_k}\} \subseteq (I(G):x_1)$ for some $i_1, ..., i_k$. Let $x_{i_j} y_l \in I(G)$ for some $1 \le j \le k$. As $x_1 y_{i_j} \in I(G)$ by the condition (c), $x_1 y_l \in I(G)$ and hence $l \in \{1, i_1, ..., i_k\}$. So $(I(G):x_1) = (I(G''), y_1, ..., y_{i_k})$, where G'' is the graph obtained from G by deleting $x_1, y_1, x_{i_2}, y_{i_2}, \ldots, x_{i_k}, y_{i_k}$. But by induction, $S/I(G')$ is Cohen-Macaulay of dimension $n-k$. Hence $S/(I(G):x_1)$ is Cohen-Macaulay of dimension n. Now in (1), both $S/(I(G):x_1)$ and $S/(I(G), x_1)$ are Cohen-Macaulay of dimension n, we have $S/I(G)$ is also Cohen-Macaulay of dimension n ((Proposition 1.2.9, [7]) and the fact that dimension of $S/I(G)$ is the maximum of the dimensions of $S/(I(G):x_1)$ and $S/(I(G), x_1)$).

The implication $(1) \Rightarrow (3)$ is a consequence of (Corollary 2.4, [6]).

We finally show $(3) \Rightarrow (2)$. We first observe that $n = n'$ as $I(G)$ is unmixed and both $(x_1, ..., x_n)$ and $(y_1, ..., y_{n'})$ are minimal primes. Next, we prove that the existence of conditions (a) and (b) by induction. Let $\emptyset \ne L \subset \{1, ..., n\}$ and define

$$y^L = \prod_{i \in L} y_i \qquad x^L = \prod_{i \in L} x_i \qquad T_L = \{j \mid x_j y_i \notin I(G) \text{ for any } i \in L\} \qquad u^L = y^L x^{T_L}.$$

Note that $u^L \notin I(G)$ for any subset $S \subseteq \{1, \ldots, n\}$. We now consider the ideals $(I(G):u^L)$. If $L' = \{1, ..., n\}$ then $(I(G):u^{L'}) = (x_1, ..., x_n)$ which shows that $(x_1, \ldots, x_n) \in \mathrm{Ass}(I(G))$. Since $I(G)$ is unmixed, we have $\mathrm{ht}\, I(G) = n$. Clearly for any $L \subseteq \{1, ..., n\}$, $(I(G):u^L) = (x_{j_1}, ..., x_{j_t}, y_{l_1}, ..., y_{l_{t'}})$ where for each $1 \le i \le t$, $x_{j_i} y_{r_i} \in I(G)$ for some $r_i \in S$ and for each $1 \le k \le t'$, $x_{w_k} y_{l_k} \in I(G)$ for some $w_k \in T_s$. Since $I(G)$ is unmixed of height n and $(x_{j_1}, ..., x_{j_t}, y_{l_1}, ..., y_{l_{t'}}) \in \mathrm{Ass}(I(G))$, we have $t + t' = n$.

Now choose y_i with minimum vertex degree. Without loss of generality we may assume $i = 1$. Let $x_1, ..., x_t$ be neighbors of y_1 and $L = \{1\}$. Then as in the previous paragraph, consider $(I(G):u^L) = (x_1, \ldots, x_t, y_{l_1}, \ldots, y_{l_{n-t}})$. After relabeling, we may assume $y_1, \ldots, y_t$ are only connected to $x_1, \ldots, x_t$. Let G' be the induced subgraph on $x_1, ..., x_t, y_1, ..., y_t$. By our choice of y_1, of minimal vertex degree t, notice that every other vertex y_j has to have vertex degree at least t. In other words, since t is minimal, each vertex $y_i, 1 \le i \le t$ in G' has at least t neighbors and hence G' is a complete bipartite graph.

Since $S/I(G)$ is connected in codimension one and $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathrm{Ass}(I(G))$, there exists a sequence of minimal primes $(x_1, \ldots, x_n) = P_1, \ldots, P_r = (y_1, \ldots, y_n)$ such that $\mathrm{ht}(P_i + P_{i+1}) = 1$ in $S/I(G)$. If any minimal prime P_l of $I(G)$ does not contain some $x_i, 1 \le i \le t$ then it has to contain every $y_j, 1 \le j \le t$ (as G',as defined in the previous paragraph, is a complete bipartite graph). Let $1 \le l \le r$ such that for all $1 \le i \le l$, P_i contains all of $x_1, \ldots, x_t$ (alternatively, P_i's do not contain any of $y_1, \ldots, y_t$). Now P_{l+1} does not contain at least one of $x_1, \ldots, x_t$, hence it has to contain all $y_1, \ldots, y_t$. So $\mathrm{ht}(P_l + P_{l+1}) \ge t$ in $S/I(G)$. Thus $t = 1$ and hence y_1 is only connected to x_1.

Now consider $(I(G), x_1)$. Since $I(G)$ is an intersection of minimal primes, $(I(G), x_1)$ is an intersection of minimal primes of $I(G)$ containing x_1. Thus any minimal prime of $(I(G), x_1)$ is a minimal prime of $I(G)$, and so $(I(G), x_1)$ is unmixed. We now show that $(I(G), x_1)$ is connected at codimension one. Any minimal prime of $I(G)$ has to contain either x_1 or y_1 (as it is minimal it cannot contain both as y_1 is only connected to x_1). Let $P', P'' \in \mathrm{Min}(I(G), x_1)$. As $P', P'' \in \mathrm{Min}\, I(G)$, there exists a sequence of minimal primes $P' = P_1, \ldots, P_r = P''$ such that $\operatorname{ht} P_i + P_{i+1} = 1$. For any $1 \le i \le r$,

$$P_i' = \begin{cases} P_i & \text{if } x_1 \in P_i \\ (P_i \backslash \{y_1\}) \cup \{x_1\} & \text{if } x_1 \notin P_i \end{cases}.$$

The sequence $P' = P_1', \ldots, P_r' = P''$ defined as before has the property that $\operatorname{ht} P_i' + P_{i+1}' = 1$ and hence $(I(G), x_1)$ is connected in codimension one. Now notice that $(I(G), x_1) = (I(G''), x_1)$ where G'' is the graph obtained from G by deleting x_1. By induction hypothesis, $S/I(G'')$ is Cohen-Macaulay. So there exists an ordering $\{x_2, ..., x_n\}$ and $\{y_2, ..., y_n\}$ satisfying $(a) - (b)$ of (2). As y_1 is only connected to x_1, G also satisfies $(a) - (b)$ of (2).

To prove that condition (c) holds, take x_iy_j and x_jy_k in $E(G)$ such that i, j, k are distinct. Assume that x_iy_k is not an edge. Then there is a minimal prime P that does not contain either x_i or y_k as the ideal generated by all x-variables except x_i and all y-variables except y_k is a prime ideal that contains $I(G)$ and does not contain x_i or y_k. Now because $I(G)$ is unmixed, height of this prime has to be n. Since x_i and y_k are not in P, we get that y_j and x_j are both in P. As P contains at least one of x_m or y_m for all m, one observes that height of P is strictly bigger than n, which is a contradiction. □

The following remark is extremely crucial for our work.

Remark 1. *If G is a bipartite graph and ab is an edge then from (Theorem 6.7, [4]) we get $(I(G)^2 : ab)) = I(G) + (uv | u \in N(a), v \in N(b))$.*

Theorem 5. *Let G be a bipartite graph with partition $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_{n'}\}$. Then the following are equivalent*

1. *$S/I(G)$ is Cohen-Macaulay*
2. *$n = n'$ and there exists exactly n edges $e_1, \ldots, e_n$ such that $(I(G)^2 : e_i) = I(G)$ and for $i \neq j$, e_i and e_j are disjoint.*
3. *$n = n'$ and there exists exactly n edges $e_1, \ldots, e_n$ such that $S/(I(G)^2 : e_i)$ is Cohen-Macaulay and for $i \neq j$, e_i and e_j are disjoint.*

Proof. First, we show (1) ⇔ (2). If $S/I(G)$ is Cohen-Macaulay, we have ordering $x_1, ..., x_n$ and $y_1, ..., y_n$ of the vertices of G which satisfies the conditions of Theorem 4. Condition (c) implies for all i, $I(G)^2 : x_iy_i = I(G)$ and conditions (a) and (b) implies for $i \neq j$ $(I(G)^2 : x_iy_j) \neq I(G)$.

Now suppose there exist, after possible reordering, $e_1 = x_1y_1, ..., e_n = x_ny_n$ which satisfied the conditions of (2). First, we show that if G_i is the induced subgraph obtained by deleting x_i and y_i then the edge ideal J_i related to G_i satisfies the condition with $e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n$. Without loss of generality, we prove this for G_1. Clearly $(J_1^2 : e_i) = J_1$ for $2 \le i \le n$. Suppose there exists an edge $x_iy_j, i \neq j$ such that $(J_1^2 : x_iy_j) = J_1$. Without loss of generality we may assume $i = 2, j = 3$. As $(I(G)^2 : x_2y_3) \neq I(G)$ and x_1y_1 is an edge we can conclude that there exists a minimal generator of $(I(G)^2 : x_2y_3)$ which is an edge that is either of the form x_1y_l or x_my_1 (Theorem 6.7, [4]). Again without loss of generality we may assume it is of the form x_1y_l as the proof for the other follows simply by interchanging roles of x and y. So x_1y_3 and x_2y_l are edges in G (Theorem 6.7, [4]). As $(J_1^2 : x_2y_3) = J_1$ we conclude x_3y_2 is an edge in G. As $(I(G)^2 : x_3y_3) = I(G)$ we observe that x_1y_2 has to be an edge in G. So $l \neq 2, 3$. Without loss of generality we may assume $l = 4$. Now $(I(G)^2 : x_2y_2) = I(G)$ so x_3y_4 has to be an edge in G. Again $(I(G)^2 : x_3y_3) = I(G)$ hence x_1y_4 is an edge in G contradicting the assumption. So we may assume for all i the edge ideal $I(G_i)$ of the graph G_i obtained by deleting x_i and y_i satisfies the conditions in (2).

Now by induction we may assume the result holds for $n - 1$. Pick $e_i = x_iy_i$ such that y_i has minimum degree. Let G' be the induced subgraph on vertices other than x_i, y_i with edge ideal $I(G')$. As $I(G')$ satisfies the condition it is Cohen-Macaulay by induction. Without loss of generality we may assume $i = 1$ and ordering that gives ordering of previous theorem for $I(G')$ is $x_2, ..., x_n, y_2, ..., y_n$. As y_2 has degree one in G' it can have at most degree 2 in G. If x_1y_2 is not an edge, due to minimality degree of y_1 is at most 1. If x_1y_2 is an edge in G and x_iy_1 is an edge in G for $i > 2$, as $(I(G)^2 : x_1y_1) = I(G)$, we have x_iy_2 is an edge in G and hence in G' contradicting the assumption. Now if x_1y_2 and x_2y_1 both are edges in G. Notice that x_2y_1 also satisfies the hypothesis $(I(G)^2 : x_2y_1 = I(G))$. For, x_1 has to be connected to any neighbor of x_2 as x_1y_2 is an edge and x_2y_2 satisfies the hypothesis $(I(G)^2 : x_2y_2 = I(G))$. This leads to a contradiction and hence no x_i for $i > 1$ is connected to y_1. This guarantees that conditions (a) and (b) of Theorem 4(2) is satisfied. The condition (c) is satisfied as for all i, $(I(G)^2 : x_iy_i) = I(G)$.

Next we show (1) $\Leftrightarrow$ (3). To prove the if part, we pick, without loss of generality, y_1 with minimum degree and the corresponding edge $e_1 = x_1y_1$. If degree of y_1 more than one then degree of any other vertex is more than one; as $(I(G)^2 : e_1)$ is Cohen-Macaulay this will be a contradiction to the fact that any Cohen-Macaulay bipartite graph should have a y-vertex of degree 1 (Theorem 3). So y_1 has degree one. Hence $(I(G)^2 : e_1) = I(G)$ and $I(G)$ is Cohen-Macaulay.

For the only if part let $e_1 = (x_1y_1), ..., e_n = (x_ny_n)$ be as the ordering prescribed by the Herzog-Hibi (Theorem 3) characterization. All we need to show is that $J = (I(G)^2 : x_iy_j)$ is not Cohen-Macaulay for $i > j$. This follows as $(J^2 : e) = J$ for $e = x_jy_i$ (which is a minimal monomial generator of J) as well as for $e_1, ..., e_n$. To see this first we show that $(J^2 : e_k) = J$ for all k. Here at every step we use the description of colon ideal provided by (Theorem 6.7, [4]). If x_ly_m is a minimal monomial generator of $(J^2 : e_k)$ which is not in J then x_ly_k and x_ky_m are in J. Both of them cannot belong to $I(G)$ as from $(I(G)^2 : e_k) = I(G)$ that will imply x_ly_m belongs to $I(G)$ and as a result will belong to J, contradicting the assumption. Without loss of generality assume x_ky_m does not belong to $I(G)$. Then x_ky_j and x_iy_m is in $I(G)$. If x_ly_k does not belong to $I(G)$ then x_ly_j and x_iy_k belong to $I(G)$. If x_ly_k is in $I(G)$ as x_ky_j is in $I(G)$ and $(I(G)^2 : e_k) = I(G)$ we have x_ly_j is in $I(G)$. In either case we have x_ly_j and x_iy_m belong to $I(G)$. Hence x_ly_m belongs to J contradicting our assumption.

Next we show that $(J^2 : x_jy_i) = J$. If x_ly_k is a minimal monomial generator of $(J^2 : x_jy_i)$ which is not in J then x_jy_k and x_ly_i is in J. As x_jy_k is in J it is either in $I(G)$ or y_k is a neighbor of x_i in G. If x_jy_k is in $I(G)$ as $(I(G)^2 : x_jy_j) = I(G)$ we have x_iy_k is in $I(G)$. By symmetry x_ly_j is in $I(G)$. Hence x_ly_k is in J contrary to the assumption. Hence J is not Cohen-Macaulay. □

The next theorem gives insight into the associated graded ring of a Cohen-Macaulay bipartite edge ideal. The proof of this theorem uses the description of the colon of the nth power of an edge ideal with $n - 1$ edges introduced in [4].

Theorem 6. *Let $I(G)$ be Cohen-Macaulay bipartite edge ideal with an ordering of vertices satisfying Theorem 3(2) and $e_i = x_iy_i$ for $1 \leq i \leq n$. Then for all i and for all k, $(I(G)^k : e_i) = I(G)^{k-1}$. Hence e_is are non zero divisors in the associated graded ring of $I(G)$.*

Proof. Let $f \in (I(G)^k : e_i) \subset (I(G)^{k-1} : e_i)$ be a minimal monomial generator of $(I(G)^k : e_i)$. By induction $(I(G)^{k-1} : e_i) = I(G)^{k-2}$. So $f = gh_1....h_{k-2}$ where h_js are minimal monomial generators of $I(G)$ and g any monomial. So $e_ih_1....h_{k-2}g \in I(G)^k$. As f is a minimal monomial generator, without loss of generality we may assume g is of degree 2 and $e_ih_1..h_{k-2}g$ is a minimal monomial generator of $I(G)^k$. Let $g = x_ky_l, k \leq l$. If g is an edge we are done. Otherwise by ([4], Theorem 6.7), x_k and y_l are even connected with respect to $e_ih_1...h_{k-2}$. If x_iy_l is an edge and for some $j, m, p, h_j = x_my_p$ and x_my_i is an edge. Then by Theorem 4(2(c)) x_my_l is an edge and hence proceeding inductively we show g is an edge and the result follows. □

We illustrate this theorem for $k = 3, 4$.

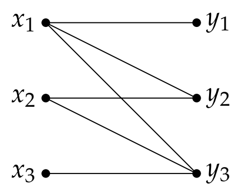
Example 1. *Let $S = \mathbf{k}[x_1, x_2, x_3, y_1, y_2, y_3]$ and $I = (x_1y_1, x_2y_2, x_3y_3, x_1y_2, x_1y_3, x_2y_3)$. One can check using Macaulay 2, that $(I^3 : x_1y_1) = (I^3 : x_2y_2) = (I^3 : x_3y_3) = I^2$ and $(I^4 : x_1y_1) = (I^4 : x_2y_2) = (I^4 : x_3y_3) = I^3$.*

In a private communication, Prof. Villarreal mentioned that results similar to Theorems 5 and 6 can be found in [8,9].

3. Regular Elements in Powers of Bipartite Edge Ideals

This section presents methods to recognize regular elements on the power of bipartite edge ideals based on the combinatorics of the graph. We first present some examples to motivate the definition and the results.

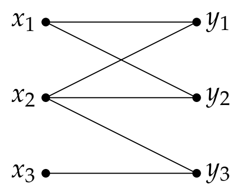
Example 2. *Consider the ring* $S = \mathbf{k}[x_1, x_2, x_3, y_1, y_2, y_3]$ *and the bipartite edge ideal* $I(G) = (x_1y_1, x_2y_2, x_3y_3, x_1y_2, x_1y_3, x_2y_3)$ *corresponding to*



Macaulay2 computations show that $x_3 - y_1$ *is a regular element on* $I(G)^s$ *for* $1 \le s \le 10$*. Notice that* $I(G)$ *is Cohen-Macaulay. This can also be recovered from (Theorem 3.8, [3]).*

One would be tempted to generalize that $x_n - y_1$ is always a regular element for bipartite graphs. But it is not always the case as it is shown in this example.

Example 3. *Consider the ring* $S = \mathbf{k}[x_1, x_2, x_3, y_1, y_2, y_3]$ *and the bipartite edge ideal* $I(G) = (x_1y_1, x_2y_2, x_3y_3, x_1y_2, x_2y_1, x_2y_3)$ *corresponding to*



Macaulay2 computations show that $x_3 - y_1$ *is not a regular element* $S/I(G)$ *or* $S/I(G)^2$*.*

Studying more such examples, we came up with the following definition involving the combinatorial nature of the graphs.

Definition 2. *Let G be a bipartite graph. Then* $x_\mu \in V_1, y_\nu \in V_2$ *satisfies the neighborhood condition if*

$$N(x_\mu) \subseteq N(x_{a_i}) \text{ for all } i, 1 \le i \le p \text{ where } N(y_\nu) = \{x_{a_1}, \dots, x_{a_p}\}. \tag{2}$$

Remark 2. *Condition (2) of Definition 2 is equivalent to the following condition*

$$N(y_\nu) \subseteq N(y_{b_j}) \text{ for all } i, 1 \le j \le q \text{ where } N(x_\mu) = \{y_{b_1}, \dots, y_{b_q}\}.$$

Suppose (2) of Definition 2 is true. Then $\{y_{b_1}, \dots, y_{b_q}\} = N(x_\mu) \subseteq N(x_{a_i})$*, where* $N(y_\nu) = \{x_{a_1}, \dots, x_{a_p}\}$*. This means* $x_{a_i} \in N(y_{b_j})$ *where* $1 \le i \le p, 1 \le j \le q$*. In other words,* $N(y_\nu) \subseteq N(y_{b_j})$*, where* $1 \le j \le q$*. The other direction is analogous.*

We show in [3] that $x_n - y_1$ is a regular element on $S/I(G)^s$ for all $s \ge 1$ when G is an unmixed bipartite graph. Of course, when G is unmixed bipartite, x_n and y_1 satisfies the neighborhood conditions. In this section, we show that the difference of vertices which satisfies the neighborhood condition are the right candidates for being a regular element on $S/I(G)^s$ for any bipartite graph G.

Theorem 8 is the main theorem we study in this section. We break up the proof of this theorem into three main parts, where Theorem 7, Lemma 1 provide all the tools required to prove Theorem 8.

Theorem 7. *Let G be a bipartite graph and suppose that $x_\mu \in V_1$ and $y_\nu \in V_2$ satisfies the neighborhood properties. If m is a monomial such that $mx_\mu^k, my_\nu^k \in I(G)^s$, then $m \in I(G)^s$ for $s, k \geq 1$.*

Proof. We prove by induction on k. Suppose $k = 1$. Then $mx_\mu, my_\nu \in I(G)^s$. As $mx_\mu \in I(G)^s$, then either $m \in I(G)^s$ or $m = m'y_t$ for some $y_t \in N_G(x_\mu)$ and $m' \in I(G)^{s-1}$. If $m \in I(G)^s$, then the claim is obviously true.

Suppose $m = m'y_t$ with $m' \in I(G)^{s-1} \backslash I(G)^s$. Let $m' = ae_1 \cdots e_{s-1}$ for $e_1, \ldots, e_k \in I(G), a \in S$. We assume $e_i = (x_{u_i} y_{v_i})$, $1 \leq i \leq s-1$. Since $my_\nu \in I(G)^s$, we have $m'y_t y_\nu \in I(G)^s$. Thus

$$\begin{aligned} m'y_t y_\nu &= ae_1 \cdots e_{s-1} y_t y_\nu \in I(G)^s \\ &= bf_1 \ldots f_s \text{ for } f_1, \ldots, f_s \in I(G), b \in S \end{aligned} \tag{3}$$

Suppose a neighbor of y_t divides a, then clearly $m = m'y_t \in I(G)^s$. Now suppose a neighbor of y_ν divides a. Since x_μ and y_ν satisfies the neighborhood properties, any neighbor of y_ν is also a neighbor of y_t and hence $m = m'y_t \in I(G)^s$.

Suppose that no neighbor of y_t or y_ν divide a. Now in the decomposition in (3), if y_ν does not divide $f_1 \cdots f_s$, then $f_1 \cdots f_s$ divides $m'y_t = m$ and hence $m \in I(G)^s$. Now if y_t does not divide $f_1 \cdots f_s$, then $f_1 \cdots f_s$ divides $m'y_\nu$. Thus $m'y_\nu = b_1 f_1 \cdots f_s$. If y_ν divides b_1, then $f_1 \cdots f_s$ divides m' and hence $m \in I(G)^s$. Now suppose y_ν divides, say $f_1 = (x_\delta y_\nu)$. Again, since x_μ and y_ν satisfy the neighborhood properties, any neighbor of y_ν is a neighbor of y_t and hence $m = m'y_t = b_1(x_\delta y_t) f_2 \cdots f_s \in I(G)^s$.

Now suppose that $y_t y_\nu$ divides $f_1 \cdots f_s$. Since $y_t y_\nu$ divides $f_1 \cdots f_s$, we assume, without loss of generality, $f_1 = x_{u_1} y_t$ and $f_2 = (x_{u_2} y_\nu)$. Thus we have

$$\begin{aligned} m'y_t y_\nu &= af_1 f_2 e_3 \cdots e_{s-1} y_{v_1} y_{v_2} \in I(G)^s \\ &= bf_1 \ldots f_s \text{ for } f_1, \ldots, f_s \in I(G), b \in S \end{aligned} \tag{4}$$

Now a neighbor of y_{v_1}, say x_0, divides a, then

$$m = m'y_t = a'(x_0 y_{v_1})(x_{u_1} y_t) e_2 \cdots e_{s-1} \in I(G)^s \text{ where } a = a'x_0$$

Similarly if a neighbor of y_{v_2}, say x_0, divides a, then

$$m = m'y_t = a''(x_0 y_{v_2}) e_1 (x_{u_2} y_t) e_2 \cdots e_{s-1} \in I(G)^s \text{ where } a = a''x_0$$

Now suppose no neighbor of y_{v_1} or y_{v_2} divides a. Consider (4). If y_{v_1} does not divide $f_3 \cdots f_s$, then

$$\begin{aligned} m'y_t y_\nu &= af_1 f_2 e_3 \cdots e_{s-1} y_{v_1} y_{v_2} \\ &= b' y_{v_1} f_1 \ldots f_s = b' y_{v_1} (x_{u_1} y_t)(x_{u_2} y_\nu) f_3 \cdots f_s \\ &= b'(x_{u_1} y_{v_1})(x_{u_2} y_t) f_3 \cdots f_s y_\nu \\ &= b' e_1 (x_{u_2} y_t) f_3 \cdots f_s y_\nu \end{aligned}$$

Deleting y_ν on both sides, we get $m = m'y_t = b'(x_{u_1} y_{v_1})(x_{u_2} y_t) f_3 \cdots f_s \in I(G)^s$. Thus we assume y_{v_1} divides $f_3 \cdots f_s$ and hence assume, without loss of generality $f_3 = (x_{u_3} y_{v_1})$. Now we have

$$\begin{aligned} m'y_t y_\nu &= af_1 f_2 f_3 e_4 \cdots e_{s-1} y_{v_2} y_{v_3} \in I(G)^s \\ &= bf_1 \ldots f_s \text{ for } f_1, \ldots, f_s \in I(G), b \in S \end{aligned} \tag{5}$$

Now if y_{v_2} does not divides $f_4 \cdots f_s$, then

$$\begin{aligned} m'y_t y_v &= af_1 f_2 f_3 e_4 \cdots e_{s-1} y_{v_2} y_{v_3} \\ &= b' y_{v_2} f_1 \ldots f_s = b' y_{v_2} (x_{u_1} y_t)(x_{u_2} y_v) f_3 \cdots f_s \\ &= b'(x_{u_1} y_t)(x_{u_2} y_{v_2}) f_3 \cdots f_s y_v \\ &= b' f_1 e_2 f_3 \cdots f_s y_v \end{aligned}$$

Deleting y_v on both sides we get $m = m'y_t = b' f_1 e_2 f_3 \cdots f_s \in I(G)^s$.

Thus we assume y_{v_2} divide $f_4 \cdots f_s$ and hence assume, without loss of generality, $f_4 = (x_{u_4} y_{v_2})$. We now have

$$\begin{aligned} m'y_t y_v &= af_1 f_2 f_3 f_4 e_5 \cdots e_{s-1} y_{v_3} y_{v_4} \in I(G)^s \\ &= bf_1 \ldots f_s \text{ for } f_1, \ldots, f_s \in I(G), b \in S \end{aligned} \tag{6}$$

We continue in the same fashion and arrive at the j-th decomposition

$$\begin{aligned} m'y_t y_v &= af_1 \cdots f_{2j-1} f_{2j} e_{2j+1} \cdots e_{s-1} y_{v_{2j-1}} y_{v_{2j}} \in I(G)^s \\ &= bf_1 \ldots f_s \text{ for } f_1, \ldots, f_s \in I(G), b \in S \end{aligned} \tag{7}$$

Also $f_{2r-1} = (x_{u_{2r-1}} y_{v_{2r-3}})$ and $f_{2r} = (x_{u_{2r}} y_{v_{2r-2}})$ for $2 \le r \le j$. Now if a neighbor of $y_{v_{2j-1}}$, say x_0, divides a, then

$$\begin{aligned} m = m'y_t = a'(x_0 y_{v_{2j-1}}) f_1 e_2 f_3 e_4 \cdots e_{2j-2} f_{2j-1} e_{2j} e_{2j+1} \cdots e_{s-1} \in I(G)^s \\ \text{where } a = a' x_0 \end{aligned} \tag{8}$$

If a neighbor of $y_{v_{2j}}$, say (x_0), divides a, then

$$\begin{aligned} m = m'y_t = a'(x_0 y_{v_{2j}}) e_1 (x_{u_2} y_t) e_3 f_4 e_5 f_6 \cdots e_{2j-1} f_{2j} e_{2j+1} e_{2j+2} \cdots e_{s-1} \in I(G)^s \\ \text{where } a = a' x_0 \end{aligned} \tag{9}$$

Now suppose no neighbor of $y_{v_{2j-1}}$ or $y_{v_{2j}}$ divides a. Now consider (7). If $y_{v_{2j-1}}$ does not divide $f_{2j+1} \cdots f_s$, then

$$\begin{aligned} m'y_t y_v &= af_1 \cdots f_{2j-1} f_{2j} e_{2j+1} \cdots e_{s-1} y_{v_{2j-1}} y_{v_{2j}} \\ &= b' y_{v_{2j-1}} f_1 \ldots f_s \\ &= b' e_1 (x_{u_2} y_t) e_3 f_4 e_5 f_6 \cdots f_{2j-2} e_{2j-1} f_{2j} f_{2j+2} \cdots f_s y_v \end{aligned} \tag{10}$$

Deleting y_v on both sides we have $m = m'y_t = b' e_1 (x_{u_2} y_t) e_3 f_4 e_5 f_6 \cdots f_{2j-2} e_{2j-1} f_{2j} f_{2j+2} \cdots f_s \in I(G)^s$. Thus we assume $y_{v_{2j-1}}$ divides $f_{2j+1} \cdots f_s$ and hence assume, without loss of generality, $f_{2j+1} = (x_{u_{2j+1}} y_{v_{2j-1}})$. We now have

$$\begin{aligned} m'y_t y_v &= af_1 \cdots f_{2j-1} f_{2j} f_{2j+1} e_{2j+2} \cdots e_{s-1} y_{v_{2j-1}} y_{v_{2j}} \in I(G)^s \\ &= bf_1 \ldots f_s \text{ for } f_1, \ldots, f_s \in I(G), b \in S \end{aligned} \tag{11}$$

Again, if $y_{v_{2j}}$ does not divide $f_{2j+2}\cdots f_s$, then

$$\begin{aligned} m'y_t y_\nu &= af_1\cdots f_{2j-1}f_{2j}f_{2j+1}e_{2j+2}\cdots e_{s-1}y_{v_{2j-1}}y_{v_{2j}} \\ &= b'y_{v_{2j}}f_1\dots f_s \\ &= b'f_1e_2f_3e_4f_5\cdots f_{2j-1}e_{2j}f_{2j+2}f_{2j+2}\cdots f_s y_\nu \end{aligned} \tag{12}$$

Deleting y_ν on both sides, we get $m = m'y_t = b'f_1e_2f_3e_4f_5\cdots f_{2j-1}e_{2j}f_{2j+2}f_{2j+2}\cdots f_s \in I(G)^s$. Thus we assume $y_{v_{2j}}$ divides $f_{2j+2}\cdots f_s$ and hence assume, without loss of generality, $f_{2j+2} = (x_{u_{2j+2}}y_{v_{2j}})$.

Continuing in the same fashion we may reach the final decomposition

$$\begin{aligned} m'y_t y_\nu &= af_1\cdots f_{s-1}y_{v_{s-2}}y_{v_{s-1}} \in I(G)^s \\ &= bf_1\dots f_s \text{ for } f_1,\dots,f_s \in I(G), b\in S \end{aligned} \tag{13}$$

Recall that every stage we make sure that none of the neighbors of the y's appearing in $f_1,\dots,f_{s-1}$ divide a. Thus a neighbor of $y_{v_{s-2}}$ or $y_{v_{s-1}}$ divides a. Now we can use the decomposition in (8) and (9) to show that $m \in I(G)^s$ depending on whether $s-2$ or $s-1$ is odd or even. This concludes the proof of claim of this theorem in $k=1$ case.

Now assume by induction, that if $mx_\mu^l, my_\nu^l \in I(G)^s$ for $1\le l\le k-1$, then $m\in I(G)^s$. Suppose $mx_\mu^k, my_\nu^k \in I(G)^s$. We also assume that $k\le s$. For, if $k>s$, then $mx\mu^s, my_\nu^s \in I(G)^s$ and hence by induction hypothesis, we have $m\in I(G)^s$.

We claim that it is enough to show that $mx_\mu^{k-1}\in I(G)^s$ or $my_\nu^{k-1}\in I(G)^s$. Suppose we show that $mx_\mu^{k-1}\in I(G)^s$. We now have $mx_\mu^{k-1}, my_\nu^k \in I(G)^s$. Thus $mx_\mu^{k-1}y_\nu, my_\nu^{k-1}y_\nu \in I(G)^s$ and hence $(my_\nu)x_\mu^{k-1}, (my_\nu)y_\nu^{k-1}\in I(G)^s$. Since my_ν is a monomial, we use induction hypothesis to conclude that $my_\nu \in I(G)^s$. Thus we now have $mx_\mu^{k-1}, my_\nu \in I(G)^s$. As before, we have $(mx_\mu^{k-2})x_\mu, (mx_\mu^{k-2})y_\nu \in I(G)^s$. Again, since mx_μ^{k-2} is a monomial, we use induction hypothesis to conclude that $mx_\mu^{k-2}\in I(G)^s$. We now have $mx_\mu^{k-2}, my_\nu \in I(G)^s$. We continue the process to get $mx_\mu^2, my_\nu\in I(G)^s$. We still have $(mx_\mu)x_\mu, (mx_\mu)y_\nu \in I(G)^s$. Since mx_μ is a monomial, by induction hypothesis, we get $mx_\mu\in I(G)^s$. We now have $mx_\mu, my_\nu \in I(G)^s$. This is the $k=1$ case. We now use the induction hypothesis to get $m\in I(G)^s$. On the other hand, if we show that $my_\nu^{k-1}\in I(G)^s$, then we can analogously show that $m\in I(G)^s$.

Now we go to the induction step. We have $mx_\mu^k, my_\nu^k\in I(G)^s$. Since $mx_\mu^k \in I(G)^s$ and $mx_\mu^l \notin I(G)^s$ for any $l<k$, we have $m = m'y_{t_1}\cdots y_{t_k}$ where $m'\in I(G)^{s-k}, y_{t_1},\dots,y_{t_k}\in N_G(x_\mu)$ and not all $y_{t_1},\dots,y_{t_k}$ may be distinct. Suppose a neighbor of $y_{t_1},\dots,y_{t_k}$ divides a, then $mx_\mu^{k-1}\in I(G)^s$.

Now suppose no neighbor of $y_{t_1},\dots,y_{t_k}$ divide a. Since $my_\nu^k\in I(G)^s$ we have

$$\begin{aligned} my_\nu^k &= m'y_{t_1}\cdots y_{t_k}y_\nu^k \in I(G)^s \\ &= bf_1\cdots f_s \text{ where } f_1,\dots,f_s\in I(G), b\in S \end{aligned} \tag{14}$$

We observe that m' may be written divisible by many minimal monomial generators of $I(G)^{s-k}$. We can take $m' = ae_1...e_{s-k}$ such that $\frac{m'}{e_1...e_{s-k}}$ has smallest number of x variables in common with $f_1....f_s$.

It is clear that y_ν^k must divide $f_1....f_s$, otherwise $my_\nu^l\in I(G)^s$ for some $l<k$ and hence $my_\nu^{k-1}\in I(G)^s$. Recall the no neighbor of $y_{t_1},\dots,y_{t_k}$ divides a. Thus we can assume that no neighbor of y_ν, divides a as that will make $mx_\mu^{k-1}\in I(G)^s$. So without loss of generality we may assume for $1\le i\le k$, $f_i = x_{u_i}y_\nu$ where for every j, $e_j = x_{u_j}y_{v_j}$.

Now we observe that if any neighbor of y_{v_i} for $1\le i\le k$ divide a then, clearly, $mx_\mu^{k-1}\in I(G)^s$. For, without loss of generality, say $x_0y_{v_1}$ is an edge where x_0 divides a. As $x_{u_1}y_\nu$ is an edge, so is $x_{u_1}y_{t_1}$ (by neighborhood properties). Thus we have $m = (\frac{a}{x_0})(x_0y_{v_1})e_2....e_{s-k}(x_{u_1}y_{t_1})....y_{t_k}\in I(G)^{s-k+1}$. Hence this will force $mx_\mu^{k-1}\in I(G)^s$. So we assume no neighbor of y_{v_i} for $1\le i\le k$ divide a.

As there are s many x variables in $f_1 \cdots f_s$ and $k < s$, some of the x variables of $f_1 \cdots f_s$ divides a. We also have that no neighbor of any y_{t_i} divides a and y_ν^k divides $f_1 \cdots f_s$. Let $f_{k+1} = x_0 y_{v_{k+1}}$ where x_0 divides a and $e_{k+1} = x_{u_{k+1}} y_{v_{k+1}}$. We may write $m' = a' e_1 ... e_k f_{k+1} e_{k+2} .... e_{s-k}$ where $a' = (\frac{a}{x_0} x_{u_{k+1}})$. But this is an expression of m' with a' having less number of x variables in common with $f_1 ... f_s$ than a which is a contradiction. Thus, one of the neighbors of y_{v_i} for some $1 \leq i \leq k$ divides a and hence $m \in I(G)^s$. □

Lemma 1. *Let G be a bipartite graph and suppose that $x_\mu \in V_1$ and $y_\nu \in V_2$ satisfies the neighborhood properties. Now assume $m_1, \ldots, m_k \in S$ are monomials of the same degree such that $(m_1 + \cdots + m_k)(x_\mu - y_\nu) \in I(G)^s$. Further suppose,*

$$m_1 x_\mu = m_2 y_\nu \tag{15}$$

$$m_i x_\mu = m_{i+1} y_\nu \text{ for } 2 \leq i \leq k-1 \tag{16}$$

$$m_1 y_\nu, m_k x_\mu \in I(G)^s \tag{17}$$

Then $m_j \in I(G)^s$ for $1 \leq j \leq k$.

Proof. First, assume that $N_G(y_\nu) = \{x_{\nu_1}, \ldots, x_{\nu_p}\}$. We prove by induction on k. If $k = 1$, then clearly the claim is true by Theorem 7. By induction, assume the claim is true for $(m_1 + \cdots + m_l)(x_\mu - y_\nu) \in I(G)^s$ satisfying (15)–(17) and $l \leq k-1$. Now suppose we have

$$(m_1 + \cdots + m_k)(x_\mu - y_\nu) \in I(G)^s$$

satisfying (15)–(17). We show that $m_1 \in I(G)^s$. This will show that $(m_2 + \cdots + m_k)(x_\mu - y_\nu) \in I(G)^s$ satisfying (15)–(17). Thus by induction hypothesis we have $m_j \in I(G)^s$ for $2 \leq j \leq k$ proving the claim.

From (15), we have $m_1 = m y_\nu$ and $m_2 = m x_\nu$ where $m \in S$, a monomial. From (16), we have $m_3 = \frac{m_2 x_\mu}{y_\nu} = \frac{m x_\mu^2}{y_\nu}$. Subsequently, we show that

$$m_i = \frac{m x_\mu^{i-1}}{y_\nu^{i-2}} \text{ for } 2 \leq i \leq k \tag{18}$$

Since $m_1 y_\nu \in I(G)^s$, we have $m_1 \in I(G)^s$ or $m_1 = a e_1 \cdots e_{s-1} x_{\nu_t}$ for some $t \in \{1, \ldots, p\}$ where $N_G(y_\nu) = \{x_{\nu_1}, \ldots, x_{\nu_p}\}$.

Suppose $m_1 = a e_1 \cdots e_{s-1} x_{\nu_1}$. Since $m_1 = m y_\nu$, y_ν divides a or one of the e_i's. If y_ν divides a, then $m_1 \in I(G)^s$.

Now suppose y_ν divides, say $e_1 = x_{\nu_b} y_\nu$ for some $b \in \{1, \ldots, p\}$. Since $m_1 = m y_\nu$, we have $m = a e_2 \cdots e_{s-1} x_{\nu_1} x_{\nu_b}$. Using this equality in (18), we have

$$m_k = \frac{m x_\mu^{k-1}}{y_\nu^{k-2}} = \frac{a e_2 \cdots e_{s-1} x_{\nu_1} x_{\nu_b} x_\mu^{k-1}}{y_\nu^{k-2}}$$

Since y_ν does not divide a, then y_ν divides some of the $e_1, \ldots, e_{s-k}$ and hence we have $k-2 \leq s-2$ or $k \leq s$. Without loss of generality, assume y_ν divides $e_2, \ldots, e_{k-1}$. Thus

$$m_k = a e_k \cdots e_{s-1} x_{\nu_1}^{l_1} \cdots x_{\nu_p}^{l_p} x_\mu^{k-1} \text{ where } \sum_{j=1}^{p} l_j = k$$

Let $u = a e_k \cdots e_{s-1} x_{\nu_1}^{l_1} \cdots x_{\nu_p}^{l_p}$. Now as $m_k x_\mu \in I(G)^s$, we have $u x_\mu^k \in I(G)^s$. Also, notice that

$$u y_\nu^k = \frac{m_k}{x_\mu^{k-1}} y_\nu^k = \frac{m}{y_\nu^{k-2}} y_\nu^k = m y_\nu^2 = m_1 y_\nu \in I(G)^s$$

Since u is a monomial, we have $u \in I(G)^s$, by Theorem 7. Now $m_1 = uy_\nu^{k-1} \in I(G)^s$ and hence we are done. □

We now prove one of the main results of this section. In this theorem, we attempt to rearrange the sum $m_1 + \cdots + m_k$ into a configuration shown in the previous lemma.

Theorem 8. *Let G be a bipartite graph and suppose that $x_\mu \in V_1$ and $y_\nu \in V_2$ satisfies the neighborhood properties. Then $x_\mu - y_\nu$ is an regular element on $S/I(G)^s$ for all s.*

Proof. Consider $(m_1 + \cdots + m_k)(x_\mu - y_\nu) \in I(G)^s$ where m_i's are monomials of the same degree. We prove $m_1, \ldots, m_k \in I(G)^s$ by induction on k.

Suppose $k = 1$ and $m_1(x_\mu - y_\nu) = m_1x_\mu - m_1y_\nu \in I(G)^s$. Thus $m_1x_\mu, m_1y_\nu \in I(G)^s$. Now we use Theorem 7, to show that $m_1 \in I(G)^s$ proving the base case of induction.

Suppose $(m_1 + \cdots + m_l)(x_\mu - y_\nu) \in I(G)^s$ for $l \leq k-1$ implies $m_1, \ldots, m_l \in I(G)^s$. Now consider

$$(m_1 + \cdots + m_k)(x_\mu - y_\nu) \;=\; m_1x_\mu - m_1y_\nu + m_2x_\mu - m_2y_\nu + \cdots + m_kx_\mu - m_ky_\nu \;\in\; I(G)^s. \quad (19)$$

where all m_i's are distinct. We show $m_i \in I(G)^s$ for $1 \leq i \leq k$.

Observe that if $m_1x_\mu, m_1y_\nu \in I(G)^s$, then we have $m_1(x_\mu - y_\nu) \in I(G)^s$ and $(m_2 + \cdots + m_k)(x_\mu - y_\nu) \in I(G)^s$. Now we use induction hypothesis to show that $m_i \in I(G)^s$ for $1 \leq i \leq k$.

Now we first consider the following configuration, i.e., after possible re-ordering of m_i's we have

$$m_1x_\mu = m_2y_\nu \quad (20)$$

$$m_ix_\mu = m_{i+1}y_\nu, \text{ for } 2 \leq i \leq k-1 \quad (21)$$

$$m_kx_\mu = m_1y_\nu \quad (22)$$

We refer to this case as the *k-cancellation* case. Using (20), we get $m_1 = my_\nu$ and $m_2 = mx_\mu$. Using this and (21), we get

$$m_i = \frac{mx_\mu^{i-1}}{y_\nu^{i-2}} \text{ for } 3 \leq i \leq k \quad (23)$$

Thus $m_k = \frac{mx_\mu^{k-1}}{y_\nu^{k-2}}$. Using this description in (22) we get $x_\mu^k = y_\nu^k$, a contradiction.

Now consider (19). Without loss of generality, after possible reordering, assume that $m_1x_\mu = m_2y_\nu$. If $m_1y_\nu = m_2x_\mu$, then we get $(m_1 + m_2)(x_\mu - y_\nu) \in I(G)^s$ and $(m_3 + \cdots + m_k)(x_\mu - y_\nu) \in I(G)^s$. Now using induction hypothesis, we get $m_i \in I(G)^s$.

Suppose, if $m_1y_\nu = m_3x_\mu$ we introduce the re-ordering

$$m_1^{(1)} = m_3, m_2^{(1)} = m_1, m_3^{(1)} = m_2$$
$$m_i^{(1)} = m_i \text{ for } 4 \leq i \leq k-1$$

Notice that $(m_1 + \cdots + m_k)(x_\mu - y_\nu) = \left(m_1^{(1)} + \cdots + m_k^{(1)}\right)(x_\mu - y_\nu)$. Thus it is enough to show that $m_i^{(1)} \in I(G)^s$. Under this re-ordering $m_1^{(1)}x_\mu = m_2^{(1)}y_\nu$ and $m_2^{(1)}x_\mu = m_3^{(1)}y_\nu$. If $m_1^{(1)}y_\nu = m_3^{(1)}x_\mu$, then we get $(m_1^{(1)} + m_2^{(1)} + m_3^{(1)})(x_\mu - y_\nu) \in I(G)^s$ and $(m_4^{(1)} + \cdots + m_k^{(1)})(x_\mu - y_\nu) \in I(G)^s$. Now using induction hypothesis, we get $m_i^{(1)} \in I(G)^s$ and hence $m_i \in I(G)^s$.

Now if $m_1^{(1)} y_\nu = m_4^{(1)} x_\mu$, we introduce a new ordering

$$\begin{aligned} m_1^{(2)} &= m_4^{(1)} \\ m_l^{(2)} &= m_{l-1}^{(1)} \text{ for } 2 \le l \le 4 \\ m_q^{(2)} &= m_q^{(1)} \text{ for } 5 \le q \le k \end{aligned}$$

As before we consider if $m_1^{(2)} y_\nu = m_4^{(2)} x_\mu$ or $m_1^{(2)} y_\nu = m_5^{(2)} x_\mu$ and introduce new ordering, if necessary.

We now continue this process and arrive at the j-th re-ordering defined as follows

$$\begin{aligned} m_1^{(j)} &= m_{j+2}^{(j-1)} \\ m_l^{(j)} &= m_{l-1}^{(j-1)} \text{ for } 2 \le l \le j+2 \\ m_q^{(j)} &= m_q^{(j-1)} \text{ for } j+3 \le q \le k \end{aligned}$$

with the following configuration

$$m_i^{(j)} x_\mu = m_{i+1}^{(j)} y_\nu \text{ for } 1 \le i \le j+1$$

First, suppose $j = k-2$. As before, we consider two cases $m_1^{(j)} y_\nu = m_{j+2}^{(j)} x_\mu$ or $m_1^{(j)} y_\nu \neq m_{j+2}^{(j)} x_\mu$. If $m_1^{(j)} y_\nu = m_{j+2}^{(j)} x_\mu$, then we arrive at the k-cancellation case discussed above, which leads to a contradiction. So we have $m_1^{(j)} y_\nu \neq m_{j+2}^{(j)} x_\mu$ which is discussed separately in Lemma 1, showing that $m_i \in I(G)^s$.

Now we assume $j < k-2$ and $m_1^{(j)} y_\nu \neq m_t^{(j)} x_\mu$ for $2 \le t \le k$. If $m_{j+2}^{(j)} x_\mu \neq m_t^{(j)} y_\nu$ for $j+3 \le t \le k$, then we have $(m_1^{(j)} + \cdots + m_{j+2}^{(j)})(x_\mu - y_\nu) \in I(G)^s$ and $(m_{j+2}^{(j)} + \cdots + m_k^{(j)})(x_\mu - y_\nu) \in I(G)^s$ and we use induction hypothesis to conclude that $m_i^{(j)} \in I(G)^s$ and hence $m_i \in I(G)^s$ for $1 \le i \le k$.

Thus assume $m_{j+2}^{(j)} x_\mu = m_t^{(j)} y_\nu$ for some $j+3 \le t \le k$. Now we use the ordering

$$\begin{aligned} m_{j+3}^{(j,1)} &= m_t^{(j)}, m_t^{(j,1)} = m_{j+3}^{(j)} \\ m_i^{(j,1)} &= m_i^{(j)} \text{ for } i \neq j+3, t \end{aligned}$$

with the configuration $m_i^{(j,1)} x_\mu = m_{i+1}^{(j,1)} y_\nu$ for $1 \le i \le j+2$.

Now if $m_{j+3}^{(j,1)} x_\mu \neq m_a^{(j,1)} y_\nu$ for $j+4 \le a \le k$, then $(m_1^{(j,1)} + \cdots + m_{j+3}^{(j,1)})(x_\mu - y_\nu) \in I(G)^s$ and $(m_{j+4}^{(j,1)} + \cdots + m_k^{(j,1)})(x_\mu - y_\nu) \in I(G)^s$ and we use induction hypothesis to conclude that $m_i^{(j,1)} \in I(G)^s$ and hence $m_i \in I(G)^s$ for $1 \le i \le k$.

Now if $m_{j+3}^{(j,1)} x_\mu = m_a^{(j,1)} y_\nu$ for some $j+4 \le a \le k$, then we use the ordering as before

$$\begin{aligned} m_{j+4}^{(j,2)} &= m_a^{(j,1)}, m_a^{(j,2)} = m_{j+4}^{(j,1)} \\ m_i^{(j,2)} &= m_i^{(j,1)} \text{ for } i \neq j+4, a \end{aligned}$$

with the configuration $m_i^{(j,2)} x_\mu = m_{i+1}^{(j,2)} y_\nu$ for $1 \le i \le j+3$.

We continue in the same fashion to reach (j,l)-th re-ordering to get

$$(m_1^{(j,l)} + \cdots m_k^{(j,l)})(x_\mu - y_\nu) \in I(G)^s$$

with the following configuration

$$m_i^{(j,l)} x_\mu = m_{i+1}^{(j,l)} y_\nu \text{ for } 1 \leq i \leq j+l+1$$

Suppose $j+l = k-2$, then $m_1^{(j,l)} y_\nu, m_k^{(j,l)} x_\mu \in I(G)^s$. Now using Lemma 1 we have $m_i^{(j,l)} \in I(G)^s, 1 \leq i \leq k$ and hence $m_i \in I(G)^s$ for $1 \leq i \leq k$.

If $j+l < k-2$, then there exists a term $m_b^{(j,l)}$ such that $m_b^{(j,l)}(x_\mu - y_\nu) \in I(G)^s$ and $\left(\sum_{t \neq b} m_t^{(j,l)}\right)(x_\mu - y_\nu) \in I(G)^s$ and hence we are done by induction. □

Corollary 1. *Let G be a bipartite graph. Suppose $x_\mu \in V_1, y_\nu \in V_2$. Then x_μ and y_ν satisfies the neighborhood properties, if and only if $x_\mu - y_\nu$ is regular on $S/I(G)^s$ for all s.*

Proof. Suppose x_μ and y_ν satisfies the neighborhood properties, then $x_\mu - y_\nu$ is regular on $S/I(G)^s$ for all s by Theorem 8.

Now if x_μ and y_ν does not satisfy the neighborhood properties, then there exists y_p such that $x_\mu y_p \in E(G)$ and $x_{\nu 1} y_p \notin E(G)$ where $x_{\nu 1} \in N(y_\nu)$. Thus for all s and $e = x_{\nu 1} y_{\nu 1} \in I(G)$,

$$\begin{aligned} e^{s-1}(x_{\nu 1} y_p)(x_\mu - y_\nu) &= e^{s-1}((x_{\nu 1} y_p) x_\mu - (x_{\nu 1} y_p) y_\nu) \\ &= e^{s-1}(x_{\nu 1}(y_p x_\mu) - (x_{\nu 1} y_\nu) y_p) \end{aligned}$$

Since $y_p x_\mu, x_{\nu 1} y_\nu \in I(G)$, we get $e^{s-1}(x_{\nu 1} y_p)(x_\mu - y_\nu) \in I(G)^s$. Thus $x_\mu - y_\nu$ is not a regular element on $I(G)^s$. □

Author Contributions: Both the authors contributed equally to the creation of this article.

Funding: The first author was partially supported by DST INSPIRE (India) research grant (DST/INSPIRE/04/2017/000752) and he would like to acknowledge that.

Acknowledgments: The first author is very thankful to Professor C. Huneke for his valuable suggestions especially regarding the proof of Theorem 5.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Villarreal, R.H. Unmixed bipartite graphs. *Rev. Colomb. Mat.* **2007**, *41*, 393–395.
2. Herzog, J.; Hibi, T. Distributive lattices, bipartite graphs and Alexander duality. *J. Algebr. Combin.* **2005**, *22*, 289–302. [CrossRef]
3. Banerjee, A.; Mukundan, V. On the Powers of Unmixed Bipartite Edge Ideals. *J. Algebra Appl.* **2018**. [CrossRef]
4. Banerjee, A. The regularity of powers of edge ideals. *J. Algebr. Combin.* **2015**, *41*, 303–321. [CrossRef]
5. Herzog, J.; Hibi, T. Monomial ideals. In *Graduate Texts in Mathematics*; Springer: London, UK, 2011; Volume 260, p. xvi+305.
6. Hartshorne, R. Complete intersections and connectedness. *Amer. J. Math.* **1962**, *84*, 497–508. [CrossRef]
7. Bruns, W.; Herzog, J. Cohen-Macaulay rings. In *Cambridge Studies in Advanced Mathematics*; Cambridge University Press: Cambridge, UK, 1993; Volume 39, p. xii+403.
8. Morey, S.; Reyes, E.; Villarreal, R.H. Cohen-Macaulay, shellable and unmixed clutters with a perfect matching of König type. *J. Pure Appl. Algebra* **2008**, *212*, 1770–1786. [CrossRef]
9. Zaare-Nahandi, R. Cohen-Macaulayness of bipartite graphs, revisited. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 1601–1607. [CrossRef]