Ostrowski Type Inequalities Involving ψ-Hilfer Fractional Integrals

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Abstract: In this study we introduce several new Ostrowski-type inequalities for both left and right sided fractional integrals of a function g with respect to another function ψ. Our results generalized the ones presented previously by Farid. Furthermore, two illustrative examples are presented to support our results.

Keywords: fractional calculus; fractional integrals; Ostrowski type inequality

MSC: 26A24; 26A33; 26B15

1. Introduction and Preliminaries

Since 1695 [1–3], fractional calculus has been studied by many researchers from both theoretical and applied viewpoints [4,5]. Particularly, fractional calculus is used to generalize classical inequalities. Studies involving integral inequalities are important in several areas such as mathematics, physics, chemistry, biology, engineering and others [6–15]. We recall that there are many definitions of fractional operators, including Riemann–Liouville (RL), Hadamard, Liouville, Weyl (see [16–19]). From such fractional integrals, one can obtain generalizations of the inequalities: Hadamard, Hermite–Hadamard, Hardy, Opial, Gruss, and Montgomery, among others [20–32].

We mention that the following inequality was developed by Ostrowski [33]:

Theorem 1. Let g : I → R be a mapping differentiable in I° such that I is an interval in R, I° is the interior of I and a1, b1 ∈ I°, a1 < b1. If |g′(ξ)| ≤ M for all ξ ∈ [a1, b1], then the integral inequality holds

\[\left| g(x) - \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} g(\xi) \, d\xi \right| \leq \frac{1}{4} + \left( x - \frac{a_1 + b_1}{2} \right)^2 \frac{M}{(b_1 - a_1)^2} (b_1 - a_1), \tag{1} \]

for all x ∈ [a1, b1].

In the literature, the inequality (1) is called the Ostrowski inequality, see [34]. This inequality has a great importance while studying the error bounds of different numerical quadrature rules. In recent years, such inequalities have been generalized and developed by many researchers. Various authors obtained new Ostrowski-type inequalities for different fractional operators, see [16–19,35–47] and the references therein.

In 2009, Anastassiou et al. [20] obtained Montgomery identities for fractional integrals and a generalization for double fractional integrals. For fractional integrals they discussed both Ostrowski
Theorem 2. Assume that the conditions of the Theorem 1 are satisfied. Also, suppose that the function $\psi \in C^1(I)$ is increasing and positive monotone, and $\psi'(x) \geq 1$ ($\forall x \in I$). Let $I_{a_1}^{\alpha_1;\psi} g(x)$ and $I_{b_1}^{\alpha_1;\psi} g(x)$ be defined as (2) and (3), respectively. Then the following inequality holds:

$$
\left| \left( (\psi(b_1) - \psi(x))^\beta_1 + (\psi(x) - \psi(a_1))^\alpha_1 \right) g(x) \right|
$$

and Grüss inequalities. In 2010, Alomari and Darus [36] presented some Ostrowski-type inequalities for the class of convex (or concave) functions. In 2012, Set [41] obtained some new fractional Ostrowski-type inequalities. In the same year, Liu [40] established some Ostrowski-type inequalities involving RL fractional integrals for the h-convex function. His results are generalizations of [41,42]. He also provided new estimates on Ostrowski-type inequalities for fractional integrals.

In 2013, Yue [38] obtained Ostrowski inequalities for both fractional integrals and associated fractional integrals. In 2014, Aljinević [16] studied Montgomery identities for fractional integrals of a function $\psi$ with respect to another function $g$. Also, he gave the Ostrowski inequality for fractional integrals for functions whose first derivatives belong to $L_p$ spaces. In the same year, Yıldız and Kirtay [46] established new generalizations for Ostrowski inequalities by using the generalized RL fractional integral.

Yıldız et al. [47] used the RL fractional integrals to obtain several new generalizations of Ostrowski type inequalities. Farid [35] found a new version of Ostrowski type inequalities in a very simple way for RL fractional integrals. In 2014, Aljinević [16] studied Montgomery identities for fractional integrals of a function $\psi$ with respect to another function $g$. Also, he gave the Ostrowski inequality for fractional integrals for functions whose first derivatives belong to $L_p$ spaces. In the same year, Yıldız et al. [47] used the RL fractional integrals to obtain several new generalizations of Ostrowski inequalities. Farid [35] found a new version of Ostrowski type inequalities in a very simple way for RL fractional integrals. In 2014, Aljinević [16] studied Montgomery identities for fractional integrals of a function $\psi$ with respect to another function $g$. Also, he gave the Ostrowski inequality for fractional integrals for functions whose first derivatives belong to $L_p$ spaces.

**Section 2. Main Results**

Below, we will show several new Ostrowski-type inequalities for both left and right sided fractional integrals of a function $g$ with respect to another function $\psi$.

**Theorem 2.** Assume that the conditions of the Theorem 1 are satisfied. Also, suppose that the function $\psi \in C^1(I)$ is increasing and positive monotone, and $\psi'(x) \geq 1$ ($\forall x \in I$). Let $I_{a_1}^{\alpha_1;\psi} g(x)$ and $I_{b_1}^{\alpha_1;\psi} g(x)$ be defined as (2) and (3), respectively. Then the following inequality holds:

$$
\left| \left( (\psi(b_1) - \psi(x))^\beta_1 + (\psi(x) - \psi(a_1))^\alpha_1 \right) g(x) \right|
$$
\[
- \left( \Gamma (\beta_1 + 1) I_{\beta_1}^{\alpha_1; \psi} g(x) + \Gamma (\alpha_1 + 1) I_{\alpha_1}^{\psi} g(x) \right) \\
\leq M \left( \frac{\beta_1}{\beta_1 + 1} (\psi (b_1) - \psi (x))^{\beta_1 + 1} + \frac{\alpha_1}{\alpha_1 + 1} (\psi (x) - \psi (a_1))^{\alpha_1 + 1} \right),
\]
where \( \alpha_1, \beta_1 > 0 \) and \( x \in [a_1, b_1] \).

**Proof of Theorem 2.** Taking into account that \( \psi \) is an increasing and positive monotone function, for \( \alpha_1 > 0 \) and \( \zeta \in [a_1, x] \) we get
\[
(\psi (x) - \psi (\zeta))^{\alpha_1} \leq (\psi (x) - \psi (a_1))^{\alpha_1}.
\]
Utilizing (5) and the given condition on \( g' \), we obtain
\[
\int_{a_1}^{x} (M \psi' (\zeta) - g' (\zeta)) (\psi (x) - \psi (\zeta))^{\alpha_1} d\zeta \leq (\psi (x) - \psi (a_1))^{\alpha_1} \int_{a_1}^{x} (M \psi' (\zeta) - g' (\zeta)) d\zeta
\]
and
\[
\int_{a_1}^{x} (M \psi' (\zeta) + g' (\zeta)) (\psi (x) - \psi (\zeta))^{\alpha_1} d\zeta \leq (\psi (x) - \psi (a_1))^{\alpha_1} \int_{a_1}^{x} (M \psi' (\zeta) + g' (\zeta)) d\zeta.
\]
If the above integrals are calculated, we obtain the following inequalities, respectively:
\[
(\psi (x) - \psi (a_1))^{\alpha_1} g(x) - \Gamma (\alpha_1 + 1) I_{\alpha_1}^{\psi} g(x) \leq \frac{M \alpha_1}{\alpha_1 + 1} (\psi (x) - \psi (a_1))^{\alpha_1 + 1}
\]
and
\[
\Gamma (\alpha_1 + 1) I_{\alpha_1}^{\psi} g(x) - (\psi (x) - \psi (a_1))^{\alpha_1} g(x) \leq \frac{M \alpha_1}{\alpha_1 + 1} (\psi (x) - \psi (a_1))^{\alpha_1 + 1}.
\]
By using (6) and (7), we report the following inequality:
\[
\left| (\psi (x) - \psi (a_1))^{\alpha_1} g(x) - \Gamma (\alpha_1 + 1) I_{\alpha_1}^{\psi} g(x) \right| \leq \frac{M \alpha_1}{\alpha_1 + 1} (\psi (x) - \psi (a_1))^{\alpha_1 + 1}.
\]
On the other hand, if \( \psi \) is an increasing and positive function, for \( \zeta \in [x, b_1] \) and \( \beta_1 > 0 \) we get
\[
(\psi (\zeta) - \psi (x))^{\beta_1} \leq (\psi (b_1) - \psi (x))^{\beta_1}.
\]
By using (9) and the given condition on \( g' \), we conclude
\[
\int_{x}^{b_1} (M \psi' (\zeta) - g' (\zeta)) (\psi (x) - \psi (\zeta))^{\beta_1} d\zeta \leq (\psi (b_1) - \psi (x))^{\beta_1} \int_{x}^{b_1} (M \psi' (\zeta) - g' (\zeta)) d\zeta
\]
and
\[
\int_{x}^{b_1} (M \psi' (\zeta) + g' (\zeta)) (\psi (x) - \psi (\zeta))^{\beta_1} d\zeta \leq (\psi (b_1) - \psi (x))^{\beta_1} \int_{x}^{b_1} (M \psi' (\zeta) + g' (\zeta)) d\zeta.
\]
If the above integrals are calculated, we obtain the following inequalities, respectively:
\[
\Gamma (\beta_1 + 1) I_{\beta_1}^{\psi} g(x) - (\psi (b_1) - \psi (x))^{\beta_1} g(x) \leq \frac{M \beta_1}{\beta_1 + 1} (\psi (b_1) - \psi (x))^{\beta_1 + 1}
\]
and
\[
(\psi (b_1) - \psi (x))^{\beta_1} g(x) - \Gamma (\beta_1 + 1) I_{\beta_1}^{\psi} g(x) \leq \frac{M \beta_1}{\beta_1 + 1} (\psi (b_1) - \psi (x))^{\beta_1 + 1}.
\]
By using (10) and (11), the following inequality will appear:

\[
\left| (\psi(b_1) - \psi(x))^{\beta_1} g(x) - \Gamma(\beta_1 + 1) I_{b_1}^{\beta_1;\psi} g(x) \right| \leq \frac{M\beta_1}{\beta_1 + 1} (\psi(b_1) - \psi(x))^{\beta_1 + 1}.
\]

(12)

So, by utilizing (8) and (12), we obtain (4). \(\square\)

**Theorem 3.** Let \( g : I \to \mathbb{R} \) be a mapping differentiable in \( I^o \) (the interior of \( I \)) such that \( I \) is an interval in \( \mathbb{R} \) and \( a_1, b_1 \in I^o \), \( a_1 < b_1 \). Assume that the function \( \psi \in \mathcal{C}^1(I) \) is increasing and positive monotone, and \( \psi'(x) \geq 1 \) (\( \forall x \in I \)). Also, let \( I_{a_1}^{\alpha_1;\psi} \) and \( I_{b_1}^{\beta_1;\psi} \) be defined as (2) and (3), respectively. If \( m \leq g'(t) \leq M \) for \( M \geq 0, m \leq 0 \) and all \( x \in [a_1, b_1] \), then the following inequalities hold:

\[
(\psi(x) - \psi(a_1))^{\alpha_1} - (\psi(x) - \psi(a_1)) \leq \left( \Gamma(\alpha_1 + 1) I_{a_1}^{\alpha_1;\psi} g(x) - \Gamma(\beta_1 + 1) I_{b_1}^{\beta_1;\psi} g(x) \right)
\]

\[
\leq M \left( \frac{\alpha_1}{\alpha_1 + 1} (\psi(a_1) - \psi(x))^{\alpha_1 + 1} + \frac{\beta_1}{\beta_1 + 1} (\psi(b_1) - \psi(x))^{\beta_1 + 1} \right)
\]

(13)

and

\[
(\psi(x) - \psi(a_1))^{\beta_1} - (\psi(x) - \psi(a_1)) \geq \left( \Gamma(\alpha_1 + 1) I_{a_1}^{\alpha_1;\psi} g(x) - \Gamma(\beta_1 + 1) I_{b_1}^{\beta_1;\psi} g(x) \right)
\]

\[
\leq -m \left( \frac{\beta_1}{\beta_1 + 1} (\psi(b_1) - \psi(x))^{\beta_1 + 1} + \frac{\alpha_1}{\alpha_1 + 1} (\psi(a_1) - \psi(x))^{\alpha_1 + 1} \right),
\]

(14)

where \( \alpha_1, \beta_1 > 0 \) and \( x \in [a_1, b_1] \).

**Proof of Theorem 3.** Using the given comparing conditions on \( g' \), the proof is similar to one of Theorem 2. That is, from (5) and by using the given condition on \( g' \), we conclude

\[
\int_{a_1}^{x} (M\psi'(\xi) - g'(\xi)) (\psi(x) - \psi(\xi))^{\alpha_1} d\xi \leq (\psi(x) - \psi(a_1))^{\alpha_1} \int_{a_1}^{x} (M\psi'(\xi) - g'(\xi)) d\xi
\]

and

\[
\int_{a_1}^{x} g'(\xi) - m\psi'(\xi) (\psi(x) - \psi(\xi))^{\alpha_1} d\xi \leq (\psi(x) - \psi(a_1))^{\alpha_1} \int_{a_1}^{x} (g'(\xi) - m\psi'(\xi)) d\xi.
\]

If the above integrals are calculated, we obtain the following inequalities, namely:

\[
(\psi(x) - \psi(a_1))^{\alpha_1} g(x) - \Gamma(\alpha_1 + 1) I_{a_1}^{\alpha_1;\psi} g(x) \leq \frac{M\alpha_1}{\alpha_1 + 1} (\psi(x) - \psi(a_1))^{\alpha_1 + 1}
\]

(15)

and

\[
\Gamma(\alpha_1 + 1) I_{a_1}^{\alpha_1;\psi} g(x) - (\psi(x) - \psi(a_1))^{\alpha_1} g(x) \leq -\frac{m\alpha_1}{\alpha_1 + 1} (\psi(x) - \psi(a_1))^{\alpha_1 + 1}.
\]

(16)

On the other hand, by using (9) and the given condition on \( g' \), we have

\[
\int_{x}^{b_1} (M\psi'(\xi) - g'(\xi)) (\psi(\xi) - \psi(x))^{\beta_1} d\xi \leq (\psi(b_1) - \psi(x))^{\beta_1} \int_{x}^{b_1} (M\psi'(\xi) - g'(\xi)) d\xi
\]
and
\[ \int_x^{b_1} (g'(\xi) - m\psi'(\xi)) \left( \psi(\xi) - \psi(x) \right)^{\beta_1} d\xi \leq \left( \psi(b_1) - \psi(x) \right)^{\beta_1} \int_x^{b_1} (g'(\xi) - m\psi'(\xi)) d\xi. \]

If the above integrals are calculated, we obtain the following inequalities, namely:
\[ \Gamma(\beta_1 + 1) \int_{a_1}^{b_1} g(x) - (\psi(b_1) - \psi(x))^{\beta_1} g(x) \leq \frac{M\beta_1}{\beta_1 + 1} \left( \psi(b_1) - \psi(x) \right)^{\beta_1 + 1} \] (17)
and
\[ (\psi(b_1) - \psi(x))^{\beta_1} g(x) - \Gamma(\beta_1 + 1) \int_{a_1}^{b_1} g(x) \leq \frac{m\beta_1}{\beta_1 + 1} \left( \psi(b_1) - \psi(x) \right)^{\beta_1 + 1}, \] (18)
respectively. By using (15) and (17), we obtain (13). In addition, by using (16) and (18), we provide (14).

**Theorem 4.** Let \( g : I \to \mathbb{R} \) be a mapping differentiable in \( I^0 \) (the interior of I) such that I is an interval in \( \mathbb{R} \) and \( a_1, b_1 \in I^0, a_1 < b_1 \). Assume that the function \( \psi \in C^1(I) \) is increasing and positive monotone, and \( \psi'(x) \geq 1 \) (\( \forall x \in I \)). Also, let \( t^{\alpha_1;\beta_1}_{a_1} \) and \( t^{\beta_1;\psi}_{b_1} \) be defined as (2) and (3), respectively. If \( m \leq g'(t) \leq M \) for \( M \geq 0, m \leq 0 \) and all \( \xi \in [a_1, b_1] \), then the following inequalities hold:
\[
\left( (\psi(x) - \psi(a_1))^{\alpha_1} + (\psi(b_1) - \psi(x))^{\beta_1} \right) g(x)
- \left( \Gamma(\alpha_1 + 1) \int_{a_1}^{\alpha_1;\beta_1} g(x) + \Gamma(\beta_1 + 1) \int_{b_1}^{\beta_1;\psi} g(x) \right)
\leq \frac{M\alpha_1}{\alpha_1 + 1} (\psi(x) - \psi(a_1))^{\alpha_1 + 1} - \frac{m\beta_1}{\beta_1 + 1} (\psi(b_1) - \psi(x))^{\beta_1 + 1},
\] (19)
and
\[
- \left( (\psi(b_1) - \psi(x))^{\beta_1} + (\psi(x) - \psi(a_1))^{\alpha_1} \right) g(x)
+ \left( \Gamma(\alpha_1 + 1) \int_{a_1}^{\alpha_1;\beta_1} g(x) + \Gamma(\beta_1 + 1) \int_{b_1}^{\beta_1;\psi} g(x) \right)
\leq \frac{M\beta_1}{\beta_1 + 1} (\psi(b_1) - \psi(x))^{\beta_1 + 1} - \frac{m\alpha_1}{\alpha_1 + 1} (\psi(x) - \psi(a_1))^{\alpha_1 + 1},
\] (20)

where \( \alpha_1, \beta_1 > 0 \) and \( x \in [a_1, b_1] \).

**Proof of Theorem 4.** Proof is constructed in the same line as the proof of Theorem 3. By using (15) and (18), we obtain (19). In addition, from (16) and (17), we get (20). \( \square \)

**Theorem 5.** Suppose that the conditions of the Theorem 2 are satisfied. Also, assume that the function \( \psi \in C^1(I) \) is increasing and positive monotone, and \( \psi'(x) \geq 1 \) (\( \forall x \in I \)). Let \( t^{\alpha_1;\beta_1}_{a_1} \) and \( t^{\beta_1;\psi}_{b_1} \) be defined as (2) and (3), respectively. Then the following inequality holds:
\[
\left| \left( (\psi(b_1) - \psi(x))^{\beta_1} g(b_1) + (\psi(x) - \psi(a_1))^{\alpha_1} g(a_1) \right) \right|
- \left( \Gamma(\beta_1 + 1) \int_{x=\beta_1}^{\alpha_1;\beta_1} g(b_1) + \Gamma(\alpha_1 + 1) \int_{x=\alpha_1;\psi} g(a_1) \right)
\leq M \left( \frac{\beta_1}{\beta_1 + 1} (\psi(b_1) - \psi(x))^{\beta_1 + 1} + \frac{\alpha_1}{\alpha_1 + 1} (\psi(x) - \psi(a_1))^{\alpha_1 + 1} \right),
\] (21)

where \( \alpha_1, \beta_1 > 0 \) and \( x \in [a_1, b_1] \).
Proof of Theorem 5. Recalling \( \psi \) is an increasing and positive monotone function, for \( x_1 > 0 \) and \( \xi \in [a_1, x] \) we obtain
\[
(\psi (\xi) - \psi (a_1))^{a_1} \leq (\psi (x) - \psi (a_1))^{a_1}.
\] (22)

By using (22) and the given condition on \( g' \), we have
\[
\int_{a_1}^{x} (M\psi' (\xi) - g' (\xi))(\psi (\xi) - \psi (a_1))^{a_1} d\xi \leq (\psi (x) - \psi (a_1))^{a_1} \int_{a_1}^{x} (M\psi' (\xi) - g' (\xi)) d\xi
\]
and
\[
\int_{a_1}^{x} (M\psi' (\xi) + g' (\xi))(\psi (\xi) - \psi (a_1))^{a_1} d\xi \leq (\psi (x) - \psi (a_1))^{a_1} \int_{a_1}^{x} (M\psi' (\xi) + g' (\xi)) d\xi.
\]

If the above integrals are calculated, we obtain the following inequalities, respectively:
\[
\Gamma (a_1 + 1) \frac{\alpha_1}{x^\alpha} \bar{g} (a_1) - (\psi (x) - \psi (a_1))^{a_1} g (a_1) \leq \frac{M \alpha_1}{a_1 + 1} (\psi (x) - \psi (a_1))^{a_1 + 1}
\] (23)
and
\[
(\psi (x) - \psi (a_1))^{a_1} g (a_1) - \Gamma (a_1 + 1) \frac{\alpha_1}{x^\alpha} \bar{g} (a_1) \leq \frac{M \alpha_1}{a_1 + 1} (\psi (x) - \psi (a_1))^{a_1 + 1}.
\] (24)

By utilizing (23) and (24), the following inequality holds:
\[
\left| (\psi (x) - \psi (a_1))^{a_1} g (a_1) - \Gamma (a_1 + 1) \frac{\alpha_1}{x^\alpha} \bar{g} (a_1) \right| \leq \frac{M \alpha_1}{a_1 + 1} (\psi (x) - \psi (a_1))^{a_1 + 1}.
\] (25)

Using the fact that \( \psi \) is an increasing and positive monotone function, for \( \xi \in [x, b_1] \) and \( \beta_1 > 0 \) we get
\[
(\psi (b_1) - \psi (\xi))^{\beta_1} \leq (\psi (b_1) - \psi (x))^{\beta_1}.
\] (26)

By using (26) and the given condition on \( g' \), we have
\[
\int_{x}^{b_1} (M\psi' (\xi) - g' (\xi))(\psi (b_1) - \psi (\xi))^{\beta_1} d\xi \leq (\psi (b_1) - \psi (x))^{\beta_1} \int_{x}^{b_1} (M\psi' (\xi) - g' (\xi)) d\xi
\]
and
\[
\int_{x}^{b_1} (M\psi' (\xi) + g' (\xi))(\psi (b_1) - \psi (\xi))^{\beta_1} d\xi \leq (\psi (b_1) - \psi (x))^{\beta_1} \int_{x}^{b_1} (M\psi' (\xi) + g' (\xi)) d\xi.
\]

If the above integrals are calculated, we obtain the following inequalities, respectively:
\[
(\psi (b_1) - \psi (x))^{\beta_1} g (b_1) - \Gamma (\beta_1 + 1) \frac{\beta_1}{x^\beta} \bar{g} (b_1) \leq \frac{M \beta_1}{\beta_1 + 1} (\psi (b_1) - \psi (x))^{\beta_1 + 1}
\] (27)
and
\[
\Gamma (\beta_1 + 1) \frac{\beta_1}{x^\beta} \bar{g} (b_1) - (\psi (b_1) - \psi (x))^{\beta_1} g (b_1) \leq \frac{M \beta_1}{\beta_1 + 1} (\psi (b_1) - \psi (x))^{\beta_1 + 1}.
\] (28)

By making use of (27) and (28), the following inequality holds:
\[
| (\psi (b_1) - \psi (x))^{\beta_1} g (b_1) - \Gamma (\beta_1 + 1) \frac{\beta_1}{x^\beta} \bar{g} (b_1) | \leq \frac{M \beta_1}{\beta_1 + 1} (\psi (b_1) - \psi (x))^{\beta_1 + 1}.
\] (29)

So, from (25) and (29), we obtain (21). □
Corollary 1. If $\beta_1 = \alpha_1$ in Theorem 2, then the following fractional Ostrowski inequality holds:
\[
\left| ((\psi (b_1) - \psi (x))^{a_1} + (\psi (x) - \psi (a_1))^{a_1}) g(x) \right|
\leq \frac{\frac{M\alpha_1}{\alpha_1 + 1}}{\alpha_1 + 1} \left( (\psi (b_1) - \psi (x))^{a_1+1} + (\psi (x) - \psi (a_1))^{a_1+1} \right),
\]
where $\alpha_1 > 0$ and $x \in [a_1, b_1]$.

Corollary 2. If $\alpha_1 = \beta_1 = 1$ and $\psi(x) = x$, then we lead to the Ostrowski inequality (1).

Corollary 3. If $\alpha_1 = \beta_1$ in Theorem 5, then we obtain the following fractional Ostrowski inequality:
\[
\left| ((\psi (b_1) - \psi (x))^{a_1} g (b_1) + (\psi (x) - \psi (a_1))^{a_1} g (a_1)) \right|
\leq \frac{\frac{M\alpha_1}{\alpha_1 + 1}}{\alpha_1 + 1} \left( (\psi (b_1) - \psi (x))^{a_1+1} + (\psi (x) - \psi (a_1))^{a_1+1} \right),
\]
where $\alpha_1 > 0$ and $x \in [a_1, b_1]$.

Remark 1. If we take $\psi (x) = x$, then Theorem 2, Theorem 3, and Theorem 5 reduce to Theorem 1.2–Theorem 1.4 in Farid [35], respectively. But, in [35], $-m$ should be $M$ in the first inequality in Theorem 1.3. Also, $M$ should be $-m$ in the second inequality.

Remark 2. After following the steps of the proof of Theorem 2 with $\psi (x) = x$ and $\alpha_1 = \beta_1 = 1$, an alternative proof of the Ostrowski inequality is obtained (see [37]).

3. Examples

In this section, we support our main results by presenting two examples.

Example 1. Let $\alpha_1 = 0.5$, $\beta_1 = 2.2$, $\psi(x) = e^x$, $g(x) = \sin x$ and $[a_1, b_1] = [0, \pi]$. Then, we obtain $|g'(x)| = |\cos x| \leq 1$, that is, $M = 1$. Also, $\psi(x) = e^x$ is an increasing continuous derivative and positive monotone function with $\psi'(x) = e^x \geq 1$ for all $x \in [0, \pi]$. Then, using Theorem 2, for $[0, \pi]$ we obtain the following Ostrowski type inequality:
\[
\left| \Gamma (3.2) I_{\alpha_1}^{2.2\psi} \sin x - \frac{3}{5} \Gamma (1.5) I_{b_2}^{0.5\psi} \sin x \right|
\leq \frac{11}{16} (e^x - e^3)^{3.2} + \frac{1}{3} (e^x - 1)^{1.5}.
\]

Example 2. Let $\alpha_1 = 0.5$, $\beta_1 = 2.2$, $\psi(x) = 6\sqrt{x + 2}$, $g(x) = (x - 1)^2$ and $[a_1, b_1] = [0, 2]$. Then, we get $g'(x) = 2(x - 1)$. Let $m = -2$ and $M = 2$. Also, $\psi(x) = 6\sqrt{x + 2}$ is an increasing continuous derivative and positive monotone function with $\psi'(x) = \frac{3}{\sqrt{x + 2}} \geq 1$ for all $x \in [0, 2]$. Then, using Theorem 3, for $x \in [0, 2]$ we obtain the following Ostrowski type inequality:
\[
\left( \left( \frac{6\sqrt{x + 2} - 6\sqrt{2}}{\sqrt{x + 2}} \right)^{0.5} - \left( 12 - 6\sqrt{x + 2} \right)^{2.2} \right) (x - 1)^2 - \left( \Gamma (1.5) I_{b_2}^{0.5\psi} (x - 1)^2 - \Gamma (3.2) I_{b_2}^{2.2\psi} (x - 1)^2 \right)
\leq 2 \left( \frac{1}{3} \left( 6\sqrt{x + 2} - 6\sqrt{2} \right)^{1.5} + \frac{11}{16} \left( 12 - 6\sqrt{x + 2} \right)^{3.2} \right).
\]
and
\[
\left(12 - 6\sqrt{x + 2}\right)^{2.2} - \left(6\sqrt{x + 2} - 6\sqrt{2}\right)^{1.5}(x - 1)^2 + \left(\Gamma (1.5) I_{0+}^{0.5, \psi}(x - 1)^2 - \Gamma (3.2) I_{2}^{2, \psi}(x - 1)^2\right)
\leq 2\left(\frac{11}{16} (6\sqrt{2} - 6\sqrt{x + 2})^{3.2} + \frac{1}{3} (6\sqrt{x + 2} - 6\sqrt{2})^{1.5}\right).
\]

4. Conclusions

Studies involving integral inequalities play an important role in several areas of science and engineering. During recent years, such inequalities have been generalized and developed by many researchers. Ostrowski inequalities have a great importance while studying the error bounds of different numerical quadrature rules, for example the midpoint rule, Simpson’s rule, the trapezoidal rule and other generalized Riemann types. In this paper, by generalizing the inequalities in [35], we proposed, within four theorems and their related corollaries, several new Ostrowski-type integral inequalities for the left and right sided fractional integrals of a function \(g\) with respect to another function \(\psi\). Finally, we investigated in detail two examples to show the reported results.

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