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A Group Replacement Decision Support System Based on Internet of Things

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Abstract: This paper combines computer-based monitoring technologies and Internet of things (IoT) technology to develop IoT condition-based group replacement decision support system for a production/service system with numerous parallel independent operating servers. This proposed IoT conditioned-based group replacement decision support system first develops the discounted cost model for a service/production system with numerous independent working servers. The original discounted cost model is further revised into an equivalent model to stimulate the proof procedure by applying the uniformization approach. Several significant theoretical properties are proved and many numerical examples are conducted for two kinds of group replacement policies, respectively. The first class of group replacement policy is developed and proved theoretically that there is a threshold of amount of customers existed to activate the group replacement depending on various amount of operating servers; numerical examples conducted in this study can also illustrate the above theoretical outcomes already derived for the first class of group replacement policy. Besides, for the second class of group replacement policy, the results of numerical examples definitely demonstrate that there is a threshold of the amount of operating servers needed to start the group replacement according to distinct amount of customers in the system. This proposed IoT condition-based group replacement decision support system derives the structure and detailed procedure flow to actually conduct the group replacement operations for many practical service or production systems.

Keywords: group replacement; condition-based replacement; decision support system; unreliable Markovian service systems; Internet of things

1. Introduction

To keep production/service system manipulating normally to run production line successfully or offer suitable services for customers is very significant in such customer-oriented business world. On the contrary, the delay of production flow or the loss of customers would have occurred and caused a lot of cost. In addition to built-in reliability design, a desirable maintenance policy also plays an important part in avoiding the breakdown of an operating system. Recently, there has been much research concerning maintenance and replacement problems. With regard to single-unit systems, different replacement policies including age-replacement [1–10], preventive replacement [11–14], failure limit replacement, and block replacement [5] are proposed, combined with minimal repair [5], unplanned replacement, warranty policy [5,13,14], reliability criteria and other options depending on different conditions.

Age replacement implies that a system is replaced at a specified replacement age or at failure, whichever occurs first. Chien [3] establishes a completely renewable age-replacement policy combined with a proportional warranty policy. Park and Pham [5] deals with the optimal maintenance policy, block replacement policy and age replacement policy under renewable and non-renewable warranty policies. Lim et al. [4] presents age replacement policy depending on imperfect repair with random probability. The developed policy comprises the situation that such failure can be either perfectly repaired or minimally repaired with random probabilities. Concerning the on-line information, an adaptive age replacement policy is proposed for systems under changeable manipulating conditions using a cumulative exposure model [6]. Preventive replacement means that the care and service for maintaining equipment in approving operating condition by conducting systematic detection, and correction of failures either before they occur or before they become major defects. Dimitrakos and Kyriakidis [11] conducts the preventive maintenance installation as early as the degree of the system deterioration surpasses an extreme standard. Berthaut et al. [12] proposes a maintenance policy to establish the preventive maintenance schedule depending on the stock quantity. Wu et al. [13] considers a revenue-generating system to derive warranty contracts for a periodic preventive maintenance policy. Huang et al. [14] surveys an imperfect and periodic preventive maintenance strategy a product handled by base warranty and extended warranty region from the perspective of manufacturers.

Many of these replacement models for single-unit systems generally assume that all malfunctions will be immediately detected, repaired and replaced. However, it is not true in the real world. What frequently occurred is that the single-unit system must cease the production/service process for the malfunction recognition and further cause production loss or customer loss. A multi-unit system is the other option to prevent the pausing of related production/service process. According to our opinions, the replacement policies for multi-unit systems are usually derived from those for single-unit systems but more complex. Shafiee and Finkelstein [15] develops an age group replacement policy for a multi-server series system; the component is replaced by a new one when the degradation level reaches a given critical size, and the other components undergo a preventive maintenance action. A planned group preventive maintenance is conducted for the whole system at the proposed operational age. Babishin and Taghipour [16] considers a multicomponent system subject to hard and hidden soft failures and propose an optimal maintenance policy for each part and the optimal periodic examination for the system; this policy makes the maintenance decision according to the optimum age before replacement for the hard-type components and the optimum amount of minimal repairs until replacement for the soft-type components. Zhao et al. [7] presents a summary of review of age replacement models for a parallel multi-unit system with random number of units.

Several previous studies assume that the machines manufacture products with a consistent rate [7,15,17–19]. It means that related production loss cost incurred with a consistent rate when a failed machine is still kept un-repaired. Contrarily, this study considers that the production loss cost relies on the amount of products waiting for service in the system. Therefore, more accurate production loss cost can be acquired with this assumption. The group replacement problems focused in this study clearly belong to the field of unreliable queuing systems, which can be resolved by the matrix-geometric method [20,21]. Moreover, Liu [22] presents the group m-failure maintenance policies to deal with failure/repair process of servers and arrival/service process of customers at the same time; in total, three models are developed, two with positive repair time and one with instantaneous repair; the matrix geometric approach is employed to achieve the steady state distribution and further obtain the expected average cost for these three models. For the theoretical analysis, this paper demonstrates that there exists an optimal group replacement parameter m^*, which can acquire the minimal average cost for all three models. Moreover, several mathematical properties and sensitivity analyses are numerically demonstrated regarding different values of parameters. Consequently, the comparisons of these three proposed models in different situations are also discussed.

The above papers implementing the steady state analysis with matrix-geometric method can only get the expected average amount of customers, rather than let the replacement decision to dynamically depend on the amount of operating servers and customers in system [23]. Therefore, it is very important to be aware of the actual status of operating machines and the number of jobs waiting for process instantly. For achieving this goal, computer-based monitoring technologies and Internet of things technology can be applied in the maintenance model proposed in this paper. In recent years, research on condition-based maintenance has been speedily increasing because of the fast development of computer-based monitoring technologies. Condition-based maintenance is a maintenance strategy that gathers and evaluates real-time information, and suggests maintenance decisions based on the present status of the system. Olde Keizer et al. [18] conduct a literature review on condition-based maintenance policies focusing on multi-component systems subject to different dependences. Alaswad and Xiang [19] presents a review of condition-based maintenance literature with emphasis on various important aspects of optimization criteria, inspection frequency, maintenance degree, and solution methodology. Aizpurua et al. [24] proposes an online system maintenance method, which conducts an online predictive diagnosis algorithm to differentiate between critical and non-critical assets. A prognostics-updated method for estimating the system health is then conducted to yield well-informed, more precise, condition-based suggestions for the maintenance of critical components and for the group-based reactive repair of non-critical components. To take advantage of recent fast growing development of Internet of things, Li et al. [25] establishes predictive maintenance system which is depending on Internet of things technology to secure the mine safety monitoring and maintenance of equipment system.

Consequently, this paper tries to combine computer-based monitoring technologies and Internet of things (IoT) technology to develop IoT condition-based group replacement decision support system for an unreliable Markovian service system. Based on the structure and detailed procedure flow developed in this paper, the proposed IoT condition-based group replacement decision support system can be definitely conducted to solve the replacement problems of some designated service or production systems with several machines or servers manipulating in parallel.

The paper is organized as follows. Section 1 contains the background introduction and literature review. Section 2 describes the mathematical model formulation, and Section 3 depicts the related discounted cost formulation, and Section 4 illustrates the modified uniformization model. Furthermore, Section 5 demonstrates the related proofs of optimal policy and Section 6 contains numerical examples and results. Finally, Section 7 summarizes with the concluding remarks.

2. Mathematical Model Formulation

2.1. Problem Description

This study considers a production system with M_{qn} servers connected in parallel in a single waiting line. Products enter in production line with a Poisson process of arrival rate σ, and the process time per product follows an exponential distribution with rate ξ. The servers fail with identically exponential distribution times and the breakdown rate of each server is β. In the replacement process, replacements are conducted by a crew of j repairmen. The replacement time for each server by using one repairman follows exponential distribution with rate γ. The overall replacement rate equals $\gamma_{m_{qn}} = \frac{j\gamma}{M_{qn}-m_{qn}}$, when m_{qn} servers are normal and $M_{qn} - m_{qn}$ servers under replacement. The objective of this research is to build an IoT condition-based group replacement decision support system. The structure and the detailed procedure flow of our proposed decision support system (DSS) are shown in Figures 1 and 2, respectively.

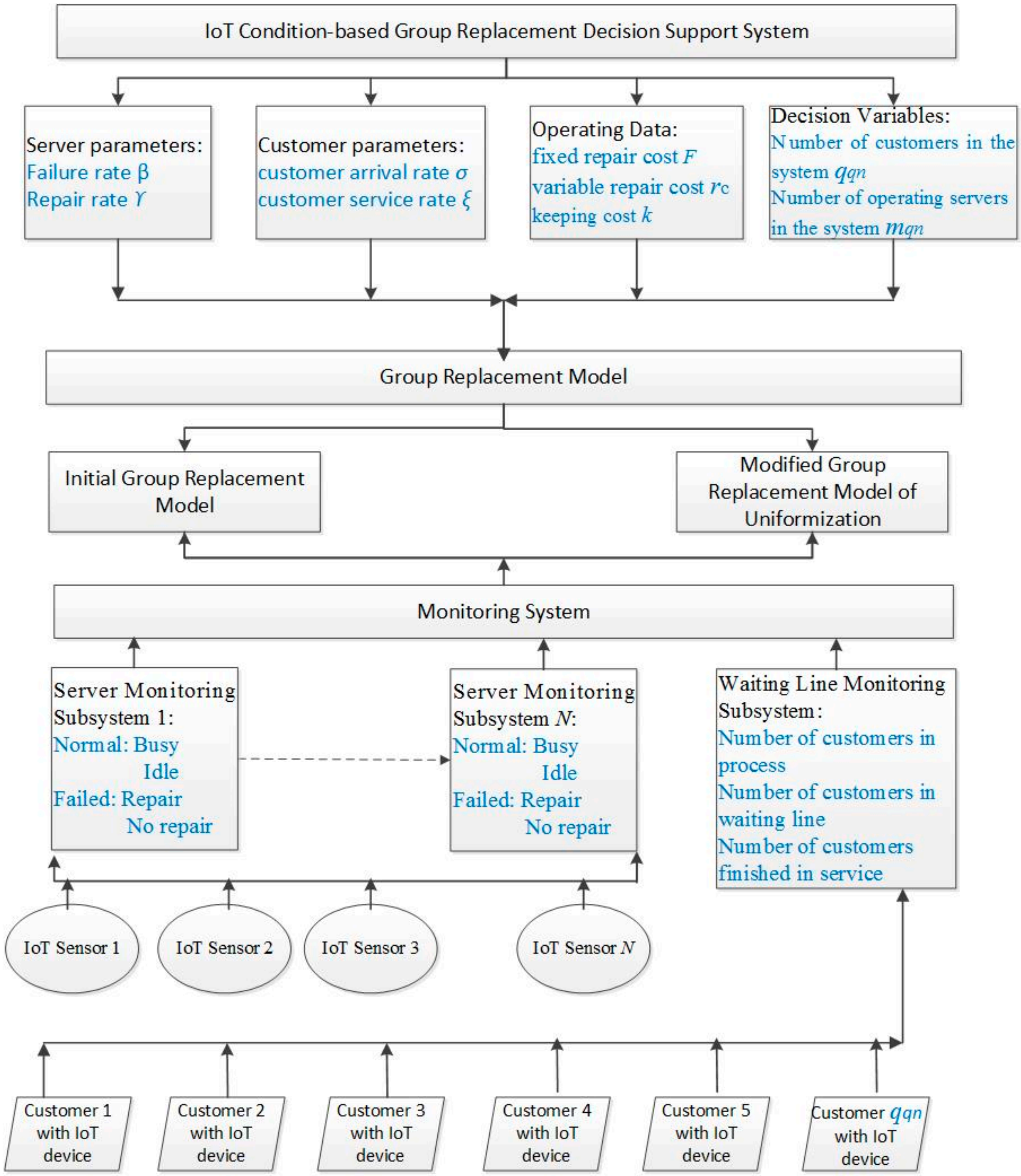

Figure 1. The structure of decision support system.

The main structure of our proposed decision support system shown in Figure 1 includes (1) Group Replacement Model Module: Initial Group Replacement Model and Modified Group Replacement Model of Uniformation; (2) Computer Monitoring System: Server Monitoring Subsystem and Waiting Line Monitoring Subsystem; (3) IoT Sensor Module: Servers with IoT devices and Customers with IoT devices.

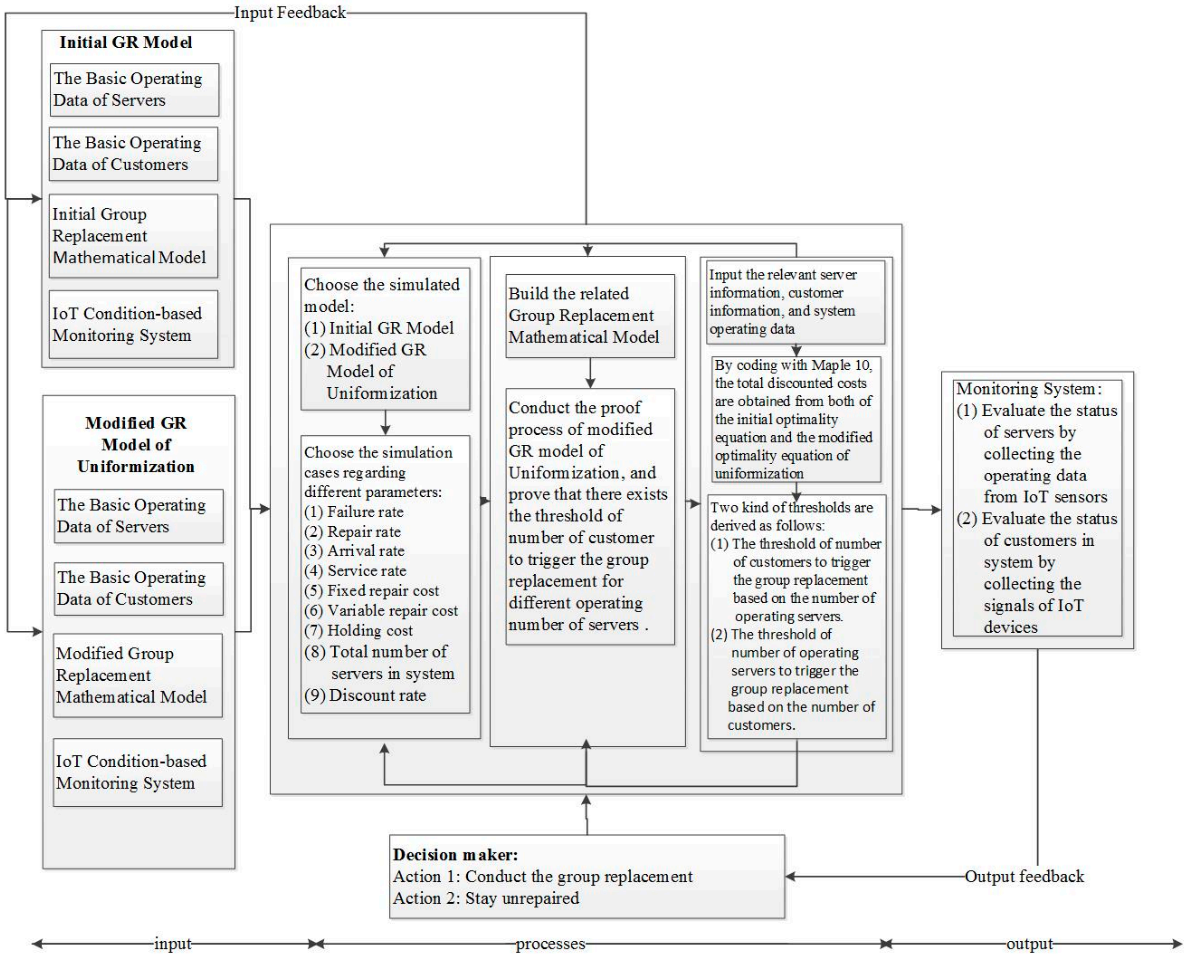

Figure 2. The procedure flow of decision support system.

The main procedure of our proposed decision support system is as follows:

Step 1: Choose the simulated group replacement model. Step 2: Choose the numerical cases regarding different parameters. Step 3: Build the related group replacement mathematical model. Step 4: Conduct the proof process of the Modified Group Replacement Model of Uniformation. Step 5: Several numerical cases are conducted to find the corresponding thresholds to trigger group replacements based on different operating servers or specific amount of customers in system for different values of parameters. Step 6: Computer monitoring system instantaneously evaluates the status of servers by collecting the precise operating data from desired IoT sensors, and the status of customers in system by collecting the signals of built-in IoT devices from these customers, and further decides whether the group replacement should be conducted based on the group replacement threshold.

2.2. Group Replacement Mathematical Model Formulation

The group replacement mathematical model of this proposed IoT condition-based decision support system can be modeled as a continuous-time Markov decision process. The parameters of the continuous-time Markov decision model can be specified as follows. The state space of the process is the set: $\mathrm{S} = \{q,\ m\}$, where the state of the process is the state $(q_{qn}, m_{qn}), q_{qn} \geq 0$, and $0 \leq m_{qn} \leq M_{qn}$, where q_{qn} indicates the amount of products in system, and m_{qn} the amount of normal servers in system. The action set $\mathrm{A}(q_{qn},\ m_{qn}) = \{1,2\}$, in which $a = 1$ represents the production process is keeping on and $a = 2$ indicates that the group replacement of breakdown servers is carried out. The transition probabilities for each action can be specified as follows.

The mechanism of transition process for action $a = 1$ (Proceed the production process) is $(q_{qn}, m_{qn}) \to (q_{qn}+1, m_{qn})$ with arrival rate σ; $(q_{qn}, m_{qn}) \to (q_{qn}, m_{qn}-1)$ with overall breakdown rate $m_{qn}\beta$; $(q_{qn}, m_{qn}) \to (q_{qn}-1, m_{qn})$ with overall process rate $\xi\min(q_{qn}, m_{qn})$.

The mechanism of transition process for action $a = 2$ (Cease the production process and conduct the group replacement) is $(q_{qn},\ m_{qn}) \to (q_{qn}+1, m_{qn})$ with arrival rate σ; $(q_{qn}, m_{qn}) \to (q_{qn}-1, m_{qn})$ with overall process rate $\xi\min(q_{qn}, m_{qn})$; $(q_{qn}, m_{qn}) \to (q_{qn}, m_{qn}-1)$ with overall breakdown rate $m_{qn}\beta$; $(q_{qn}, m_{qn}) \to (q_{qn}, M_{qn})$ the overall replacement rate $\gamma_{m_{qn}}$ equals $\frac{j\gamma}{M_{qn}-m_{qn}}$.

An equivalent transition mechanism can also be described as follows:

The transition mechanism for action $a = 1$ means that the process will remain in (q_{qn}, m_{qn}) for an exponential distribution time with overall rate $\sigma(q_{qn}, m_{qn}) = \sigma + m_{qn}\beta + \xi\min(q_{qn}, m_{qn})$. After a transition takes place, it will head to one of $(q_{qn}+1, m_{qn})$, $(q_{qn}, m_{qn}-1)$, $(q_{qn}-1, m_{qn})$ with following specific probabilities:

$$P_{(q_{qn},m_{qn})(q_{qn}+1,m_{qn})}(a=1) = \frac{\sigma}{\sigma(q_{qn}, m_{qn})},$$

$$P_{(q_{qn},m_{qn})(q_{qn},m_{qn}-1)}(a=1) = \frac{m_{qn}\beta}{\sigma(q_{qn}, m_{qn})},$$

$$P_{(q_{qn},m_{qn})(q_{qn}-1,m_{qn})}(a=1) = \frac{\xi\min(q_{qn}, m_{qn})}{\sigma(q_{qn}, m_{qn})}.$$

The transition mechanism for action $a = 2$ indicates that the process will remain in state (q_{qn}, m_{qn}) for an exponentially distributed time with rate $\sigma + m_{qn}\beta + \xi\min(q_{qn}, m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}$. After a transition takes place, it will head to one of $(q_{qn}+1, m_{qn})$, $(q_{qn}, m_{qn}-1)$, $(q_{qn}-1, m_{qn})$, (q_{qn}, M_{qn}) with following probabilities:

$$P_{(q_{qn},m_{qn})(q_{qn}+1,m_{qn})}(a=2) = \frac{\sigma}{\sigma + m_{qn}\beta + \xi\min(q_{qn}, m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}},$$

$$P_{(q_{qn},m_{qn})(q_{qn},m_{qn}-1))}(a=2) = \frac{m_{qn}\beta}{\sigma + m_{qn}\beta + \xi\min(q_{qn}, m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}},$$

$$P_{(q_{qn},m_{qn})(q_{qn}-1,m_{qn})}(a=2) = \frac{\xi\min(q_{qn}, m_{qn})}{\sigma + m_{qn}\beta + \xi\min(q_{qn}, m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}},$$

$$P_{(q_{qn}m_{qn})(q_{qn},M_{qn})}(a=2) = \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\sigma + m_{qn}\beta + \xi\min(q_{qn}, m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}}.$$

Given the condition of next state is $(q_{qn}, m_{qn})'$, the duration until the transition from (q_{qn}, m_{qn}) to $(q_{qn}, m)'$ follows exponential distribution $F_{(q_{qn},m_{qn})(q_{qn},m_{qn})'}(\cdot|a)$:

$$F_{(q_{qn},m_{qn})(q_{qn}+1,m_{qn})}(t|a=1) = F_{(q_{qn},m_{qn})(q_{qn},m_{qn}-1)}(t|a=1) = F_{(q_{qn},m_{qn})(q_{qn}-1,m_{qn})}(t|a=1) = 1 - e^{-(\sigma+\xi\min(q_{qn},\ m_{qn})+m_{qn}\beta)t}.$$

$$F_{(q_{qn},m_{qn})(q_{qn}+1,m_{qn})}(t|a=2) = F_{(q_{qn},m_{qn})(q_{qn}-1,m_{qn})}(t|a=2) = F_{(q_{qn},m_{qn})(q_{qn},m_{qn}-1)}(t|a=2) = F_{(q_{qn},m_{qn})(q_{qn},M_{qn})}(t|a=2)$$
$$= 1 - e^{-(\sigma+\xi\min(q_{qn},m_{qn})+m_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}})t}.$$

Finally, the cost structure of the model can be specified as follows.

If action a is implemented in (q_{qn}, m_{qn}), accordingly the direct cost $C((q_{qn}, m_{qn}), a)$ is caused, and moreover, the cost rate $c((q_{qn}, m_{qn}), a)$ is exerted till the latter transition arises:

$$C((q_{qn}, m_{qn}), a=1) = 0, c((q_{qn}, m_{qn}), a=1) = kq_{qn}, c((q_{qn}, m_{qn}), a=1) = kq_{qn}.$$

Furthermore, based on the above mechanism of transition, notice that the probability to complete the replacement is $\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}}$.

Therefore, the expected replacement cost of action $a = 2$ is equal to

$$C((q_{qn}, m_{qn}), a = 2) = \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\alpha + \sigma(q_{qn}, m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}} * [F_c + (M_{qn} - m_{qn})r_c],$$

which represents the property that the replacement cost is caused unless the replacement is accomplished until the next transition. F_c represents the fixed replacement cost for triggering replacements, r_c indicates the variable replacement cost of one replaced server, and k denotes the keeping cost per unit time for each product. If the transition happens for t units of time, then the total cost will be $C((q_{qn}, m_{qn}), a) + t^{*}c((q_{qn}, m_{qn}), a)$.

3. Formulation under the Discounted Cost Criterion

It is assumed that costs are continuously discounted with rate $\alpha \geq 0$, and minimize the expected total discounted cost $TU_\alpha(q_{qn}, m_{qn})$. Applying the principle of continuous time Markov Decision Process (MDP) [24], the proposed model can be converted into a discrete time MDP as follows:

The one-stage cost $\overline{C_\alpha}((q_{qn}, m_{qn}), a)$ is derived as the following discrete time equation:

$$\overline{C_\alpha}((q_{qn}, m_{qn}), a) = C((q_{qn}, m_{qn}), a) + \sum_{(q_{qn},m_{qn})'} P_{(q_{qn},m_{qn})(q_{qn},m_{qn})'}(a) \int_0^\infty \int_0^t e^{-\alpha s} c((q_{qn}, m_{qn}), a) ds dF_{(q_{qn},m_{qn})(q_{qn},m_{qn})'}(t|a).$$

Specially, when $a = 1$ (Proceed the production process), then

$$\overline{C_\alpha}((q_{qn}, m_{qn}), 1) = 0 + \int_0^\infty \int_0^t e^{-\alpha s} ds kq\sigma(q_{qn}, m_{qn}) e^{-\sigma(q_{qn},m_{qn})t} dt = \frac{kq_{qn}}{\alpha + \sigma(q_{qn}, m_{qn})}. \tag{1}$$

Also, when $a = 2$ (Conduct the group replacement),

$$\begin{aligned} \overline{C_\alpha}((q_{qn}, m_{qn}), 2) &= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} [F + (M_{qn} - m_{qn})r_c] + \int_0^\infty \int_0^t e^{-\alpha s} ds kq e^{-[\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}]t} dt \\ &= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} [F + (M_{qn} - m_{qn})r_c] + \frac{kq_{qn}}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}}. \end{aligned} \tag{2}$$

The future discounted cost function can be derived as follows:

$$\overline{TU}((q_{qn}, m_{qn}), a) = \sum_{(q_{qn},m_{qn})'} P_{(q_{qn},m_{qn}(q_{qn},m_{qn})'}(a) \int_0^\infty e^{-\alpha t} TU_\alpha((q_{qn}, m_{qn})') dF_{(q_{qn},m_{qn})(q_{qn},m_{qn})'}(t|a).$$

Specially, when $a = 1$, through some simple intermediary integration:

$$\begin{aligned} \overline{TU}((q_{qn}, m_{qn}), 1) &= \sum_{(q_{qn},m_{qn})'} P_{(q_{qn},m_{qn})(q_{qn},m_{qn})'}(1) \int_0^\infty e^{-\alpha t} TU_\alpha((q_{qn}, m_{qn})') \sigma(q_{qn}, m_{qn}) e^{-\sigma(q_{qn},m_{qn})t} dt \\ &= \sum_{(q_{qn},m_{qn})'} P_{(q_{qn},m_{qn})(q_{qn},m_{qn})'}(1) TU_\alpha((q_{qn}, m_{qn})') (\frac{\sigma(q_{qn},m_{qn})}{\alpha+\sigma(q_{qn},m_{qn})}) \\ &= \frac{\sigma}{\alpha+\sigma(q_{qn},m_{qn})} TU_\alpha(q_{qn} + 1, m_{qn}) + \frac{m_{qn}\beta}{\alpha+\sigma(q_{qn},m_{qn})} TU_\alpha(q_{qn}, m_{qn} - 1) + \frac{\zeta\min(q_{qn},m_{qn})}{\alpha+\sigma(q_{qn},m_{qn})} * \\ &\quad TU_\alpha(q_{qn} - 1, m_{qn}). \end{aligned} \tag{3}$$

Similarly, when $a = 2$,

$$\begin{aligned}
&\overline{TU}((q_{qn}, m_{qn}), 2) \\
&= \sum_{(q_{qn},m_{qn})'} P_{(q_{qn},m_{qn})(q_{qn},m_{qn})'}(1) \int_0^\infty e^{-\alpha t} TU_\alpha((q_{qn}, m_{qn})')[\sigma(q_{qn}, m_{qn}) + \tfrac{j\gamma}{M_{qn}-m_{qn}}] e^{-[\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}]t} dt \\
&= \sum_{(q_{qn},m_{qn})'} P_{(q_{qn},m_{qn})(q_{qn},m_{qn})'}(1) TU_\alpha((q_{qn}, m_{qn})')\Big(\frac{\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}}\Big) \\
&= \frac{\sigma}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} * TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} * TU_\alpha(q_{qn}, m_{qn}-1) \\
&+ \frac{\zeta\min(q_{qn},m_{qn})}{\alpha+\sigma(q_{qn},m_{qn}+\frac{j\gamma}{M_{qn}-m_{qn}}} * TU_\alpha(q_{qn}-1, m_{qn}) + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} * TU_\alpha(q_{qn}, M_{qn}).
\end{aligned} \tag{4}$$

Based on the above outcomes, the discrete time version of optimality equations can be denoted as $TU_\alpha(q_{qn}, m_{qn}) = \min\{TU^1_\alpha(q_{qn}, m_{qn}), TU^2_\alpha(q_{qn}, m_{qn})\}$, where

$$\begin{aligned}
&TU^1_\alpha(q_{qn}, m_{qn}) = \overline{C_\alpha}((q_{qn}, m_{qn}), 1) + \overline{TU}((q_{qn}, m_{qn}), 1) \\
&= \frac{kq_{qn}}{\alpha+\sigma(q_{qn},m_{qn})} + \frac{\sigma}{\alpha+\sigma(q_{qn},m_{qn})} TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\alpha+\sigma(q_{qn},m_{qn})} TU_\alpha(q_{qn} m_{qn}-1) + \frac{\zeta\min(q_{qn},m_{qn})}{\alpha+\sigma(q_{qn},m_{qn})} * \\
&\quad TU_\alpha(q_{qn}-1, m_{qn}). \\
&TU^2_\alpha(q_{qn}, m_{qn}) = \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}}[F + (M_{qn}-m_{qn})r_c] + \frac{kq_{qn}}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} + \\
&\frac{\sigma}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} TU_\alpha(q_{qn}, m_{qn}-1) + \\
&\frac{\zeta\min(q_{qn},m_{qn})}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} TU_\alpha(q_{qn}-1, m_{qn}) + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}} TU_\alpha(q_{qn}, M_{qn}).
\end{aligned} \tag{5}$$

The state space is not countable and the one-stage cost $\overline{C}((q_{qn}, m_{qn}), a)$ is unbounded. Applying standard results from unbounded cost MDPs [26] (Th. 4.2.3, Prop. 4.3.1, (c)), the subsequent Lemma can be obtained:

Lemma 1.

(i) *$TU_\alpha(q_{qn}, m_{qn})$ is the sole solution to the equations of optimality in (5).*

(ii) *There exists an optimal stationary policy, which conducts action to minimize the right side of Equation (5).*

(iii) *Assume $TU_\alpha(q_{qn}, m_{qn}; n = 0) = 0$,*

$$TU_\alpha(q_{qn}, m_{qn}; n) = \min_{a\in A(q_{qn},m_{qn})} \{\overline{C}((q_{qn}, m_{qn}), a) + \overline{TU}((q_{qn}, m_{qn}), a; n-1)\}.$$

Consequently, $\lim_{n\to\infty} TU_\alpha(q_{qn}, m_{qn}; n) = TU_\alpha(q_{qn}, m_{qn})$ for any preliminary state (q_{qn}, m_{qn}).

4. The Modified Uniformization Model

Unfortunately, the discounted cost Equation (5) developed in Section 3 is very difficult to prove that there exists a threshold of amount of customer to trigger the group replacement based on different amount of operational servers. Therefore, this proposed IoT conditioned-based group replacement decision support system further revises the original discounted cost model into an equivalent model to facilitate the proof procedure by applying the uniformization approach. Several significant theoretical properties are proven and many numerical examples are conducted for two kinds of group replacement policies, respectively.

Section 3 shows that all transition rates are limited by $\alpha + \sigma + M_{qn}\beta + M_{qn}\zeta + j\gamma$ in state $(q_{qn},\ m_{qn})$. In this section, the uniformization approach proposed by [27,28] is carried out to convert the proposed model in Section 3 into a similar model where all exponentially distributed sojourn times abide by the identical rate $\overline{\sigma} \geq \alpha + \sigma + M_{qn}\beta + M_{qn}\zeta + j\gamma$. This model can be modified as follows:

All transitions from each state happen at a common rate $\overline{\sigma}$. While the process stays in (q_{qn}, m_{qn}), however, only a fraction $\frac{\sigma(q_{qn},m_{qn})}{\overline{\sigma}}$ are those transitions from (q_{qn}, m_{qn}) to state $(q_{qn},\ m_{qn})'$ not equal to (q_{qn}, m_{qn}) and the others are transitions returning to (q_{qn}, m_{qn}), which are called virtual transitions. Specially, we derive a similar discrete-time Markov chain in which transitions are described as follows:

(1) When $a = 1$ (proceed the production process)
$(q_{qn}, m_{qn}) \to (q_{qn}+1, m_{qn})$ with probability $\frac{g}{\overline{\sigma}}$, $(q_{qn}, m_{qn}) \to (q_{qn}, m_{qn}-1)$ with probability $\frac{m\beta}{\overline{\sigma}}$, $(q_{qn}, m_{qn}) \to (q_{qn}-1, m_{qn})$ with probability $\frac{\zeta \min(q_{qn},m_{qn})}{\overline{\sigma}}$, $(q_{qn},\ m_{qn}) \to$ absorbing with probability $\frac{\alpha}{\overline{\sigma}}$, $(q_{qn}, m_{qn}) \to (q_{qn},\ m_{qn})$ with probability $(1 - \frac{\alpha+\sigma(q_{qn},m_{qn})}{\overline{\sigma}})$.

(2) When $a = 2$ (conduct the group replacement)
$(q_{qn},\ m_{qn}) \to (q_{qn}+1, m_{qn})$ with probability $\frac{g}{\overline{\sigma}}$, $(q_{qn}, m_{qn}) \to (q_{qn}, m_{qn}-1)$ with probability $\frac{m_{qn}\beta}{\overline{\sigma}}$, $(q_{qn}, m_{qn}) \to (q_{qn}-1,\ m_{qn})$ with probability $\frac{\zeta \min(q_{qn},m_{qn})}{\overline{\sigma}}$, $(q_{qn}, m_{qn}) \to (q_{qn}, M_{qn})$ with probability $\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}$, $(q_{qn}, m_{qn})\ m_{qn}) \to$ absorbing state with probability $\frac{\alpha}{\overline{\sigma}}$, $(q_{qn}, m_{qn}) \to (q_{qn}, m_{qn})$ with probability $(1 - \frac{\alpha + \frac{j\beta}{M_{qn}-m_{qn}} + \sigma(q_{qn},m_{qn})}{\overline{\sigma}})$.

By applying this modified Markov decision process, the equation of optimality can be converted as follows:

Recollect the optimality Equation (5), while action $a = 1$, $TU_\alpha(q_{qn}, m_{qn}) = U_\alpha^1(q_{qn}, m_{qn})$ is acquired. By multiplying both parts of the equation with $\frac{\alpha+\sigma(q_{qn},m_{qn})}{\overline{\sigma}}$, then increasing both parts of the above equation with $V_\alpha(q_{qn}, m_{qn})$ and moving $\frac{\alpha+\sigma(q_{qn},m_{qn})}{\overline{\sigma}} TU_\alpha(q_{qn}, m_{qn})$ from left part of equation to right part of equation, we have

$$\begin{aligned} TU_\alpha^1(q_{qn}, m_{qn}) = {} & \frac{kq_{qn}}{\overline{\sigma}} + \frac{g}{\overline{\sigma}} TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}} U_\alpha(q_{qn}, m_{qn}-1) + \frac{\zeta \min(q_{qn},m_{qn})}{\overline{\sigma}} TU_\alpha(q_{qn}-1, m_{qn}) \\ & + (1 - \frac{\alpha+\sigma(q_{qn},m_{qn})}{\overline{\sigma}}) TU_\alpha(q_{qn}, m_{qn}). \end{aligned} \tag{6}$$

While action $a = 2$, $TU_\alpha(q_{qn}, m_{qn}) = TU^2(q_{qn}, m_{qn})$ is obtained. Through multiplying both parts of the above equation with $\frac{\alpha+\sigma(q_{qn},m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}$, next adding both parts of the above equation with $TU_\alpha(q_{qn}, m_{qn})$ and moving $\frac{\alpha+\sigma(q_{qn},m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}} TU_\alpha(q_{qn}, m_{qn})$.

From left part of equation to right part of equation, we obtain

$$\begin{aligned} & TU_\alpha^2(q_{qn}, m_{qn}) = \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}} [F + (M_{qn} - m_{qn}) r_c] + \frac{kq_{qn}}{\overline{\sigma}} + \frac{g}{\overline{\sigma}} TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}} TU_\alpha(q_{qn}, m_{qn}-1) + \\ & \frac{\zeta \min(q_{qn},m_{qn})}{\overline{\sigma}} TU_\alpha(q_{qn}-1, m_{qn}) + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}} TU_\alpha(q_{qn}, M_{qn}) + (1 - \frac{\alpha+\sigma(q_{qn},m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}) U_\alpha(q_{qn}.m_{qn}). \end{aligned} \tag{7}$$

According to the outcomes of (6) and (7), the modified uniformization optimality equations can be denoted as

$$TU_\alpha^U(q_{qn}, m_{qn}) = \min\{TU_\alpha^1(q_{qn}, m_{qn}), TU_\alpha^2(q_{qn}, m_{qn})\},$$

where

$$\begin{aligned} & TU_\alpha^1(q_{qn}, m_{qn}) = \frac{kq_{qn}}{\overline{\sigma}} + \frac{g}{\overline{\sigma}} TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}} TU_\alpha(q_{qn}, m_{qn}-1) + \frac{\zeta \min(q_{qn},m_{qn})}{\overline{\sigma}} TU_\alpha(q_{qn}-1, m_{qn}) \\ & + (1 - \frac{\alpha+\sigma(q_{qn},m_{qn})}{\overline{\sigma}}) TU_\alpha(q_{qn}, m_{qn}). \\ & TU_\alpha^2(q_{qn}, m_{qn}) = \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}} [F + (M_{qn} - m_{qn}) r_c] + \frac{kq_{qn}}{\overline{\sigma}} + \frac{g}{\overline{\sigma}} TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}} TU_\alpha(q_{qn}, m_{qn}-1) + \\ & \frac{\zeta \min(q_{qn},m_{qn})}{\overline{\sigma}} TU_\alpha(q_{qn}-1, m_{qn}) + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}} TU_\alpha(q_{qn}, M_{qn}) + (1 - \frac{\alpha+\sigma(q_{qn},m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\lambda}}) TU_\alpha(q_{qn}.m_{qn}). \end{aligned} \tag{8}$$

Actually, the optimality equations in (5) and (8) are similar.

In the continuation Equation (8) will be applied to verify properties of the total cost function and the optimal replacement policy. Now the sequential approximated Equation (8) is described as follows:

$$\begin{aligned}
&TU_\alpha(q_{qn}, m_{qn}; 0) = 0,\\
&TU_\alpha(q_{qn}, m_{qn}; n) = \min\{\tfrac{kq_{qn}}{\overline{\sigma}} + \tfrac{g}{\overline{\sigma}}TU_\alpha(q_{qn}+1, m_{qn}; n-1) + \tfrac{m_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}-1; n-1)+\\
&\tfrac{\zeta\min(q_{qn},m_{qn})}{\overline{\sigma}}TU_\alpha(q_{qn}-1m_{qn}; n-1) + (1-\tfrac{\alpha+\sigma(q_{qn},m_{qn})}{\overline{\sigma}})TU_\alpha(q_{qn}, m_{qn}; n-1),\\
&\tfrac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn}-m_{qn})r_c] + \tfrac{kq_{qn}}{\overline{\sigma}} + \tfrac{g}{\overline{\sigma}}TU_\alpha(q_{qn}+1, m_{qn};\ n-1) + \tfrac{m_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}-1;\ n-1)+\\
&\tfrac{\zeta\min(q_{qn},m_{qn})}{\overline{\sigma}}TU_\alpha(q_{qn}-1, m_{qn};\ n-1) + \tfrac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn};\ n-1)+\\
&(1-\tfrac{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})TU_\alpha(q_{qn},\ m_{qn};\ n-1).
\end{aligned} \tag{9}$$

5. The Theoretical Properties of Optimal Policy

In this section, we establish some monotonicity properties for the total expected cost function $TU_\alpha(q_{qn}, m_{qn})$ of the modified model, which will be applied to construct the framework of corresponding optimal policies. The next theorem shows the monotonicity of the total expected cost function on the number of customer in system.

Theorem 1. *$TU_\alpha(q_{qn}, m_{qn})$ is an increasing function of q_{qn}, the amount of customers in system.*

Proof. See Appendix A. □

With the result we have in Theorem 1, we can derive the following Theorem about the monotonicity property of the total expected cost function on the number of working machines. First, the monotonicity properties for the difference function between $TU^1_\alpha(q_{qn}, m_{qn})$ and $TU^2_\alpha(q_{qn}, m_{qn})$ will be derived. These properties are important for proving the threshold structure of the optimal policy. Before doing so, we need to prove the following Lemma 2.

Lemma 2. *$TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}, m_{qn}; n) \le \frac{k}{\alpha}$ for all (q_{qn}, m_{qn}).*

Proof. It is noted that $TU_\alpha(q_{qn}, m_{qn}; n)$ indicates the minimum expected discounted cost for an n-period group replacement problem, in which n transitions are permitted till the finish of the horizon. The difference between $TU_\alpha(q_{qn}+1, m_{qn}; n)$ and $TU_\alpha(q_{qn}, m_{qn}; n)$ is that one more customer present in $TU_\alpha(q_{qn}+1, m_{qn}; n)$ than $TU_\alpha(q_{qn}, m_{qn}; n)$. Thus, an upper bound for the difference $TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}, m_{qn}; n)$ is provided by the total discounted keeping cost incurred because of that one more customer within n transitions. If the additional customer stays in system for all n transitions, in the light of the uniformization model, the time between transitions complies with exponential distribution with rate $\overline{\sigma}$. Thus, the total time T_n follows a Gamma distribution with parameters $(n, \overline{\sigma})$:

$$f(t) = \begin{cases} \frac{\overline{\sigma}^n t^{n-1} e^{-\overline{\sigma}t}}{(n-1)!} & t \ge 0 \\ 0 & t < 0 \end{cases}.$$

Consequently, the expected discounted keeping cost can be denoted as:

$$E[k\int_0^t e^{\alpha s}ds] = k\int_0^\infty\int_0^t e^{-\alpha s}dsf(t)dt = k\int_0^\infty \frac{1}{\alpha}(1-e^{-\alpha t})f(t)dt = \frac{k}{\alpha}[1-\int_0^\infty e^{-\alpha t}f(t)dt] \le \frac{k}{\alpha}.$$

The proof is completed. □

Theorem 2. *$TU_\alpha(q_{qn}, m_{qn}) - TU_\alpha(q_{qn}, M_{qn})$ is increasing in q_{qn}.*

Proof. See Appendix A. □

Theorem 3. *$TU_\alpha^1(q_{qn}, m_{qn}) - TU_\alpha^2(q_{qn}, m_{qn})$ is increasing in q_{qn}.*

Proof. The proof will be conducted by induction on n. From (9), for $n = 1$, we have

$$TU_\alpha^1(q_{qn}, m_{qn}; n = 1) - TU_\alpha^2(q_{qn}, m_{qn}; n = 1) = -\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c].$$

It is obvious that $TU_\alpha^1(q_{qn}, m_{qn}; n = 1) - TU_\alpha^2(q_{qn}, m_{qn}; n = 1)$ is non-decreasing in q_{qn}.

According to the above outcome, it can be assumed that $TU_\alpha^1(q_{qn}, m_{qn}; n) - TU_\alpha^2(q_{qn}, m_{qn}; n)$ is increasing in q_{qn}, then it need to be proved $TU_\alpha^1(q_{qn}, m_{qn}; n+1) - TU_\alpha^2(q_{qn}, m_{qn}; n+1)$ is also increasing in q_{qn}. From (9), we have

$$\begin{aligned} &TU_\alpha^1(q_{qn}, m_{qn}; n+1) - U_\alpha^2(q_{qn}, m_{qn}; n+1) \\ &= -\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]. \end{aligned}$$

According the above result and Theorem 2, it can be concluded that $TU_\alpha^1(q_{qn}, m_{qn}; n+1) - TU_\alpha^2(q_{qn}, m_{qn}; n+1)$ is increasing in q_{qn}.

Based on regular theses applying successive approximations of discounted MDPs, $TU_\alpha^1(q_{qn}, m_{qn}) - TU_\alpha^2(q_{qn}, m_{qn}) = \lim_n\{TU_\alpha^1(q_{qn}, m_{qn}; n) - TU_\alpha^2(q_{qn}, m_{qn}; n)\}$ is increasing in q_{qn}. The proof is thus carried through. □

Now we present the optimal policy for this model.

Theorem 4. *There exist thresholds $q^*_{qn}(m_{qn}) \geq 0$, in which an α-optimal policy is do the group replacement if $q_{qn} \geq q^*_{qn}(m)$ while the system is in state (q_{qn}, m_{qn}).*

Proof. According to (8), it can be obtained that

$$\begin{aligned} &TU_\alpha(q_{qn}, m_{qn}) = \min\{\frac{kq_{qn}}{\overline{\sigma}} + \frac{g}{\overline{\sigma}}TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}-1) + \frac{\zeta\min(q_{qn}, m_{qn})}{\overline{\sigma}} * \\ &TU_\alpha(q_{qn}-1, m_{qn}) + (1 - \frac{\alpha+\sigma(q_{qn}, m_{qn})}{\overline{\sigma}})TU_\alpha(q_{qn}, m_{qn}),\ \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] \\ &+ \frac{kq_{qn}}{\overline{\sigma}} + \frac{g}{\overline{\sigma}}TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}-1) + \frac{\zeta\min(q_{qn}, m_{qn})}{\overline{\sigma}}TU_\alpha(q_{qn}-1, m_{qn}) + \\ &\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}) + (1 - \frac{\alpha+\sigma(q_{qn}, m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\lambda}})TU_\alpha(q_{qn}, m_{qn}). \end{aligned}$$

Let

$$\begin{aligned} &q^*_{qn}(m_{qn}) = \min\{q_{qn} : \frac{kq_{qn}}{\overline{\sigma}} + \frac{g}{\overline{\sigma}}TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}-1) + \frac{\zeta\min(q_{qn}, m_{qn})}{\overline{\sigma}} * \\ &TU_\alpha(q_{qn}-1, m_{qn}) + (1 - \frac{\alpha+\sigma(q_{qn}, m_{qn})}{\overline{\sigma}})TU_\alpha(q_{qn}, m_{qn}) > \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{kq_{qn}}{\overline{\sigma}} + \\ &\frac{g}{\overline{\sigma}}TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}-1) + \frac{\zeta\min(q_{qn}, m_{qn})}{\overline{\sigma}} * TU_\alpha(q_{qn}-1, m_{qn}) \\ &+ \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}) + (1 - \frac{\alpha+\sigma(q_{qn}, m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})TU_\alpha(q_{qn}, m_{qn})\}. \end{aligned}$$

Now, according to Theorem 3, it confirms that $TU_\alpha^1(q_{qn}, m_{qn}) - TU_\alpha^2(q_{qn}, m_{qn})$ is increasing in q_{qn}, and thus it can be obtained that

$$TU_\alpha(q_{qn}, m_{qn}) = \begin{cases} TU_\alpha^1(q_{qn}, m_{qn}) & for\ q_{qn} < q^*_{qn}(m_{qn}) \\ TU_\alpha^2(q_{qn}, m_{qn}) & for\ q_{qn} \geq q^*_{qn}(m) \end{cases},$$

where

$TU_\alpha^1(q_{qn}, m_{qn}) = \frac{kq_{qn}}{\overline{\sigma}} + \frac{g}{\overline{\sigma}} TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}} TU_\alpha(q_{qn}, m_{qn}-1) + \frac{\zeta \min(q_{qn}, m_{qn})}{\overline{\sigma}} TU_\alpha(q_{qn}-1, m_{qn})$
$+(1 - \frac{\alpha+\sigma(q_{qn}, m_{qn})}{\overline{\sigma}}) TU_\alpha(q_{qn}, m_{qn}).$

$TU_\alpha^2(q_{qn}, m_{qn}) = \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{kq_{qn}}{\overline{\sigma}} + \frac{g}{\overline{\sigma}} TU_\alpha(q_{qn}+1, m_{qn}) + \frac{m_{qn}\beta}{\overline{\sigma}} TU_\alpha(q_{qn}, m_{qn}-1)$
$+ \frac{\zeta \min(q_{qn}, m_{qn})}{\overline{\sigma}} TU_\alpha(q_{qn}-1, m_{qn}) + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}} TU_\alpha(q_{qn}, M_{qn}) + (1 - \frac{\alpha+\sigma(q_{qn}, m_{qn}) + \frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})*$
$TU_\alpha(q_{qn}, m_{qn}).$

The proof is hence completed. □

6. Numerical Results

This section conducts different numerical examples to demonstrate the related theorems and lemmas, which have been proved previously in Section 5. Several numerical examples were programmed in Maple 10 (Version 10, Maplesoft, Waterloo, ON, Canada) and were run on Asus TX201L (Asus, Taipei, Taiwan) with Intel(R) Core(TM) i7-4500U CPU (Intel Corporation, Santa Clara, CA, USA)@ 2.40GHz, 4.00 GB RAM.

First, in Example 1, the total discounted costs acquired from the successive approximation version of initial optimality Equation (5) is compared with the modified optimality equation of uniformization (8) by running 300 iterations and the outcomes are presented in the following Table 1.

Table 1. Total discounted costs of initial and modified optimality equation. $M = 8, \sigma_m = 5, m = 0, \ldots, 8$, $\zeta = 3$, $\beta = 1$, $F = 25$, $r_c = 2$, $k = 20$, $\gamma = 4$, $j = 4$, $\alpha = 0.8$, iteration = 300.

m	0	1	2	3	4	5	6	7	8
$q = 0$									
init	82.843	73.144	66.434	61.573	57.924	55.100	52.859	51.042	49.540
unif	81.696	71.998	65.288	60.428	56.778	53.953	51.712	49.895	48.393
dev	1.147	1.146	1.146	1.145	1.146	1.147	1.147	1.147	1.147
dev.%	1.4%	1.59%	1.76%	1.86%	1.98%	2.08%	2.17%	2.25%	2.32%
$q = 1$									
init	96.097	82.907	74.379	68.509	64.272	61.092	58.627	56.664	55.065
unif	94.951	81.761	73.233	67.363	63.126	59.946	57.480	55.517	53.918
dev	1.146	1.146	1.146	1.146	1.146	1.146	1.147	1.147	1.147
dev.%	1.19%	1.38%	1.54%	1.67%	1.78%	1.88%	1.96%	2.02%	2.08%
$q = 2$									
init	107.486	95.392	83.636	76.170	70.975	67.282	64.510	62.355	60.633
unif	106.339	94.245	82.490	74.964	69.829	66.136	63.363	61.208	59.486
dev	1.147	1.147	1.146	1.206	1.146	1.146	1.147	1.147	1.147
dev.%	1.07%	1.20%	1.37%	1.58%	1.61%	1.70%	1.78%	1.84%	1.89%
$q = 3$									
init	119.025	107.727	95.423	84.892	78.267	73.783	70.564	68.145	66.259
unif	117.878	106.580	94.277	83.746	77.121	72.636	69.418	66.998	65.112
dev	1.147	1.147	1.146	1.146	1.146	1.147	1.146	1.147	1.147
dev.%	0.96%	1.06%	1.2%	1.34%	1.46%	1.55%	1.62%	1.68%	1.73%
$q = 4$									
init	130.775	119.232	107.915	95.965	86.616	80.804	76.890	74.083	71.970
unif	129.629	118.085	106.769	94.819	85.470	79.657	75.743	72.937	70.823
dev	1.146	1.147	1.146	1.146	1.146	1.147	1.147	1.146	1.147
dev.%	0.88%	0.96%	1.06%	1.19%	1.32%	1.42%	1.49%	1.54%	1.59%

Table 1. *Cont.*

m	0	1	2	3	4	5	6	7	8
$q = 5$									
init	142.822	130.681	119.645	108.079	97.032	88.771	83.677	80.259	77.809
unif	141.675	129.534	118.498	106.933	95.886	87.625	82.530	79.113	76.663
dev	1.147	1.147	1.147	1.146	1.146	1.146	1.147	1.146	1.146
dev.%	0.80%	0.88%	0.96%	1.06%	1.18%	1.29%	1.37%	1.43%	1.47%
$q = 6$									
init	155.278	142.372	131.028	120.239	108.719	98.615	91.313	86.845	83.858
unif	154.131	141.225	129.881	119.093	107.573	97.469	90.167	85.698	82.711
dev	1.147	1.147	1.147	1.146	1.146	1.146	1.146	1.147	1.147
dev.%	0.73%	0.81%	0.88%	0.95%	1.05%	1.16%	1.26%	1.32%	1.37%
$q = 7$									
init	168.291	154.506	142.537	131.777	120.946	109.821	100.651	94.193	90.270
unif	167.144	153.359	141.391	130.631	119.800	108.674	99.505	93.047	89.123
dev	1.147	1.147	1.146	1.146	1.146	1.147	1.146	1.146	1.147
dev.%	0.68%	0.74%	0.8%	0.87%	0.95%	1.04%	1.34%	1.22%	1.27%
$q = 8$									
init	182.026	167.286	154.511	143.338	132.977	121.883	111.371	103.086	97.368
unif	180.879	166.140	153.364	142.191	131.831	120.737	110.225	101.940	96.221
dev	1.147	1.146	1.147	1.147	1.146	1.146	1.146	1.146	1.147
dev.%	0.63%	0.69%	0.74%	0.80%	0.86%	0.94%	1.03%	1.11%	1.18%
$q = 9$									
init	196.613	180.907	167.232	155.399	144.928	134.479	123.167	113.331	105.865
unif	195.466	179.760	166.086	154.252	143.781	133.333	122.020	112.185	104.717
dev	1.147	1.147	1.146	1.147	1.147	1.146	1.147	1.146	1.148
dev.%	0.58%	0.63%	0.69%	0.74%	0.79%	0.85%	0.93%	1.01%	1.08%

From the above table, the succeeding outcomes are summarized as follows:

(1) The total costs calculated from initial optimality Equation (5) are nearly approximated to the outcomes acquired by modified optimality equation of uniformization (8) with a tiny bias less than 1.147.

(2) The outcomes demonstrate that the total discount cost function $\mathrm{U}_\alpha(q, m)$ obtained Equations (5) and (8) are all increasing in the amount of customers, which has been demonstrated theoretically in Theorem 1.

(3) The outcomes demonstrate that the total discount cost function $U_\alpha(q, m)$ acquired from Equations (5) and (8) are all decreasing in the amount of working servers.

Therefore, it can be concluded that there exist similar properties and outcomes of the total discounted cost acquired from (5) and (8), respectively.

Afterwards, several examples are conducted to demonstrate the associated characteristics of cost difference function $\Delta(q_{qn}, m_{qn}) = TU^1_\alpha(q_{qn}, m_{qn}) - TU^2_\alpha(q_{qn}, m_{qn})$ and find the corresponding thresholds to trigger group replacements based on different operating servers or specific amount of customers in system. The outcomes are presented in the following Table 2.

Table 2. Cost difference function $\Delta(q_{qn}, m_{qn})$ and thresholds to trigger group replacements $M = 8$, $\sigma_m = 5, m = 0, \ldots, 8$, $\zeta = 3$, $\beta = 1$, $F = 25$, $r_c = 2$, $k = 20$, $\gamma = 4$, $j = 4$, $\alpha = 0.8$, iteration = 300.

m	0	1	2	3	4	5	6	7	8
$q = 0$									
act1	81.696	71.998	65.288	60.428	56.778	53.954	51.712	49.895	48.393
act2	81.982	72.652	66.285	61.794	58.608	56.476	55.531	57.478	-
dif	−0.286	−0.654	−0.997	−1.366	−1.83	−2.522	−3.819	−7.583	-
unif	81.696	71.998	65.288	60.428	56.778	53.954	51.712	49.895	48.393

Table 2. *Cont*

m	0	1	2	3	4	5	6	7	8
$q = 1$									
act1	94.952	81.761	73.233	67.363	63.126	59.946	57.480	55.517	53.918
act2	94.951	82.235	74.110	68.645	64.895	62.421	61.263	63.071	-
dif	0.001	−0.474	−0.877	−1.282	−1.769	−2.475	−3.783	−7.554	-
unif	94.951	81.761	73.233	67.363	63.126	59.946	57.480	55.517	53.918
$q = 3$									
act1	118.316	106.685	94.277	83.746	77.121	72.636	69.418	66.998	65.112
act2	117.878	106.580	94.666	84.720	78.682	74.963	73.090	74.467	-
dif	0.438	0.105	−0.389	−0.974	−1.561	−2.327	−3.672	−7.469	-
unif	117.878	106.580	94.277	83.746	77.121	72.636	69.418	66.998	65.112
$q = 5$									
act1	142.568	130.123	118.738	106.933	95.886	87.625	82.530	79.113	76.663
act2	141.675	129.534	118.498	107.214	96.910	89.611	85.970	86.414	-
dif	0.893	0.589	0.240	−0.281	−1.024	−1.986	−3.440	−7.301	-
unif	141.675	129.534	118.498	106.933	95.886	87.625	82.530	79.113	76.663
$q = 6$									
act1	155.262	142.054	130.385	119.175	107.573	97.469	90.167	85.698	82.711
act2	154.131	141.225	129.881	119.093	108.178	99.079	93.370	92.840	-
dif	1.131	0.829	0.504	0.082	−0.605	−1.610	−3.203	−7.142	-
unif	154.131	141.225	129.881	119.093	107.573	97.469	90.167	85.698	82.711
$q = 8$									
act1	182.502	167.453	154.362	142.844	132.025	120.737	110.225	101.94	96.221
act2	180.879	166.140	153.364	142.191	131.831	121.380	112.455	108.269	-
dif	1.623	1.313	0.998	0.653	0.194	−0.643	−2.230	−6.329	-
unif	180.879	166.140	153.364	142.191	131.831	120.737	110.225	101.940	96.221
$q = 10$									
act1	212.880	195.942	181.030	168.070	156.765	146.256	134.617	123.577	114.500
act2	210.823	194.213	179.635	167.027	156.124	146.187	135.938	128.907	-
dif	2.057	1.729	1.395	1.043	0.641	0.069	−1.321	−5.330	-
unif	210.823	194.213	179.635	167.027	156.124	146.187	134.617	123.577	114.500
$q = 18$									
act1	356.441	334.560	314.346	295.996	279.741	265.875	254.628	240.777	225.093
act2	353.203	331.688	311.878	293.980	278.243	264.993	254.558	244.142	-
dif	3.238	2.872	2.468	2.016	1.498	0.882	0.069	−3.366	-
unif	353.203	331.688	311.878	293.980	278.243	264.993	254.558	240.777	225.093
$q = 500$									
act1	12,187.7	12,158.3	12,130.2	12,103.6	12,079.0	12,056.9	12,038.5	12,022	12,001.3
act2	12,182.5	12,153.5	12,125.9	12,099.8	12,075.9	12,054.7	12,037.4	12,023.8	-
dif	5.2	4.8	4.3	3.8	3.1	2.2	1.1	−1.8	-
unif	12,182.5	12,153.5	12,125.9	12,099.8	12,075.9	12,054.7	12,037.4	12,022	12,001.3
$q = 7000$									
act1	172,872.5	172,843	172,815	172,788	172,764	172,742	172,723	172,707	172,686
act2	172,867.3	172,838.3	172,810.6	172,785	172,761	172,740	172,722	172,708.8	-
dif	5.2	4.7	4.4	3	3	2	1	−1.8	-
unif	172,867.3	172,838.3	172,810.6	172,785	172,761	172,740	172,722	172,707	172,686

From the above table, the following consequences can be summarized as:

(1) The cost difference function of $TU_{\alpha}^{1}(q_{qn}, m_{qn}) - TU_{\alpha}^{2}(q_{qn}, m_{qn})$ is increasing in the amount of customers for every specific amount of working servers, which has been shown previously in Theorem 3.

(2) In the light of the above consequence of (1), numerical results of Table 2, and Theorem 4, the amount of customers to activate the group replacement while m_{qn} servers are normally working, $q_{qn}^{*}\left(m_{qn}\right)$, is presented as follows:
$q_{qn}^{*}(0) = 1$: In the situation of all servers are broken down, the group replacement is activated

while the amount of customers raises to 1.

$q_{qn}^{*}(1) = 3$: When one servers is normally working, the group replacement is triggered while the amount of customers increases to 3.

Moreover, $q_{qn}^{*}(2) = 5, q_{qn}^{*}(3) = 6, q_{qn}^{*}(4) = 8, q_{qn}^{*}(5) = 10, q_{qn}^{*}(6) = 18, q_{qn}^{*}(7) = \infty, q_{qn}^{*}(8) = \infty$. Consequently, according to above numerical results, it can be concluded that the group replacement will never be conducted no matter how high the amount of customers in the system after the amount of working servers reaches to 7.

(3) Table 2 also demonstrates that the cost difference function of $TU_{\alpha}^{1}(q_{qn}, m_{qn}) - TU_{\alpha}^{2}(q_{qn}, m_{qn})$ is decreasing in the number of normally working servers for each respective amount of customers in the system.

(4) According the above numerical results of Table 2, the threshold of the number of working servers to activate the group replacement with q_{qn} customers in the system, $m_{qn}^{*}\,(q_{qn})$, is presented as:

$$m_{qn}^{*}(0) = \text{none}, m_{qn}^{*}(1) = \text{none}, m_{qn}^{*}(2) = \text{none}.$$

When there are from 0 to 2 customers in the system, the group replacement will not be activated even all servers are failed.

$$m_{qn}^{*}(3) = 1.$$

When there are 3 customers in the system, the group replacement is activated when the amount of operating servers is reduced to 1.

Moreover, $m_{qn}^{*}(4) = 1, m_{qn}^{*}(5) = 2, m_{qn}^{*}(6) = 3, m_{qn}^{*}(7) = 3, m_{qn}^{*}(8) = 4, m_{qn}^{*}(9) = 4, m_{qn}^{*}(10) = 5, m_{qn}^{*}(11) = 5, m_{qn}^{*}(12) = 5, m_{qn}^{*}(13) = 5, m_{qn}^{*}(14) = 5, m_{qn}^{*}(15) = 5, m_{qn}^{*}(16) = 5, m_{qn}^{*}(17) = 5$.

In these cases of 10 to 17 customers in the system, the group replacements are triggered while the amount of operating servers is dropped to 5.

$$\begin{gathered} m_{qn}^{*}(18) = 6, m_{qn}^{*}(19) = 6, m_{qn}^{*}(20) = 6, m_{qn}^{*}(30) = 6, m_{qn}^{*}(40) = 6, m_{qn}^{*}(50) = 6, m_{qn}^{*}(100) = 6, m_{qn}^{*}(200) = 6, \\ m_{qn}^{*}(300) = 6, m_{qn}^{*}(400) = 6, m_{qn}^{*}(500) = 6, m_{qn}^{*}(1000) = 6, m_{qn}^{*}(2000) = 6, m_{qn}^{*}(3000) = 6, m_{qn}^{*}(4000) = 6, \\ m_{qn}^{*}(5000) = 6, m_{qn}^{*}(6000) = 6, m_{qn}^{*}(7000) = 6. \end{gathered}$$

In these cases of from 18 to 7000 customers in the system, the group replacements are started while the amount of operating servers is decreased to 6.

Actually, according to our numerical results, the cost difference function of $TU_{\alpha}^{1}(q_{qn}, m_{qn}) - TU_{\alpha}^{2}(q_{qn}, m_{qn})$ with 7 operating servers converges to around -1.874 after the amount of customers in system reaches to 100, consequently the threshold of the amount of normally operating servers to start the group replacement is 6 while there are more than 17 customers in the system.

7. Conclusions

This proposed IoT conditioned-based group replacement decision support system first develops the discounted cost model for a service/production system with numerous independent working servers. The original discounted cost model is further revised into an equivalent model to facilitate the proof procedure by applying the uniformization approach. Several significant theoretical properties are proven and many numerical examples are conducted for two kinds of group replacement policies, respectively.

For the first kind of group replacement policy, Theorem 1, Lemma 2, Theorems 2–4 certainly prove that there is a threshold of the amount of customers necessary to activate the group replacement depending on various amount of working servers. These theorems are very difficult to be certified and are the most significant theoretical breakthroughs and the main contribution in this paper. Moreover, numerical examples conducted in Table 2 can also illustrate the above theoretical outcomes already derived for the first class of group replacement policy. Besides, for the second class of group replacement policy, the numerical results of Table 2 definitely confirm that there is a threshold of amount of working servers existed to start the group replacement according to distinct amount of customers in the system.

According to the perspective view of management, our proposed decision support system applies a computer monitoring system to instantaneously evaluate the status of servers by collecting the precise operating data from desired IoT sensors, and the status of customers in the system by collecting the signals of built-in IoT devices from these customers, and further decides whether the group replacement should be conducted based on the group replacement threshold.

Consequently, based on the structure and detailed procedure flow presented in Figures 1 and 2, this proposed IoT condition-based group replacement decision support system can be used to solve the replacement problems of some designated service or production systems with several machines or servers operating in parallel, such as telecommunication and customized production systems, internet service systems, or some parallel subsystems deployed in submarines, satellites, and space stations.

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Appendix A

Proof of Theorem 1.

The proof will be conducted by induction on n.

For $n = 1$, we have

$$TU_\alpha(q_{qn}, m_{qn}; n=1) = \min\{\frac{kq_{qn}}{\overline{\sigma}}, \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{kq_{qn}}{\overline{\sigma}}\},$$

where $\overline{\sigma} \geq \alpha + \sigma + M_{qn}\zeta + M_{qn}\beta + j\gamma$.

$TU_\alpha(q_{qn}, m_{qn}; n=1)$ is the minimum of two increasing functions of q_{qn}, therefore it is obviously increasing in q_{qn}. Suppose $TU_\alpha(q_{qn}, m_{qn}; n)$ is increasing in q_{qn}, then now we need to prove $TU_\alpha(q_{qn}, m_{qn}; n+1)$ is also increasing in q_{qn}.

As we know, $TU_\alpha(q_{qn}, m_{qn}; n+1) = \min\{TU_\alpha^1(q_{qn}, m_{qn}; n+1), TU_\alpha^2(q_{qn}, m_{qn}; n+1)\}$, and if both of $TU_\alpha^1(q_{qn}, m_{qn}; n+1)$ and $TU_\alpha^2(q_{qn}, m_{qn}; n+1)$ are increasing in q_{qn}, then the result holds. We prove these two properties next.

(1) $TU_\alpha^1(q_{qn}, m_{qn}; n+1)$ is increasing in q_{qn}.

Recall (9), it needs to be showed that $TU^1(q_{qn}, m_{qn}; n+1) - TU^1(q_{qn}+1, m_{qn}; n+1) \leq 0$, for all m_{qn}. To show the inequality, two cases for q_{qn} are considered:

Case 1. When $q_{qn} < m_{qn}$:

Then $\min(q_{qn},\ m_{qn}) = q_{qn},\ \min(q_{qn}+1,\ m_{qn}) = q_{qn}+1$, and

$$\begin{aligned}
&TU^1(q_{qn}, m_{qn}; n+1) - TU^1(q_{qn}+1, m_{qn}; n+1) \\
&= -\tfrac{k}{\overline{\sigma}} + \tfrac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+2, m_{qn}; n)] + \tfrac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - \\
&TU_\alpha(q_{qn}+1, m_{qn}-1; n)] + \tfrac{q_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}, m_{qn}; n)] - \tfrac{\zeta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}; n) \\
&+(1 - \tfrac{\alpha+\sigma+m_{qn}\beta+(q_{qn}+1)\zeta}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}+1, m_{qn}; n)] + \tfrac{\zeta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}; n) \\
&= -\tfrac{k}{\overline{\sigma}} + \tfrac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+2, m_{qn}; n)] + \tfrac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - \\
&TU_\alpha(q_{qn}+1, m_{qn}-1; n)] + \tfrac{q_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}, m_{qn}; n)] \\
&+(1 - \tfrac{\alpha+\sigma+m_{qn}\beta+(q_{qn}+1)\zeta}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}+1, m_{qn}; n)].
\end{aligned}$$

Because all items in brackets are non-positive, $TU^1(q_{qn}, m_{qn}; n+1) - TU^1(q_{qn}+1, m_{qn}; n+1) \le 0$ for $q_{qn} < m_{qn}$.

Case 2. When $q_{qn} \ge m_{qn}$:

Then $q_{qn}+1 > m_{qn}$ and min $(q_{qn}, m_{qn}) = \min(q_{qn}+1, m_{qn}) = m_{qn}$, and the difference $TU^1(q_{qn}, m_{qn}; n+1) - TU^1(q_{qn}+1, m_{qn}; n+1)$, according to the induction hypothesis, is non-positive due to all items in brackets are non-positive.

Now since $TU^1(q_{qn}, m_{qn}; n+1) - TU^1(q_{qn}+1, m_{qn}; n+1) \le 0$ for all q_{qn}, it can be concluded that $TU^1(q_{qn}, m_{qn}; n+1)$ is increasing in q_{qn}.

(2) $TU^2_\alpha(q_{qn}, m_{qn}; n+1)$ is increasing in q_{qn}.

Recall (9), it needs to show that $TU^2(q_{qn}, m_{qn}; n+1) - TU^2(q_{qn}+1, m_{qn}; n+1) \le 0$. Two cases for q_{qn} are considered.

Case 1. $q_{qn} < m_{qn}$:

It is clear that min $(q_{qn},\ m_{qn}) = q_{qn}$, min $(q_{qn}+1,\ m_{qn}) = q_{qn}+1$, and

$$\begin{aligned}
&TU^2(q_{qn}, m_{qn}; n+1) - TU^2(q_{qn}+1, m_{qn}; n+1)\\
&= -\tfrac{k}{\overline{\sigma}} + \tfrac{\sigma}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+2, m_{qn}; n)] + \tfrac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) -\\
&TU_\alpha(q_{qn}+1, m_{qn}-1; n)] + \tfrac{q_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}, m_{qn}; n)] - \tfrac{\zeta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}; n) +\\
&\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)] + \Big(1 - \frac{\alpha+\sigma+m\beta+q_{qn}\zeta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}\Big)*\\
&[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}+1, m_{qn}; n)] + \tfrac{\zeta}{\overline{\sigma}}TU_\alpha(q_{qn}+1, m_{qn}; n)\\
&= -\tfrac{k}{\overline{\sigma}} + \tfrac{\sigma}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+2, m_{qn}; n)] + \tfrac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{an}, m_{qn}-1; n) - TU_\alpha(q_{qn}+1, m_{qn}-1; n)]\\
&+ \tfrac{q_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}, m_{qn}; n)] + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)]\\
&+ \Big(1 - \frac{\alpha+\sigma+m\beta+(q+1)\zeta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}\Big)[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}+1, m_{qn}; n)] \le 0.
\end{aligned}$$

Since all items in the above brackets are not positive based on the induction hypothesis, $TU^2(q_{qn}, m_{qn}; n+1) - TU^2(q_{qn}+1, m_{qn}; n+1) \le 0$ for $q_{qn} < m_{qn}$.

Case 2. $q_{qn} \ge m_{qn}$:

It is obvious that $q_{qn}+1 > m_{qn}$ and min $(q_{qn},\ m_{qn}) =$ min $(q_{qn}+1,\ m_{qn}) = m_{qn}$, and the difference $TU^2(q_{qn}, m_{qn}; n+1) - TU^2(q_{qn}+1, m_{qn}; n+1)$ for $q_{qn} \ge m_{qn}$, according to the induction hypothesis, is non-positive due to all items in brackets are non-positive.

Since $TU^2(q_{qn}, m_{qn}; n+1) - TU^2(q_{qn}+1, m_{qn}; n+1) \le 0$ for all q_{qn}, it can be concluded that $TU^2(q_{qn}, m_{qn}; n+1)$ is increasing in q_{qn}.

(3) $TU_\alpha(q_{qn}, m_{qn}; n+1)$ is the minimum of two non-decreasing functions, and this judges that $TU_\alpha(q_{qn}, m_{qn}; n+1)$ is also non-decreasing in q_{qn} and completes the induction proof. Consequently, $TU_\alpha(q_{qn}, m_{qn}; n)$ is increasing in q_{qn} for all n, and by applying standard contentions from successive approximation for discounted Markov decision process, $TU_\alpha(q_{qn}, m_{qn}) = \lim_n TU_\alpha(q_{qn}, m_{qn}; n)$ is also increasing in q_{qn}. The proof is done. □

Proof of Theorem 2.

Since $TU_\alpha(q_{qn}, m_{qn}) = \min\{TU^1_\alpha(q_{qn}, m_{qn}), TU^2_\alpha(q_{qn}, m_{qn})\}$, two cases are considered as follows:

(1) $TU_\alpha(q_{qn}, m_{qn}) = TU^2_\alpha(q_{qn}, m_{qn})$:

The proof is conducted by induction on n. From (9), for $n = 1$, we have

$$\begin{aligned}
&TU_\alpha(q_{qn}, m_{qn}; n=1) - TU_\alpha(q_{qn}, M_{qn}; n=1) = TU^2_\alpha(q_{qn}, m_{qn}; n=1) - TU^1_\alpha(q_{qn}, M_{qn}; n=1)\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}} + \frac{kq_{qn}}{\overline{\sigma}} - \frac{kq_{qn}}{\overline{\sigma}} = \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}.
\end{aligned}$$

It is obvious that $TU_\alpha(q_{qn}, m_{qn}; n=1) - TU_\alpha(q_{qn}, M_{qn}; n=1)$ is non-decreasing in q_{qn}.

Assume that $TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)$ is increasing in q_{qn}, then it must be shown that $TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)$ is also increasing in q_{qn}. From (9), it can be seen that

$$\begin{aligned}
&TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1) = TU^2_\alpha(q_{qn}, m_{qn}; n+1) - TU^1_\alpha(q_{qn}, M_{qn}; n+1)\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)]+\\
&[\frac{\zeta\min(q_{qn},\ m_{qn})}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - \frac{\zeta\min(q_{qn},\ M_{qn})}{\overline{\sigma}}TU_\alpha(q_{qn}-1, M_{qn}; n)+\\
&[\frac{m_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}-1; n) - \frac{M_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}-1; n)] + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}; n)+\\
&[(1 - \frac{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})TU_\alpha(q_{qn}, m_{qn}; n) - (1 - \frac{\alpha+\sigma(q_{qn},M_{qn})}{\overline{\sigma}})TU_\alpha(q_{qn}, M_{qn}; n)].
\end{aligned} \tag{A1}$$

Now we consider three cases for q_{qn}:

Case 1. $q_{qn} \geq M_{qn}$

For this case, $\min(q_{qn},\ m_{qn}) = m_{qn}$ and $\min(q_{qn},\ M_{qn}) = M_{qn}$, hence (A1) becomes

$$\begin{aligned}
&TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)]+\\
&\frac{m_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}-1, M_{qn}; n)] - \frac{(M_{qn}-m_{qn})\zeta}{\overline{\sigma}}TU_\alpha(q_{qn}-1, M_{qn}; n)+\\
&\frac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}-1; n)] - \frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}-1; n)+\\
&\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}; n) + (1 - \frac{\alpha+\sigma+M_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n)-\\
&TU_\alpha(q_{qn}, M_{qn}; n)] + \frac{(M_{qn}-m_{qn})\zeta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}; n) + \frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}; n)-\\
&\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}; n)\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)]+\\
&+\frac{m_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}-1, M_{qn}; n)] + \frac{(M_{qn}-m_{qn})\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}-1, M_{qn}; n)]\\
&+\frac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}-1; n)] + \frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}-1; n)]\\
&+(1 - \frac{\alpha+\sigma+M_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)].
\end{aligned}$$

After rearranging terms, we obtain

$$\begin{aligned}
&TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (\ddot{M}_{qn} - m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)]+\\
&\frac{m_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}-1, M_{qn}; n)] + \frac{(M_{qn}-m_{qn})\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]\\
&+\frac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)] + \frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]\\
&+\{(1 - \frac{\alpha+\sigma+M_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}) * [TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{(M_{qn}-m_{qn})\zeta}{\overline{\sigma}} *\\
&[TU_\alpha(q_{qn}-1, M_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{M_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)]\}.
\end{aligned} \tag{A2}$$

According to the induction hypothesis, the six items in (A2) are increasing in q_{qn}.
To complete this proof, it needs to be showed that

$$\begin{aligned}
&\Delta^1(q_{qn}, m_{qn}; n+1) = (1 - \frac{\alpha+\sigma+M_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]\\
&-\frac{(M_{qn}-m_{qn})\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, M_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{M_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)]
\end{aligned}$$

is also increasing in q_{qn}.

Recall that parameter $\overline{\sigma}$ applied in the uniformization approach should be greater than $\overline{\sigma} = \alpha + \sigma + M_{qn}\beta + M_{qn}\zeta + j\gamma$. Now we choose $\overline{\sigma} \geq \alpha + \sigma + M^*_{qn}\beta + M^*_{qn}\zeta + j\gamma$, where M^*_{qn} is sufficiently large number. Then we rearrange $\Delta^1(q_{qn}, m_{qn}; n+1)$ as follows:

$$\Delta^1(q, m; n+1) = (1 - \frac{\alpha+\sigma+M^*_{qn}\zeta+M^*_{qn}\beta+\frac{j\gamma}{M^*_{qn}-m_{qn}}}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]$$
$$+\frac{(M^*_{qn}-M_{qn})\zeta+(M^*_{qn}-M_{qn})\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{(M_{qn}-m_{qn})\zeta}{\overline{\sigma}}*$$
$$[TU_\alpha(q_{qn}-1, M_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{M_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)].$$

In the above expression, as a result of large M^*_{qn}, the coefficient of the second increasing term $\frac{(M^*_{qn}-M_{qn})\zeta+(M^*_{qn}-M_{qn})\beta}{\overline{\sigma}}$ is almost round 1, as the coefficient of the last two decreasing terms is extremely small. According to the outcome of Lemma 2, we also know $TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}+1, m_{qn}; n)$ is bounded for every state (q_{qn}, m_{qn}). Therefore, the entire expression of $\Delta^1(q, m; n+1)$ should be increasing in q_{qn} owing to sufficiently large M^*_{qn}.

Therefore, $TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)$ is increasing in q when $q_{qn} \geq M_{qn}$.

Case 2. $m_{qn} \leq q_{qn} < M_{qn}$

In this case, $\min(q_{qn}, M_{qn}) = q_{qn}$ and $\min(q_{qn}, m_{qn}) = m_{qn}$, therefore (A2) becomes

$$\begin{aligned}
&TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn}-m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)] + \frac{m_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n)\\
&-TU_\alpha(q_{qn}-1, M_{qn}; n)] - \frac{(q_{qn}-m_{qn})\zeta}{\overline{\sigma}}TU_\alpha(q_{qn}-1, M_{qn}; n) + \frac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}-1; n)]-\\
&\frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}-1; n) + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}; n) + (1 - \frac{\alpha+\sigma+q_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})*\\
&[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] + \frac{(q_{qn}-m_{qn})\zeta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}; n) + \frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}; n)-\\
&\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}; n)\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn}-m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)] + \frac{m_{qn}\zeta}{\overline{\sigma}}*\\
&[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}-1, M_{qn}; n)] + \frac{(q_{qn}-m_{qn})\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}-1, M_{qn}; n)]+\\
&\frac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}-1; n)] + \frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}-1; n)]+\\
&(1 - \frac{\alpha+\sigma+q_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn}-m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)] + \frac{m_{qn}\zeta}{\overline{\sigma}}*\\
&[TU_\alpha(q_{qn}-1, m_{qn}; n) - U_\alpha(q_{qn}-1, M_{qn}; n)] + \frac{(q_{qn}-m_{qn})\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]+\\
&\frac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)] + \frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]+\\
&\{(1 - \frac{\alpha+\sigma+q_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\lambda}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{(q_{qn}-m_{qn})\zeta}{\overline{\sigma}}*\\
&[TU_\alpha(q_{qn}-1, M_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{M_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)]\}.
\end{aligned} \tag{A3}$$

According to the induction hypothesis, the first six terms in the above (A3) are clearly increasing in q_{qn}.

To complete this proof, it need to be showed that $\Delta^2(q_{qn}, m_{qn}; n+1)$ is also increasing in q_{qn}, where

$$\Delta^2(q_{qn}, m_{qn}; n+1) = (1 - \frac{\alpha+\sigma+q_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]$$
$$-\frac{(q_{qn}-m_{qn})\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, M_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{M_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)].$$

Here we use the same assumption $\overline{\sigma} \geq \alpha + \sigma + M^*_{qn}\beta + M^*_{qn}\zeta + j\gamma$ as case 1. Then we rearrange $\Delta^2(q_{qn}, m_{qn}; n+1)$ as follows:

$$\Delta^2(q_{qn}, m_{qn}; n+1) = (1 - \frac{\alpha+\sigma+M^*_{qn}\zeta+M^*_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]$$
$$+(1 - \frac{\left(M^*_{qn}-q_{qn})\zeta+(M^*_{qn}-M_{qn})\beta+\frac{j\gamma}{M_{qn}-m_{qn}}\right.}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]$$
$$-\frac{(q_{qn}-m_{qn})\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, M_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{M_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)].$$

In the above expression, according to the induction hypothesis, the first two items are clearly increasing in q_{qn}. Due to large M^*_{qn}, the coefficient of the second increasing item $\frac{(M^*_{qn}-q_{qn})\zeta+(M^*_{qn}-M_{qn})\beta}{\overline{\sigma}}$ is almost near 1, as the coefficients of the last two decreasing items are extremely tiny. According to the outcome of Lemma 2, it is obvious that $TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}+1, m_{qn}; n)$ is bounded for every state (q_{qn}, m_{qn}). Therefore, the entire expression of $\Delta^2(q_{qn}, m_{qn}; n+1)$ should be increasing in q_{qn} for M^*_{qn} sufficiently large.

Therefore, $TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)$ is increasing in q_{qn} when $m_{qn} \leq q_{qn} < M_{qn}$.

Case 3. $q_{qn} < m_{qn}$

In this case, $\min(q_{qn},\ M_{qn}) = \min(q_{qn},\ m_{qn}) = q_{qn}$, therefore (A2) becomes

$$\begin{aligned}
&TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)]+\\
&\frac{q_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}-1, M_{qn}; n)] + \frac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}-1; n)]\\
&-\frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}-1; n) + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}; n) + (1 - \frac{\alpha+\sigma+q_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})*\\
&[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] + \frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}; n) - \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}; n)\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)]\\
&+\frac{q_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}-1, M_{qn}; n)] + \frac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}-1; n)]\\
&+\frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}-1; n)] + (1 - \frac{\alpha+\sigma+q_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\lambda}})*\\
&[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]\\
&= \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}[F + (M_{qn} - m_{qn})r_c] + \frac{g}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)]\\
&+\frac{q_{qn}\zeta}{\overline{\sigma}}[TU_\alpha(q_{qn}-1, m_{qn}; n) - TU_\alpha(q_{qn}-1, M_{qn}; n)] + \frac{m_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)]\\
&+\frac{(M_{qn}-m_{qn})\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] + \{(1 - \frac{\alpha+\sigma+q_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\lambda}})*\\
&[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{M_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)]\}.
\end{aligned} \tag{A4}$$

According to the induction hypothesis, the first five items in above (A4) are increasing in q_{qn}. To complete the proof, it needs to be showed that $\Delta^3(q_{qn}, m_{qn}; n+1)$ is increasing in q_{qn}, where

$$\Delta^3(q_{qn}, m_{qn}; n+1) = (1 - \frac{\alpha+\sigma+q_{qn}\zeta+M_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)]$$
$$-\frac{M_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)].$$

We first use the same assumption $\overline{\sigma} \geq \alpha + \sigma + M^*_{qn}\beta + q_{qn}\zeta + j\gamma$ as case 1 and case 2, then we rearrange $\Delta^3(q_{qn}, m_{qn}; n+1)$ as follows:

$$\begin{aligned}
&\Delta^3(q_{qn}, m_{qn}; n+1) \\
&= (1 - \frac{\alpha+\sigma+q_{qn}\zeta+M^*_{qn}\beta+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\lambda}})[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] + \frac{(M^*_{qn}-M_{qn})\beta}{\overline{\sigma}} * \\
&[TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)] - \frac{M_{qn}\beta}{\overline{\sigma}}[TU_\alpha(q_{qn}, M_{qn}-1; n) - TU_\alpha(q_{qn}, M_{qn}; n)].
\end{aligned}$$

In the above expression, by the induction hypothesis, the first two terms are clearly increasing in q_{qn}. Due to extremely largest M^*_{qn}, the coefficient of the second item, $\frac{(M^*_{qn}-M_{qn})\beta}{\overline{\sigma}}$ is almost near 1, while the coefficients of the last item is extremely small. According to the outcome of Lemma 2, it is obvious that $TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}+1, m_{qn}; n)$ is bounded for every state (q_{qn}, m_{qn}). Therefore, according to a similar argument in case 1 and case 2, the entire expression of $\Delta^3(q_{qn}, m_{qn}; n+1)$ should be increasing in q_{qn} for sufficiently largest M^*_{qn}.

Depending on the outcomes of the above three cases, it can be concluded that $TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)$ is increasing in q_{qn} when $TU_\alpha(q_{qn}, m_{qn}; n+1) = TU^2_\alpha(q_{qn}, m_{qn}; n+1)$.

(2) $TU_\alpha(q_{qn}, m_{qn}) = TU^1_\alpha(q_{qn}, m_{qn})$:

This proof will be implemented by induction on n. From (9), for $n = 1$, it is noted that

$$TU_\alpha(q_{qn}, m_{qn}; n=1) - TU_\alpha(q_{qn}, M_{qn}; n=1) = \frac{kq_{qn}}{\overline{\sigma}} - \frac{kq_{qn}}{\overline{\sigma}} = 0.$$

It is clear that $TU_\alpha(q_{qn}, m_{qn}; n=1) - TU_\alpha(q_{qn}, M_{qn}; n=1)$ is non-decreasing in q_{qn}.

From the above result, it can be assumed that $TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)$ is increasing in q_{qn}, then it need to be proved that $TU_\alpha\big(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1\big)$ is also increasing in q_{qn}. From (9), we have

$$\begin{aligned}
&TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1) \\
&= \tfrac{\sigma}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)] + [\tfrac{\zeta\min(q_{qn},m_{qn})}{\overline{\sigma}}TU_\alpha(q_{qn}-1, m_{qn}; n) \\
&-\tfrac{\zeta\min(q_{qn},M_{qn})}{\overline{\sigma}}TU_\alpha(q_{qn}-1, M_{qn}; n)] + [\tfrac{m_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}-1; n) - \tfrac{M_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}-1; n)] \\
&+[(1 - \tfrac{\alpha+\sigma(q_{qn},m_{qn})}{\overline{\sigma}})TU_\alpha(q_{qn}, m_{qn}; n) - (1 - \tfrac{\alpha+\sigma(q_{qn},M_{qn})}{\overline{\sigma}})TU_\alpha(q_{qn}, M_{qn}; n)] \\
&= \tfrac{\sigma}{\overline{\sigma}}[TU_\alpha(q_{qn}+1, m_{qn}; n) - TU_\alpha(q_{qn}+1, M_{qn}; n)] + [\tfrac{\zeta\min(q_{qn},m_{qn})}{\overline{\sigma}}TU_\alpha(q_{qn}-1, m_{qn}; n) \\
&-\tfrac{\zeta\min(q_{qn},M_{qn})}{\overline{\sigma}}TU_\alpha(q_{qn}-1, M_{qn}; n)] + [\tfrac{m\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}-1; n) - \tfrac{M_{qn}\beta}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}-1; n)] \\
&+\frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}; n) + [(1 - \frac{\alpha+\sigma(q_{qn},m_{qn})+\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}})TU_\alpha(q_{qn}, m_{qn}; n) \\
&-(1 - \tfrac{\alpha+\sigma(q_{qn},M_{qn})}{\overline{\sigma}})TU_\alpha(q_{qn}, M_{qn}; n)] + \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, m_{qn}; n) - \frac{\frac{j\gamma}{M_{qn}-m_{qn}}}{\overline{\sigma}}TU_\alpha(q_{qn}, M_{qn}; n).
\end{aligned} \tag{A5}$$

Clearly, the Equation (A5) is the above Equation (A1) plus the last two items. It is obvious that the part of last two items is increasing to q_{qn}. Moreover, according to the previous result of Equation (A1), it can be concluded that $TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)$ is increasing in q when $TU_\alpha(q_{qn}, m_{qn}; n+1) = TU^1_\alpha(q_{qn}, m_{qn}; n+1)$.

Since we have first proved $TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)$ is increasing in q_{qn} when $TU_\alpha(q_{qn}, m_{qn}; n+1) = TU^2_\alpha(q_{qn}, m_{qn}; n+1)$, it can be concluded that $TU_\alpha(q_{qn}, m_{qn}; n+1) - TU_\alpha(q_{qn}, M_{qn}; n+1)$ is increasing in q_{qn}.

Based on standard arguments applying successive approximation for discounted MDPs, $TU_\alpha(q_{qn}, m_{qn}) - TU_\alpha(q_{qn}, M_{qn}) = \lim_n\{TU_\alpha(q_{qn}, m_{qn}; n) - TU_\alpha(q_{qn}, M_{qn}; n)\}$ is certainly increasing in q_{qn}. Consequently, the proof is done. □

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