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Robust Stability of Hurwitz Polynomials Associated with Modified Classical Weights

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Abstract: In this contribution, we consider sequences of orthogonal polynomials associated with a perturbation of some classical weights consisting of the introduction of a parameter t, and deduce some algebraic properties related to their zeros, such as their equations of motion with respect to t. These sequences are later used to explicitly construct families of polynomials that are stable for all values of t, i.e., robust stability on these families is guaranteed. Some illustrative examples are presented.

Keywords: Hurwitz polynomials; robust stability; orthogonal polynomials; Laguerre-type weight; Jacobi-type weight

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1. Introduction

The study of Hurwitz (or stable) polynomials, i.e., polynomials with all zeros in the left half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$, is motivated by the fact that they characterize stable linear systems. More precisely, a continuous linear system is asymptotically stable if and only if its characteristic polynomial is Hurwitz. They were introduced in [1], and since then have been widely studied in the literature. Basic information about Hurwitz polynomials can be found, for instance, in [2,3]. A problem that has received a lot of attention from the research community has been the development of methods to determine if a given polynomial with real coefficients is Hurwitz without explicitly computing its zeros. The most well-known procedure to determine the Hurwitz character of a polynomial is probably the Routh-Hurwitz criterion.

On the other hand, in real life the mathematical model of a system is usually a linear approximation that is made to simplify the analysis and design of a given problem, and thus the resulting characteristic polynomial includes some amount of uncertainty. Moreover, as the operating point of the system changes, so do the parameters of the corresponding linear approximation and, as a consequence, there is a significant amount of uncertainty in the model and it becomes necessary the use of a control system that stabilizes the operation of the system for all the expected range of variation in the parameters. This field of study is known as robust control [4].

Definition 1. *Let $n, d \in \mathbb{N}$. An uncertain polynomial in the variable x is a polynomial whose coefficients depend of the entries of a vector of uncertain parameters $\mathbf{q} \in Q \subset \mathbb{R}^d$, i.e.*

$$P(x, \mathbf{q}) = \sum_{k=0}^{n} a_k(\mathbf{q})\, x^k.$$

The set $\mathcal{P} = \{P(x, \mathbf{q}) : \mathbf{q} \in Q\}$ is called the family of uncertain polynomials [5]. If $P(x, \mathbf{q})$ is a Hurwitz polynomial for every value of $\mathbf{q}$, we say $\mathcal{P}$ is a robustly stable family.

The main results of this contribution are Theorems 1, 2, 3 and 4, where sequences of stable Hurwitz polynomials are constructed by using Laguerre-type and Jacobi-type orthogonal polynomials in such a way that they will be robustly stable with respect to a single uncertainty parameter $\mathbf{q} = t \in \mathbb{R}$, i.e., $d = 1$. This property can be potentially used in the design of control systems as a desired condition of stability. The structure of the manuscript is as follows. Section 2 is devoted to a basic introduction of orthogonal polynomials and their relationship with Hurwitz polynomials, as given in [6]. In Section 3 we introduce sequences of Laguerre-type and Jacobi-type orthogonal polynomials by means of the introduction of a parameter t on the orthogonality weight, and obtain algebraic properties related to their zeros. These sequences are later used in Section 4 to construct families of Hurwitz polynomials that will be robustly stable for all values of t in a certain interval. In Section 5 some numerical examples showing the location of the zeros for certain Hurwitz polynomials are given. Finally, the main conclusions of this work, as well as some open problems constituting future research directions, are presented in Section 6.

2. Preliminaries

2.1. Orthogonal Polynomials

Let $\mathbb{P}$ be the linear space of polynomials with real coefficients. A sequence $\{P_n\}_{n\geqslant 0}$ of monic polynomials in $\mathbb{P}$ satisfying

$$\langle P_n, P_m\rangle_\omega = \int_E P_n(x)\, P_m(x)\, \omega(x)\, dx = K_n \delta_{n,m}, \quad K_n > 0, \qquad m, n \in \mathbb{N} \cup \{0\},$$

where ω is a positive weight function supported on some interval $E \subset \mathbb{R}$, and $\delta_{n,m}$ is the Kronecker's delta, is said to be the sequence monic orthogonal polynomials (SMOP) associated with ω. Algebraic and analytic properties of orthogonal polynomials have been widely studied in the literature, since they have applications in a wide range of topics such as approximation theory, quadrature formulas, physics, signal processing, stochastical processes, control theory, among many others.

The most studied SMOP in the literature are the Jacobi, Laguerre, and Hermite polynomials, known as classical orthogonal polynomials, and are orthogonal with respect to the beta, gamma, and normal distributions, respectively (see Table 1). For a full treatment of the general properties of orthogonal polynomials, including classical polynomials, we refer the reader to [7–12].

Table 1. Classical orthogonal polynomials and their respective functions of orthogonality on the real line.

	Jacobi	**Laguerre**	**Hermite**
Parameters	$\alpha, \beta > -1$	$\alpha > -1$	none
Weight	$(1-x)^\alpha\,(x+1)^\beta$	$x^\alpha\, e^{-x}$	e^{-x^2}
Interval	$[-1, 1]$	$(0, \infty)$	$(-\infty, \infty)$

In particular, in this contribution we deal with two important properties of orthogonal polynomials:

- Three-term recurrence relation: $\{P_n\}_{n\geqslant 0}$ satisfies

$$xP_n(x) = P_{n+1} + b_n P_n(x) + a_n P_{n-1}(x), \qquad n \in \mathbb{N}, \quad P_0(x) = 1, \tag{1}$$

where $\{b_n\}_{n\geqslant 0}$ and $\{a_n\}_{n\geqslant 1}$ are sequences of real numbers that depend on the orthogonality weight.

- Zeros: For $n \geqslant 1$, the zeros of $P_n(x)$ are real, simple, and located in the convex hull of E. Moreover, the zeros of $P_n(x)$ and $P_{n+1}(x)$ interlace. Furthermore, the second kind polynomial associated with $P_n(x)$ is defined by (see [13], 1.3)

$$Q_n(x) = \int_E \frac{P_n(s) - P_n(x)}{s - x}\, \omega(s)\, ds,$$

and plays a central role in approximation theory [10]. Notice that $Q_n(x)$ has degree $n-1$ and its zeros also satisfy an interlacing property with the zeros of $P_n(x)$ [10].

In particular, we will be dealing with Laguerre and Jacobi polynomials. Laguerre polynomials are defined by the inner product

$$\langle p, q\rangle_\alpha = \int_{\mathbb{R}_+} p(x)\, q(x)\, \omega(x,\alpha)\, dx, \qquad p, q \in \mathbb{P}, \tag{2}$$

where $\omega(x,\alpha)\, dx = x^\alpha e^{-x} dx$, with $\alpha > -1$. Some algebraic properties of Laguerre (monic) orthogonal polynomials that will be used in the sequel are stated in the following Proposition.

Proposition 1. *Let $\{L_n^\alpha\}_{n\geqslant 0}$ denote the sequence of classical Laguerre monic polynomials orthogonal with respect to (2). The following statements hold [12]:*

(i) *Three-term recurrence relation. For $n \in \mathbb{N}$,*

$$xL_n^\alpha(x) = L_{n+1}^\alpha(x) + (2n+\alpha+1)\, L_n^\alpha(x) + n(n+\alpha)\, L_{n-1}^\alpha(x), \qquad L_0^\alpha(x) = 1. \tag{3}$$

(ii) *Norm. We will denote by $\|L_n^\alpha\|_\alpha^2 = \langle L_n^\alpha, L_n^\alpha\rangle_\alpha$ the corresponding squared norm. We have*

$$\|L_n^\alpha\|_\alpha^2 = n!\,\Gamma(n+\alpha+1), \qquad n \geqslant 0,$$

where Γ is the Gamma function.

(iii) *Hypergeometric function. For $n \in \mathbb{N}$,*

$$L_n^\alpha(x) = \frac{n!}{(-1)^n} \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{(n-k)!\,\Gamma(\alpha+k+1)} \frac{(-x)^k}{k!}. \tag{4}$$

On the other hand, the Jacobi polynomials are orthogonal with respect to the inner product

$$\langle p, q\rangle_{\alpha,\beta,a,b} = \int_a^b p(x)\, q(x)\, \omega(x,\alpha,\beta,a,b)\, dx, \qquad p, q \in \mathbb{P}, \tag{5}$$

where $\omega(x,\alpha,\beta,a,b)\, dx = (b-x)^\alpha\, (x-a)^\beta\, dx$ is supported on the interval $[a,b]$ (it is understood that $-\infty < a < b < \infty$) and $\alpha, \beta > -1$. Some special cases of the classical Jacobi weights are the following (see [9,10,12] for more details). For $\alpha = \beta = 0$, Legendre polynomials are generated; if $\alpha = \beta = -1/2$, we obtain Chebychev polynomials of the first kind; Chebychev polynomials of the second kind are generated when $\alpha = \beta = 1/2$; and we get Gegenbauer polynomials when $\alpha = \beta = \gamma - 1/2$ with $\gamma > -1/2$. Typically, the values $a = -1$ and $b = 1$ are considered, and the corresponding polynomials are denoted by $\{P_n^{\alpha,\beta}\}_{n\geqslant 0}$. Some of their properties are stated in the following Proposition.

Proposition 2. *Let $\{P_n^{\alpha,\beta}\}_{n\geqslant 0}$ and $\{P_n^{\alpha,\beta,a,b}\}_{n\geqslant 0}$ denote the sequences of Jacobi monic polynomials orthogonal with respect to the inner products $\langle p,q\rangle_{\alpha,\beta,-1,1}$ and $\langle p,q\rangle_{\alpha,\beta,a,b}$, respectively, defined by (5). Taking into account $P_0^{\alpha,\beta}(x) = 1$ and $P_1^{\alpha,\beta}(x) = x + \frac{\alpha-\beta}{\alpha+\beta+2}$, the following statements hold [12].*

(i) *Three-term recurrence relation. For $n \in \mathbb{N}$,*

$$xP_n^{\alpha,\beta}(x) = P_{n+1}^{\alpha,\beta}(x) + \epsilon_n^{\alpha,\beta} P_n^{\alpha,\beta}(x) + \varepsilon_n^{\alpha,\beta} P_{n-1}^{\alpha,\beta}(x), \tag{6}$$

with

$$\epsilon_n^{\alpha,\beta} = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

and

$$\varepsilon_n^{\alpha,\beta} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}.$$

(ii) *Hypergeometric function. For $n \in \mathbb{N}$, the Jacobi polynomials satisfy*

$$\begin{aligned} P_n^{\alpha,\beta,a,b}(x) &= \frac{(b-a)^n}{2^n} P_n^{\alpha,\beta}\left(2\frac{x-a}{b-a} - 1\right) \\ &= \frac{(b-a)^n}{(n+\alpha+\beta+1)_n} \times \\ &\quad \sum_{k=0}^{n} \binom{n}{k} (n+\alpha+\beta+1)_k (\alpha+k+1)_{n-k} \left(\frac{x-a}{b-a} - 1\right)^k, \end{aligned} \tag{7}$$

where $(\cdot)_n$ denotes the Pochhammer's symbol.

2.2. Relation between Hurwitz and Orthogonal Polynomials

To get our results, we follow the ideas in the recent paper [6]. Therein, the authors derived explicit connection formulas between SMOP (with respect to weights supported on positive intervals of the real line) and Hurwitz polynomials. Such a relation had been studied previously in the literature. In [14,15], the authors establish the existence of a one-to-one correspondence between a Hurwitz polynomial and a (finite) sequence of orthogonal polynomials. Furthermore, both topics have well-known connections with Padé approximants [16], the moment problem theory [2,17,18], continued fractions [19–21], total positivity of matrices [22], positive functions [23] and the stability and robust stabilization of continuous linear systems [24,25]. More precisely, the next results show how to compute a sequence of Hurwitz polynomials from a SMOP.

Proposition 3. *Let $\{P_n\}_{n\geqslant 0}$ be a sequence of monic polynomials orthogonal with respect to some positive weight function $\omega(x)$ supported in $(0,\infty)$. Let $\{Q_n\}_{n\geqslant 0}$ be the corresponding second kind polynomials. Then [6],*

$$\digamma_{2n}(x) = (-1)^n P_n(-x^2) + (-1)^{n-1} x\, Q_n(-x^2), \qquad n \geqslant 0, \tag{8}$$

is a Hurwitz polynomial of degree $2n$. On the other hand, define $G_n(x) = Q_n(x) + \mathfrak{b}_n P_n(x)$ with $\mathfrak{b}_n > 0$ such that the zeros of $P_n(x)$ and $G_n(x)$ are positive and interlace. Then,

$$\digamma_{2n+1}(x) = (-1)^n x\, P_n(-x^2) + (-1)^n G_n(-x^2), \qquad n \geqslant 0, \tag{9}$$

is a Hurwitz polynomial of degree $2n+1$. Notice that $\mathfrak{b}_n$ is the coefficient of x^{2n} of $\digamma_{2n+1}(x)$. For $\digamma_1(x)$ we can choose any $\mathfrak{b}_1 > 0$.

Notice that the parameter used to compute $G_n(x)$ in the odd degree case depends on n. In some cases, it is possible to choose a single parameter $\mathfrak{b}_n = \mathfrak{b}$ (independent of n) such that the interlacing condition holds for every $n \geqslant 1$. In such a case, the constructed sequence of Hurwitz polynomials satisfies the following recurrence relation.

Proposition 4. *If there exists a finite M such that* $-\frac{Q_n(0)}{P_n(0)} \leqslant M$ *for every* $n \in \mathbb{N}$*, then the sequence* $\{\mathcal{F}_n\}_{n\geqslant 1}$ *of Hurwitz polynomials constructed via Proposition 3 with* $\mathfrak{b}_n = \mathfrak{b} > M$ *satisfies [6]*

$$\mathcal{F}_n(x) = (x^2 + b_{\lfloor \frac{n}{2} \rfloor -1})\, \mathcal{F}_{n-2}(x) - a_{\lfloor \frac{n}{2} \rfloor -1}\, \mathcal{F}_{n-4}(x), \qquad n \geqslant 2, \tag{10}$$

with initial conditions

$$\begin{aligned} a_0\, \mathcal{F}_{-2}(x) &= -x, & \mathcal{F}_0(x) &= 1, \\ a_0\, \mathcal{F}_{-1}(x) &= 1, & \mathcal{F}_1(x) &= x + \mathfrak{b}, \end{aligned}$$

where $\lfloor y \rfloor = max\{m \in \mathbb{Z} : m \leqslant y\}$ *and* $\{a_n\}_{n\geqslant 1}$*,* $\{b_n\}_{n\geqslant 0}$ *are the coefficients in (1).*

3. Connection Formulas and Behavior Of Zeros

In this section, we consider SMOPs associated with a perturbation of a classical weight, by means of the introduction of a positive parameter t. The idea is to generate families of t-dependent SMOPs, and obtain connection formulas for these sequences in terms of the classical orthogonal polynomials. Such sequences will be used in the next Section to construct sequences of polynomials that will be Hurwitz for every value of t, by using Proposition 3. Since we require $E \subset \mathbb{R}_+$, the natural choice is to work with Laguerre polynomials. Later on, we also consider Jacobi polynomials on the interval $[0, 1]$.

3.1. Laguerre-Type Orthogonal Polynomials

Let us consider the Laguerre-type inner product

$$\langle p, q \rangle_{\alpha,t} = \int_{\mathbb{R}_+} p(x)\, q(x)\, \omega_t(x, \alpha)\, dx, \qquad p, q \in \mathbb{P}, \tag{11}$$

where $\omega_t(x, \alpha) = x^\alpha e^{-tx}$. Here, $t \in \mathbb{R}_+$ is a parameter that in certain contexts, can be interpreted as the "time" variable. We denote by $\{L_n^{\alpha,t}\}_{n\geqslant 0}$ the corresponding SMOP. Notice that $\omega_1(x, \alpha) = \omega(x, \alpha)$ is the classical Laguerre weight. We point out that another motivation for the study of the weights considered in (11) comes from random matrix theory [26]. The corresponding second kind polynomials will be denoted by $\{Q_n^{\alpha,t}\}_{n\geqslant 0}$. The $(n-1)$-th degree second kind polynomial is defined by

$$Q_n^{\alpha,t}(x) = \int_{\mathbb{R}_+} \frac{L_n^{\alpha,t}(s) - L_n^{\alpha,t}(x)}{s - x}\, \omega_t(s, \alpha)\, ds. \tag{12}$$

First, let us establish a relation between the Laguerre-type polynomials defined by the inner product (11) and the classical Laguerre orthogonal polynomials. For this purpose, we set $\xi \in \mathbb{R}$ and $t > 0$ and we expand $L_n^{\alpha,t}(\xi/t)$ in terms of the SMOP $\{L_k^\alpha(\xi)\}_{k\geqslant 0}$, which is an orthogonal basis in $\mathbb{P}$, as follows

$$L_n^{\alpha,t}(\xi/t) = t^{-n} L_n^\alpha(\xi) + \sum_{k=0}^{n-1} \Theta_{n,k}\, L_k^\alpha(\xi), \tag{13}$$

where

$$\Theta_{n,k} = \frac{\left\langle L_n^{\alpha,t}(\xi/t), L_k^\alpha(\xi) \right\rangle_\alpha}{\|L_k^\alpha\|_\alpha^2}, \qquad 0 \leqslant k \leqslant n-1.$$

Taking into account

$$t^{\alpha+1} \omega_t(x, \alpha)\, dx = t^{\alpha+1} \frac{\xi^\alpha}{t^\alpha} e^{-\xi} \frac{d\xi}{t} = \omega_1(\xi, \alpha)\, d\xi,$$

with $\xi = tx$, we get

$$\left\langle t^{\alpha+1} f(x), g(tx) \right\rangle_{\alpha,t} = \langle f(\xi/t), g(\xi) \rangle_{\alpha}, \qquad f, g \in \mathbb{P}. \tag{14}$$

As a straightforward consequence, the following generalization of the hypergeometric function in Proposition 1 for the Laguerre-type polynomials holds.

Proposition 5. *For $n \in \mathbb{N}$, $\alpha > -1$, $t > 0$, and $x \in \mathbb{R}$,*

$$L_n^{\alpha,t}(x) = t^{-n} L_n^{\alpha}(tx) = \frac{n!}{(-1)^n} \sum_{k=0}^{n} \frac{(-1)^k \Gamma(n+\alpha+1)}{(n-k)!\, \Gamma(\alpha+k+1)} \frac{t^{k-n} x^k}{k!}. \tag{15}$$

Proof. Given $n \in \mathbb{N}$, $\alpha > -1$, $t > 0$, and $x \in \mathbb{R}$, and taking $\xi = tx$, from the orthogonality and (14), we obtain

$$0 = \left\langle t^{\alpha+1} L_n^{\alpha,t}(x), L_k^{\alpha,t}(tx) \right\rangle_{\alpha,t} = \left\langle L_n^{\alpha,t}(\xi/t), L_k^{\alpha,t}(\xi) \right\rangle_{\alpha} = \|L_k^{\alpha}\|_{\alpha}^2\, \Theta_{n,k}, \qquad 0 \leqslant k \leqslant n-1.$$

Since $\|L_k^{\alpha}\|_{\alpha}^2 \neq 0$ for every $k \geqslant 0$, $\Theta_{n,k} = 0$ if $0 \leqslant k \leqslant n-1$. Combining the above equality and (4) into (13), the result follows. □

Let us mention two important consequences of the above result. The first one is the following corollary that completes the analogue of Proposition 1 for the Laguerre-type polynomials, and the second one is the description of the equations of motion for the zeros of $L_n^{\alpha,t}(x)$, which are discussed below.

Corollary 1. *Let $\{L_n^{\alpha,t}\}_{n\geqslant 0}$ denote the SMOP with respect to (11). The following statements hold taking into account $L_0^{\alpha,t}(x) = 1$.*

(*i*) *Three-term recurrence relation. For $n \in \mathbb{N}$,*

$$x L_n^{\alpha,t}(x) = L_{n+1}^{\alpha,t}(x) + \frac{2n+\alpha+1}{t} L_n^{\alpha,t}(x) + \frac{n(n+\alpha)}{t^2} L_{n-1}^{\alpha,t}(x). \tag{16}$$

(*ii*) *Norm. For $n \in \mathbb{N}$, we have*

$$\frac{\|L_n^{\alpha,t}\|_{\alpha,t}^2}{\|L_{n-1}^{\alpha,t}\|_{\alpha,t}^2} = \frac{n(n+\alpha)}{t^2}.$$

Proof. It suffices to use (15) together with the analogous formula of Proposition 1 to show the first assertion. For the second, it is known (see [9]) that

$$\frac{\|L_n^{\alpha,t}\|_{\alpha,t}^2}{\|L_{n-1}^{\alpha,t}\|_{\alpha,t}^2}$$

is equivalent to the coefficient of $L_{n-1}^{\alpha,t}(x)$ in the recurrence relation, which is the desired conclusion. □

On the other hand, notice that (15) implies that if $\ell_{n,1}^{\alpha}, \ldots, \ell_{n,n}^{\alpha}$ and $\ell_{n,1}^{\alpha}(t), \ldots, \ell_{n,n}^{\alpha}(t)$ are the n zeros of $L_n^{\alpha}(x)$ and $L_n^{\alpha,t}(x)$, respectively, then

$$0 = L_n^{\alpha,t}\big(\ell_{n,k}^{\alpha}(t)\big) = t^{-n} L_n^{\alpha}\big(t\, \ell_{n,k}^{\alpha}(t)\big), \qquad 1 \leqslant k \leqslant n.$$

Therefore, we get

$$\ell_{n,k}^{\alpha}(t) = \frac{\ell_{n,k}^{\alpha}}{t}, \qquad 1 \leqslant k \leqslant n. \tag{17}$$

We can now, by differentiating the previous equation with respect to t, formulate the following result.

Lemma 1 (Equation of motion for zeros of $L_n^{\alpha,t}(x)$)**.** *Let $n \in \mathbb{N}$, $t > 0$, and $\alpha > -1$. We have*

$$\frac{\partial \ell_{n,k}^{\alpha}(t)}{\partial t} = -\frac{\ell_{n,k}^{\alpha}}{t^2}, \qquad 1 \leqslant k \leqslant n.$$

Next, we will focus our attention on the second kind polynomial $Q_n^{\alpha,t}(x)$ to obtain an expression for its coefficients. Substituting (15) into (12) and taking $\xi = tx$ and $\zeta = ts$ with $x, s \in \mathbb{R}$, we get

$$Q_n^{\alpha,t}(x) = t^{-n-\alpha} \int_{\mathbb{R}_+} \frac{L_n^{\alpha}(\zeta) - L_n^{\alpha}(\xi)}{\zeta - \xi}\, \omega_1(\zeta, \alpha)\, d\zeta = t^{-n-\alpha} Q_n^{\alpha}(tx). \tag{18}$$

Let us denote by $l_{n,k} = l_{n,k}(\alpha)$ the coefficient of the power ξ^k in $L_n^{\alpha}(\xi)$. Then, $Q_n^{\alpha}(\xi)$ can be rewritten as

$$\begin{aligned} Q_n^{\alpha}(\xi) &= \int_{\mathbb{R}_+} \sum_{k=1}^{n} l_{n,k} \frac{\zeta^k - \xi^k}{\zeta - \xi}\, \omega_1(\zeta, \alpha)\, d\zeta \\ &= \sum_{k=1}^{n} \sum_{j=0}^{k-1} l_{n,k}\, \xi^{k-j-1}\, \Gamma(\alpha + j + 1) \\ &= \sum_{k=0}^{n-1} \left(\sum_{j=1}^{n-k} l_{n,k+j} \Gamma(\alpha + j) \right) \xi^k. \end{aligned} \tag{19}$$

Consequently, replacing (19) into (18), we get

Proposition 6. *For $n \in \mathbb{N}$, $\alpha > -1$, $t > 0$, and $x \in \mathbb{R}$,*

$$Q_n^{\alpha,t}(x) = t^{-n-\alpha} \sum_{k=0}^{n-1} \left(\sum_{j=1}^{n-k} \frac{(-1)^{n+k+j}}{(n-k-j)!} \frac{n!\,\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+j+1)} \frac{\Gamma(\alpha+j)}{(k+j)!} \right) t^k x^k, \tag{20}$$

with leading coefficient $t^{-1-\alpha}\,\Gamma(\alpha+1)$.

By using the same arguments as in Lemma 1, the evolution of the zeros of $Q_{n+1}^{\alpha}(x)$ with respect to t is given in the following result.

Lemma 2 (Equation of motion for zeros of $Q_{n+1}^{\alpha,t}(x)$)**.** *Under the hypotheses of Theorem 1, if $q_{n,1}^{\alpha}, \ldots, q_{n,n}^{\alpha}$ and $q_{n,1}^{\alpha}(t), \ldots, q_{n,n}^{\alpha}(t)$ denote the n zeros of $Q_{n+1}^{\alpha}(x)$ and $Q_{n+1}^{\alpha,t}(x)$, respectively, then*

$$\frac{\partial q_{n,k}^{\alpha}(t)}{\partial t} = -\frac{q_{n,k}^{\alpha}}{t^2}, \qquad 1 \leqslant k \leqslant n.$$

3.2. Jacobi-Type Orthogonal Polynomials

For all $t \in (-\infty, 1)$, we consider the SMOP associated with the Jacobi-type inner product defined by

$$\langle p, q \rangle_{\alpha,\beta,t} = \int_0^{1-t} p(x)\, q(x)\, (1 - t - x)^{\alpha}\, x^{\beta}\, dx, \qquad p, q \in \mathbb{P}, \tag{21}$$

i.e., when $b = 1 - t$ and $a = 0$, that we will denote by $\{J_n^{\alpha,\beta,t}\}_{n \geqslant 0}$. The corresponding polynomials of the second kind will be denoted by $\{Q_n^{\alpha,\beta,t}\}_{n \geqslant 0}$, and are defined by

$$Q_n^{\alpha,\beta,t}(x) = \int_0^{1-t} \frac{J_n^{\alpha,\beta,t}(s) - J_n^{\alpha,\beta,t}(x)}{s - x}\, \omega_t(s, \alpha, \beta)\, ds, \tag{22}$$

where $\omega_t(x,\alpha,\beta) = (1-t-x)^{\alpha}\, x^{\beta}$. The corresponding hypergeometric equation of the form (7) is given in the following result.

Proposition 7. *For $n \in \mathbb{N}$, $\alpha, \beta > -1$, $t < 1$, and $x \in \mathbb{R}$,*

$$\begin{aligned} J_n^{\alpha,\beta,t}(x) &= (1-t)^n\, J_n^{\alpha,\beta,0}\left(\frac{x}{1-t}\right) \\ &= \frac{(1-t)^n}{(n+\alpha+\beta+1)_n} \times \\ &\quad \sum_{k=0}^{n} \binom{n}{k} (n+\alpha+\beta+1)_k\, (\alpha+k+1)_{n-k} \left(\frac{x}{1-t}-1\right)^k . \end{aligned} \tag{23}$$

Proof. Let $n \in \mathbb{N}$, $\alpha, \beta > -1$, $t < 1$, and $x \in \mathbb{R}$. If $\xi = \frac{x}{1-t}$, from (7) we obtain

$$J_n^{\alpha,\beta,t}(x) = \frac{(1-t)^n}{2^n}\, P_n^{\alpha,\beta}(2\xi - 1) = (1-t)^n\, J_n^{\alpha,\beta,0}(\xi)\,,$$

which is (23). □

As a consequence, it is easy to deduce the three-term recurrence formula for the Jacobi-type polynomials, as well as the equations of motion for the zeros of $J_n^{\alpha,\beta,t}(x)$.

Corollary 2. *Taking into account $J_0^{\alpha,\beta,t}(x) = 1$ and $J_1^{\alpha,\beta,t}(x) = x - \frac{\beta+1}{\alpha+\beta+2}(1-t)$, for $n \in \mathbb{N}$ the SMOP $\{J_n^{\alpha,\beta,t}\}_{n\geqslant 0}$ satisfies the following three-term recurrence relation*

$$xJ_n^{\alpha,\beta,t}(x) = J_{n+1}^{\alpha,\beta,t}(x) + \frac{1}{2}\,(1+\epsilon_n^{\alpha,\beta})\,(1-t)\, J_n^{\alpha,\beta,t}(x) + \frac{1}{4}\,\varepsilon_n^{\alpha,\beta}\,(1-t)^2\, J_{n-1}^{\alpha,\beta,t}(x). \tag{24}$$

Lemma 3 (Equation of motion for zeros of $J_n^{\alpha,\beta,t}(x)$)**.** *Let n be a positive integer, $\alpha, \beta > -1$, and $t < 1$. If $j_{n,1}^{\alpha,\beta}, \ldots, j_{n,n}^{\alpha,\beta}$ and $j_{n,1}^{\alpha,\beta}(t), \ldots, j_{n,n}^{\alpha,\beta}(t)$ denote the n zeros of $J_n^{\alpha,\beta,0}(x)$ and $J_n^{\alpha,\beta,t}(x)$, respectively, then*

$$\frac{\partial j_{n,k}^{\alpha,\beta}(t)}{\partial t} = -j_{n,k}^{\alpha,\beta}, \qquad 1 \leqslant k \leqslant n.$$

Proof. Let $1 \leqslant k \leqslant n$ and $t < 1$. Evaluating (23) at $x = j_{n,k}^{\alpha,\beta}(t)$ we get

$$0 = J_n^{\alpha,\beta,t}(j_{n,k}^{\alpha,\beta}(t)) = (1-t)^n\, J_n^{\alpha,\beta,0}\left(\frac{j_{n,k}^{\alpha,\beta}(t)}{1-t}\right).$$

Thus,

$$j_{n,k}^{\alpha,\beta}(t) = (1-t)\, j_{n,k}^{\alpha,\beta}.$$

Since k is arbitrary, the t-derivative of the above expression completes the proof. □

Next, we give expressions for the coefficients of $Q_n^{\alpha,\beta,t}(x)$, as well as the equation of motion for their zeros with respect to t.

Proposition 8. *For $n \in \mathbb{N}$, $\alpha, \beta > -1$, $t < 1$, and $x \in \mathbb{R}$,*

$$\begin{aligned} Q_n^{\alpha,\beta,t}(x) &= (1-t)^{n+\alpha+\beta}\, Q_n^{\alpha,\beta,0}\left(\frac{x}{1-t}\right) \\ &= (1-t)^{n+\alpha+\beta} \sum_{k=0}^{n-1} \frac{\Phi_{n,k}^{\alpha,\beta}\,\Gamma(\beta+1)}{(n+\alpha+\beta+1)_n} \left(\frac{x}{1-t}-1\right)^k, \end{aligned} \tag{25}$$

with leading coefficient $(1-t)^{1+\alpha+\beta}\,\mathfrak{B}(\alpha+1,\beta+1)$. *Here,*

$$\Phi_{n,k}^{\alpha,\beta} = \sum_{i=1}^{n-k} (-1)^{i-1} \binom{n}{k+i} \frac{\Gamma(\alpha+i)}{\Gamma(\alpha+\beta+1+i)} (n+\alpha+\beta+1)_{k+i}\, (\alpha+k+i+1)_{n-k-i}$$

and $\mathfrak{B}(\,\cdot\,,\cdot\,)$ denotes the Beta function.

Proof. As in the proof of Proposition 6, if $\xi = \frac{x}{1-t}$ and $\zeta = \frac{s}{1-t}$ with $x, s \in \mathbb{R}$, from (22), (23), and the definition of the Beta function we have

$$\begin{aligned} Q_n^{\alpha,\beta,t}(x) &= (1-t)^{n+\alpha+\beta}\, Q_n^{\alpha,\beta,0}\left(\frac{x}{1-t}\right) \\ &= (1-t)^{n+\alpha+\beta} \sum_{k=0}^{n-1} \left(\sum_{i=1}^{n-k} (-1)^{i-1}\, \mathfrak{j}_{n,k+i}^{\alpha,\beta} \frac{\Gamma(\alpha+i)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1+i)} \right) (\xi-1)^k, \end{aligned}$$

where

$$\mathfrak{j}_{n,\ell}^{\alpha,\beta} = \binom{n}{\ell} \frac{(n+\alpha+\beta+1)_\ell\, (\alpha+\ell+1)_{n-\ell}}{(n+\alpha+\beta+1)_n}, \qquad \ell = 0, \ldots, n. \tag{26}$$

Hence, combining (23) with the above formula, the result follows. □

Thus, proceeding as in the previous Subsection, we have the following result.

Lemma 4 (Equation of motion for zeros of $Q_{n+1}^{\alpha,\beta,t}(x)$)**.** *Under the hypotheses of Lemma 3, if $q_{n,1}^{\alpha,\beta}, \ldots, q_{n,n}^{\alpha,\beta}$ and $q_{n,1}^{\alpha,\beta}(t), \ldots, q_{n,n}^{\alpha,\beta}(t)$ denote the n zeros of $Q_{n+1}^{\alpha,\beta}(x)$ and $Q_{n+1}^{\alpha,\beta,t}(x)$, respectively, then*

$$\frac{\partial q_{n,k}^{\alpha,\beta}(t)}{\partial t} = -q_{n,k}^{\alpha,\beta}, \qquad 1 \leqslant k \leqslant n.$$

4. Sequences of Hurwitz Polynomials Associated with Laguerre-Type and Jacobi-Type Polynomials

In this section, we use Proposition 3 to construct sequences of Hurwitz polynomials associated with both the Laguerre-type and Jacobi-type orthogonal polynomials. Notice that since these sequences are orthogonal for every value of t ($t > 0$ in the Laguerre case and $t < 1$ in the Jacobi case), and satisfy the conditions on Proposition 3, the resulting polynomials will be Hurwitz for all values of t. In other words, we obtain a family of polynomials that is robustly stable. Moreover, since the zeros of the Laguerre-type (Jacobi-type) polynomials tend to the origin as $t \to \infty$ ($t \to 1$), and the same occurs for the zeros of the corresponding second kind polynomials, it is easily deduced from Proposition 3 that the zeros of the Hurwitz polynomials (except for one zero on the odd degree case) also tend to the origin as t increases. Naturally, such zeros can approach any point in the negative real axis by means of a suitable translation on the variable.

4.1. The Laguerre-Type Case

We proceed with the construction of sequences of Hurwitz polynomials, by using Proposition 3. Notice that the even degree case does not require any conditions, and thus the following result is straightforward.

Theorem 1. *For $n \in \mathbb{N}$, $\alpha > -1$, $t > 0$, and $x \in \mathbb{R}$,*

$$\mathcal{F}_{2n}^{\alpha}(x,t) = (-1)^n \, L_n^{\alpha,t}(-x^2) + (-1)^{n-1} \, x \, Q_n^{\alpha,t}(-x^2) \tag{27}$$

is a Hurwitz polynomial with degree $2n$. In fact, the coefficients of the even and odd powers of x in $\mathcal{F}_{2n}^{\alpha}(x,t)$ are given by the coefficients of the power expansions (15) and (20), respectively.

For the odd degree case, there are some restrictions on the parameter $\mathfrak{b}$ that depends on the family of orthogonal polynomials being used. In this case, we first examine the sequence

$$\left\{ -\frac{Q_n^{\alpha,t}(0)}{L_n^{\alpha,t}(0)} \right\}_{n \geqslant 0}. \tag{28}$$

Proposition 9. *For every $t > 0$, the sequence (28)*

(i) converges to $t^{-\alpha} \, \Gamma(\alpha)$ when $\alpha > 0$, and
(ii) diverges when $-1 < \alpha < 0$.

Proof. Notice that evaluating (15) and (20) in $x = 0$, with $n \in \mathbb{N}$, we obtain

$$\frac{Q_n^{\alpha,t}(0)}{L_n^{\alpha,t}(0)} = t^{-\alpha} \, \Gamma(\alpha+1) \sum_{j=1}^{n} \frac{(-1)^j}{(n-j)!} \, \frac{n!}{j!} \, \frac{\Gamma(\alpha+j)}{\Gamma(\alpha+j+1)},$$

and taking into account well-known properties of the Gamma function, we get

$$\frac{Q_n^{\alpha,t}(0)}{L_n^{\alpha,t}(0)} = t^{-\alpha} \, \Gamma(\alpha+1) \sum_{j=1}^{n} \frac{(-1)^j \, n!}{j! \, (n-j)!} \, \frac{1}{\alpha+j}.$$

Let $\alpha \neq 0$. If we compare the above equation with the partial fraction decomposition of the reciprocal function of $\prod_{k=0}^{n} (z+k)$ evaluated at $z = \alpha$ with $n \in \mathbb{N}$, then we have

$$\frac{Q_n^{\alpha,t}(0)}{L_n^{\alpha,t}(0)} = t^{-\alpha} \, \Gamma(\alpha+1) \left[-\frac{1}{\alpha} + \frac{n!}{\prod_{k=0}^{n}(\alpha+k)} \right].$$

Thus, the convergence or divergence of (28) only depends of the convergence or divergence of

$$\left\{ \frac{n!}{\prod_{k=0}^{n}(\alpha+k)} \right\}_{n \geqslant 0}.$$

Furthermore, for $n \in \mathbb{N}$,

$$\frac{n!}{\prod_{k=0}^{n}(\alpha+k)} = \frac{\Gamma(n+1)}{(\alpha)_{n+1}} = \frac{\Gamma(\alpha)\,\Gamma(n+1)}{\Gamma(\alpha+n+1)}.$$

Consequently, taking into account (see, for instance, [27])

$$\lim_{n \to \infty} \frac{\Gamma(n+x)}{\Gamma(n+y)} \, n^{y-x} = 1, \qquad x, y \in \mathbb{R},$$

we have

$$\lim_{n\to\infty} \frac{n!}{\prod_{k=0}^{n}(\alpha+k)} = \Gamma(\alpha) \lim_{n\to\infty} \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} = \Gamma(\alpha) \lim_{n\to\infty} \frac{1}{n^{\alpha}},$$

so that both statements follow. □

With the previous result, we can now apply Proposition 3 for the odd degree case.

Theorem 2. *Let $n \in \mathbb{N}$, α, $t > 0$, and $x \in \mathbb{R}$. Write*

$$G_n^{\alpha,t}(x) = Q_n^{\alpha,t}(x) + \mathfrak{b}\, L_n^{\alpha,t}(x),$$

with $\mathfrak{b} > t^{-\alpha}\,\Gamma(\alpha)$. Then

$$F_{2n+1}^{\alpha}(x,t) = (-1)^n\, x\, L_n^{\alpha,t}(-x^2) + (-1)^n\, G_n^{\alpha,t}(-x^2) \tag{29}$$

is a Hurwitz polynomial with degree $2n+1$. Moreover,

(i) the leading coefficient of $G_n^{\alpha,t}(x)$ is $\mathfrak{b}$,
(ii) for $k = 0, \ldots, n-1$,

$$(-t)^{k-n}\,\Gamma(\alpha+n+1)\left[\frac{\binom{n}{k}\,\mathfrak{b}}{\Gamma(\alpha+k+1)} + \sum_{j=1}^{n-k}(-1)^j \binom{n}{k+j} \frac{t^{-\alpha}\,\Gamma(\alpha+j)}{\Gamma(\alpha+k+j+1)}\right]$$

is the coefficient of the power x^k of $G_n^{\alpha,t}(x)$, and
(iii) for $k = 0, \ldots, n$,

$$\binom{n}{k} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)}\, t^{k-n}$$

is the coefficient of the power x^{2k+1} of $F_{2n+1}^{\alpha}(x,t)$.

The following corollaries follow immediately from Theorems 1, 2, and Proposition 4, respectively.

Corollary 3. *For $n \in \mathbb{N}$ and $\alpha, t > 0$, let us denote by $y_{n,1}^{\alpha}(t), \ldots, y_{n,n}^{\alpha}(t)$ the n zeros of $F_n^{\alpha}(x,t)$. Then,*

(i) if n is even, for $k = 1, \ldots, n$, $\lim_{t\to\infty} y_{n,k}^{\alpha}(t) = 0$, and
(ii) if n is odd, there exists $k_0 \in \{1, \ldots, n\}$ such that for $k = 1, \ldots, n$,

$$\lim_{t\to\infty} y_{n,k}^{\alpha}(t) = -\mathfrak{b}\,\delta_{k_0,k}.$$

Proof. From (27) and (29), we get

$$\lim_{t\to\infty} F_n^{\alpha}(x,t) = \begin{cases} x^{n-1}(x+\mathfrak{b}), & \text{if } n \text{ is odd}, \\ x^n, & \text{otherwise}, \end{cases}$$

and the corollary follows. □

Corollary 4. *Given any $\alpha, t > 0$ and $\mathfrak{b} > t^{-\alpha}\,\Gamma(\alpha)$, the sequence $\{F_n^{\alpha}(\cdot,t)\}_{n\geqslant 1}$ of Hurwitz polynomials satisfies*

$$F_n^{\alpha}(x,t) = \left(x^2 + \frac{2\lfloor \frac{n}{2} \rfloor + \alpha - 1}{t}\right) F_{n-2}^{\alpha}(x,t) - \lambda_{\lfloor n/2 \rfloor}^{\alpha,t}\, F_{n-4}^{\alpha}(x,t), \qquad n \geqslant 2,$$

with initial conditions

$$\begin{aligned} \lambda_1^{\alpha,t} F_{-2}^{\alpha}(x,t) &= -t^{-1-\alpha}\,\Gamma(\alpha+1)\,x, & F_0^{\alpha}(x,t) &= 1, \\ \lambda_1^{\alpha,t} F_{-1}^{\alpha}(x,t) &= t^{-1-\alpha}\,\Gamma(\alpha+1), & F_1^{\alpha}(x,t) &= x+\mathfrak{b}, \end{aligned}$$

and $\lambda_{n+1}^{\alpha,t} = n\,(n+\alpha)\,t^{-2}$ *when* $n \geqslant 1$.

4.2. The Jacobi-Type Case

Now we deal with the Jacobi case. As before, the result for the even degree case is immediate.

Theorem 3. *For* $n \in \mathbb{N}$, $\alpha, \beta > -1$, $t < 1$, *and* $x \in \mathbb{R}$,

$$\mathcal{F}_{2n}^{\alpha,\beta}(x,t) = (-1)^n\, J_n^{\alpha,\beta,t}(-x^2) + (-1)^{n-1}\, x\, Q_n^{\alpha,\beta,t}(-x^2) \tag{30}$$

is a Hurwitz polynomial with degree $2n$. *In fact, for* $i = 0, \dots, n-1$,

$$(1-t)^{n-i} \sum_{k=0}^{n-i} (-1)^k \binom{n-k}{i}\, \mathfrak{j}_{n,n-k}^{\alpha,\beta}$$

and

$$(1-t)^{\alpha+\beta+n-i} \sum_{k=0}^{n-i-1} (-1)^k \binom{n-k-1}{i} \frac{\Phi_{n,n-k-1}^{\alpha,\beta}\,\Gamma(\beta+1)}{(n+\alpha+\beta+1)_n}$$

are the coefficients of the power of x^{2i} *and* x^{2i+1} *of* $\mathcal{F}_{2n}^{\alpha,\beta}(x,t)$, *respectively.*

To consider the odd degree case, we must study the convergence of

$$\left\{ -\frac{Q_n^{\alpha,\beta,t}(0)}{J_n^{\alpha,\beta,t}(0)} \right\}_{n\geqslant 0}. \tag{31}$$

we first need some lemmas.

Lemma 5. *Let* $\{J_n^{\beta,\alpha,t}\}_{n\geqslant 0}$ *be the sequence of monic Jacobi-type polynomials orthogonal with respect to* (21). *Let* $\{Q_n^{\beta,\alpha,t}\}_{n\geqslant 0}$ *be the corresponding second kind polynomials. Then, for* $n \in \mathbb{N}$, $\alpha, \beta > -1$, $t < 1$, *and* $x \in \mathbb{R}$,

$$J_n^{\beta,\alpha,t}(1-t-x) = (-1)^n\, J_n^{\alpha,\beta,t}(x)$$

and

$$Q_n^{\beta,\alpha,t}(1-t-x) = (-1)^{n-1}\, Q_n^{\alpha,\beta,t}(x).$$

Proof. The proof of Lemma 5 is a straightforward consequence of the fact that for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$P_n^{\beta,\alpha}(-x) = (-1)^n\, P_n^{\alpha,\beta}(x)$$

(see [10], 4.1.4). Thus, substituting the above expression in (7),

$$J_n^{\beta,\alpha,0}(1-\xi) = \frac{1}{2^n} P_n^{\beta,\alpha}(-2\xi+1) = \frac{(-1)^n}{2^n} P_n^{\alpha,\beta}(2\xi-1) = (-1)^n J_n^{\alpha,\beta,0}(\xi). \tag{32}$$

Analogously, substituting (32) into (22) when $t = 0$ and after some easy computations, we obtain

$$Q_n^{\beta,\alpha,0}(1-\xi) = (-1)^{n-1}\, Q_n^{\alpha,\beta,0}(\xi). \tag{33}$$

From (23) and (32) we have

$$J_n^{\alpha,\beta,t}(x) = (1-t)^n\, J_n^{\alpha,\beta,0}(\xi) = (-1)^n\,(1-t)^n\, J_n^{\beta,\alpha,0}(1-\xi) = (-1)^n\, J_n^{\beta,\alpha,t}(1-t-x),$$

where $\xi = \frac{x}{1-t}$. Analogously, from (25) and (33) we get

$$Q_n^{\alpha,\beta,t}(x) = (-1)^{n-1}\,(1-t)^{n+\beta+\alpha}\, Q_n^{\beta,\alpha,0}(1-\xi) = (-1)^{n-1}\, Q_n^{\beta,\alpha,t}(1-t-x),$$

where $\xi = \frac{x}{1-t}$, which is our claim. □

Lemma 6. *For $n \in \mathbb{N}$, $\alpha, \beta > -1$, and $t < 1$,*

$$\begin{aligned} J_n^{\alpha,\beta,t}(0) &= (1-t)^n \frac{(-1)^n\,(\beta+1)_n}{(\alpha+\beta+n+1)_n}, \\ Q_n^{\alpha,\beta,t}(0) &= (1-t)^{\alpha+\beta+n} \frac{(-1)^n\,\Gamma(\alpha+1)}{(\alpha+\beta+n+1)_n} \times \\ &\quad \sum_{k=1}^{n} \binom{n}{k} \frac{(\alpha+\beta+n+1)_k\,(\beta+k+1)_{n-k}}{(-1)^k\,\Gamma(\alpha+\beta+k+1)}\,\Gamma(\beta+k), \end{aligned}$$

and

$$\begin{aligned} \frac{Q_n^{\alpha,\beta,t}(0)}{J_n^{\alpha,\beta,t}(0)} &= (1-t)^{\alpha+\beta} \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+n+1)} \times \\ &\quad \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+\beta+k+1)}\,\frac{1}{\beta+k}. \end{aligned} \tag{34}$$

Proof. From (23), (32), and due to the fact that

$$P_n^{\alpha,\beta}(-1) = (-1)^n \frac{2^n\,(\beta+1)_n}{(\alpha+\beta+n+1)_n},$$

(see [10], 4.1.6), we deduce

$$J_n^{\alpha,\beta,t}(0) = \frac{(1-t)^n}{2^n}\, P_n^{\alpha,\beta}(-1) = (1-t)^n \frac{(-1)^n\,(\beta+1)_n}{(\alpha+\beta+n+1)_n}. \tag{35}$$

On the other hand, evaluating (25) at $x = 0$ and using (33) we have

$$\begin{aligned} Q_n^{\alpha,\beta,t}(0) &= (-1)^{n-1}\,(1-t)^{\alpha+\beta+n}\, Q_n^{\beta,\alpha,0}(1) \\ &= (1-t)^{\alpha+\beta+n} \frac{(-1)^n\,\Gamma(\alpha+1)}{(\alpha+\beta+n+1)_n} \times \\ &\quad \sum_{k=1}^{n} \binom{n}{k} \frac{(\alpha+\beta+n+1)_k\,(\beta+k+1)_{n-k}}{(-1)^k\,\Gamma(\alpha+\beta+k+1)}\,\Gamma(\beta+k). \end{aligned} \tag{36}$$

Finally, taking into account

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}, \qquad n \in \mathbb{N}, \quad z \in \mathbb{C}, \tag{37}$$

the quotient between (36) and (35) can be simplified as (34). □

Lemma 7. *For $n \in \mathbb{N}$, $\alpha, \beta > -1$ and $\beta \neq 0$,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+\beta+k+1)} \frac{1}{\beta+k} = \frac{n!\,(\alpha+1)_n}{\prod_{k=0}^{n}(\beta+k)}. \tag{38}$$

Proof. Fix $\beta > -1$ ($\beta \neq 0$). Notice that

$$(-1)^k \binom{n}{k} = \frac{n!}{\prod_{\substack{j=0 \\ j\neq k}}^{n} (-k+j)}, \qquad 0 \leqslant k \leqslant n.$$

Introducing the notation $\theta_k = \alpha+\beta+k+1$ with $\alpha > -1$ and $k = 0, \ldots, n$, and using (37), the left-hand side of (38) can be rewritten as

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+\beta+k+1)} \frac{1}{\beta+k} = n! \sum_{k=0}^{n} \frac{(\theta_k)_n}{\prod_{\substack{j=0 \\ j\neq k}}^{n} (-k+j)} \frac{1}{\beta+k}. \tag{39}$$

Since

$$(z)_n = \sum_{i=0}^{n} (-1)^{n-i}\, \mathfrak{s}(n,i)\, z^i, \qquad n \in \mathbb{N}, \quad z \in \mathbb{C},$$

where $\mathfrak{s}(n,i)$ denotes the Stirling number of the first kind (see [28]), we can apply the partial fraction decomposition method on the right-hand side of (39). Hence, we deduce

$$\sum_{k=0}^{n} \frac{(\theta_k)_n}{\prod_{\substack{j=0 \\ j\neq k}}^{n} (-k+j)} \frac{1}{\beta+k} = \frac{\mathfrak{q}(\beta)}{\mathfrak{p}(\beta)},$$

where $\mathfrak{p}(z) = \prod_{k=0}^{n} (z+k)$ and $\mathfrak{q}(z) = \sum_{i=0}^{n} (-1)^{n-i}\, \mathfrak{s}(n,i)\, (\alpha+\beta+1-z)^i$. This completes the proof. □

Lemmas 6 and 7 imply that if $\alpha, \beta > -1$ and $\beta \neq 0$,

$$\begin{aligned}\frac{Q_n^{\alpha,\beta,t}(0)}{J_n^{\alpha,\beta,t}(0)} = {} & (1-t)^{\alpha+\beta} \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \left[-\frac{1}{\beta} + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n+1+\beta)} \times \right. \\ & \left. \frac{n!}{\prod_{k=0}^{n}(\beta+k)} \right].\end{aligned}$$

Hence, the convergence or divergence of (31) only depends on

$$\left\{ \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n+1+\beta)} \frac{n!}{\prod_{k=0}^{n}(\beta+k)} \right\}_{n \geqslant 0}.$$

Proceeding as in the proof of Proposition 9, we see that the limit as $n \to \infty$ is zero, and we get the following result.

Proposition 10. *For every $\alpha > -1$ and $t < 1$, the sequence (31)*

(*i*) *converges to the positive number $(1-t)^{\alpha+\beta}\, \mathfrak{B}(\alpha+1, \beta)$ when $\beta > 0$, and* (*ii*) *diverges when* $-1 < \beta < 0$.

Consequently, we have

Theorem 4. *Let* $n \in \mathbb{N}$, $\alpha > -1$, $\beta > 0$, $t < 1$, *and* $x \in \mathbb{R}$. *Write*

$$G_n^{\alpha,\beta,t}(x) = (1-t)\, Q_n^{\alpha,\beta,t}(x) + \mathfrak{b}\, J_n^{\alpha,\beta,t}(x)$$

with $\mathfrak{b} > (1-t)^{\alpha+\beta+1}\,\mathfrak{B}(\alpha+1,\beta)$. *Then,*

$$\mathcal{F}_{2n+1}^{\alpha,\beta}(x,t) = (-1)^n\, x\, J_n^{\alpha,\beta,t}(-x^2) + (-1)^n\, G_n^{\alpha,\beta,t}(-x^2) \tag{40}$$

is a Hurwitz polynomial with degree $2n+1$. *Moreover,*

(*i*) *the leading coefficient of* $G_n^{\alpha,\beta,t}(x)$ *is* $\mathfrak{b}$,

(*ii*) *if* $n = 0$, $G_0^{\alpha,\beta,t}(x) = \mathfrak{b}$. *Otherwise, for* $i = 0, \ldots, n-1$,

$$(1-t)^{n-i}\left[(1-t)^{\alpha+\beta+1}\sum_{k=0}^{n-i-1}(-1)^{n-i-k-1}\binom{n-k-1}{i}\frac{\Phi_{n,n-k-1}^{\alpha,\beta}\,\Gamma(\beta+1)}{(n+\alpha+\beta+1)_n}\right.$$
$$\left. + \mathfrak{b}\sum_{k=0}^{n-i}(-1)^{n-i-k}\binom{n-k}{i}\mathfrak{j}_{n,n-k}^{\alpha,\beta}\right]$$

is the coefficient of the power x^i *of* $G_n^{\alpha,\beta,t}(x)$, *and*

(*iii*) *for* $i = 0, \ldots, n$,

$$(1-t)^{n-i}\sum_{k=0}^{n-i}(-1)^k\binom{n-k}{i}\mathfrak{j}_{n,n-k}^{\alpha,\beta}$$

is the coefficient of the power x^{2i+1} *of* $\mathcal{F}_{2n+1}^{\alpha,\beta}(x,t)$.

Remark 1. *In Theorem 4, notice that we have considered* $(1-t)\, Q_n^{\alpha,\beta,t}(x)$ *instead of* $Q_n^{\alpha,\beta,t}(x)$ *to compute* $G_n^{\alpha,\beta,t}(x)$. *This is because we want to avoid negative values in the powers of* $(1-t)$ *when* $\alpha > -1$ *and* $\beta > 0$. *On the other hand,* $\mathfrak{j}_{n,i}^{\alpha,\beta}$ *is defined in* (26).

Thus, proceeding as in the previous Subsection, Theorems 3 and 4 imply the following result about behavior for the zeros of $\mathcal{F}_n^{\alpha,\beta}(x,t)$, $n \geqslant 1$, when $t \to 1$.

Corollary 5. *For* n *a positive integer,* $\alpha > -1$, $\beta > 0$, *and* $t < 1$, *let us denote by* $y_{n,1}^{\alpha,\beta}(t), \ldots, y_{n,n}^{\alpha,\beta}(t)$ *the* n *zeros of* $\mathcal{F}_n^{\alpha,\beta}(x,t)$. *Then, there exists* $k_0 \in \{1, \ldots, n\}$ *such that the following statements hold.*

(*i*) *If* n *is odd, for* $k = 1, \ldots, n$, $\lim_{t\to 1} y_{n,k}^{\alpha,\beta}(t) = -\mathfrak{b}\,\delta_{k_0,k}$.

(*ii*) *If* n *is even, for* $k = 1, \ldots, n$, $\lim_{t\to 1} y_{n,k}^{\alpha,\beta}(t) = 0$.

On the other hand, as a consequence of Proposition 4, the sequence of Hurwitz polynomials constructed via Theorems 3 and 4 satisfies the following recurrence relation.

Corollary 6. *Let* $\alpha > -1$, $\beta > 0$, *and* $t < 1$. *If* $\mathfrak{b} > (1-t)^{\alpha+\beta+1}\,\mathfrak{B}(\alpha+1,\beta)$, *then the sequence* $\{\mathcal{F}_n^{\alpha,\beta}(\,\cdot\,,t)\}_{n\geqslant 1}$ *of Hurwitz polynomials satisfies*

$$\mathcal{F}_n^{\alpha,\beta}(x,t) = \left(x^2 + \left(1 + \epsilon_{\lfloor n/2\rfloor-1}^{\alpha,\beta}\right)\frac{1-t}{2}\right)\mathcal{F}_{n-2}^{\alpha,\beta}(x,t) - \lambda_{\lfloor n/2\rfloor-1}^{\alpha,\beta,t}\,\mathcal{F}_{n-4}^{\alpha,\beta}(x,t), \qquad n \geqslant 2,$$

with initial conditions

$$\begin{aligned} \lambda_0^{\alpha,\beta,t}\,\mathcal{F}_{-2}^{\alpha,\beta}(x,t) &= -(1-t)^{\alpha+\beta+1}\,\mathfrak{B}(\alpha+1,\beta+1)\,x, && \mathcal{F}_0^{\alpha,\beta}(x,t)=1,\\ \lambda_0^{\alpha,\beta,t}\,\mathcal{F}_{-1}^{\alpha,\beta}(x,t) &= (1-t)^{\alpha+\beta+2}\,\mathfrak{B}(\alpha+1,\beta+1), && \mathcal{F}_1^{\alpha,\beta}(x,t)=x+\mathfrak{b}, \end{aligned}$$

and $\lambda_n^{\alpha,\beta,t}=\frac{1}{4}\,(1-t)^2\,\varepsilon_n^{\alpha,\beta}$ *for* $n\geqslant 1$. $\epsilon_n^{\alpha,\beta}$ *and* $\varepsilon_n^{\alpha,\beta}$ *are given in Proposition 2.*

5. Numerical Examples

In this section, we provide some numerical examples computed using Mathematica®, to illustrate the behavior of the zeros of Hurwitz polynomials associated with Laguerre-type and Jacobi-type weights defined in (11) and (21), respectively. More specifically, we will show the location of the zeros of some elements of the sequences $\{F_n^{\alpha}(\cdot,t)\}_{n\geqslant 1}$ and $\{\mathcal{F}_n^{\alpha,\beta}(\cdot,t)\}_{n\geqslant 1}$ with respect to its parameters. In order to construct the odd degree polynomials, it is convenient to choose $\mathfrak{b}$ with explicit dependency of the parameters t and α (and also β for the Jacobi-type case) for the numerical experiments presented below.

In general, the first Laguerre-type monic polynomials and the corresponding second kind polynomials are

$$\begin{aligned} &L_0^{\alpha,t}(x)=1,\\ &L_1^{\alpha,t}(x)=x-\frac{\alpha+1}{t},\\ &L_2^{\alpha,t}(x)=x^2-\frac{2\alpha+4}{t}x+\frac{\alpha^2+3\alpha+2}{t^2},\\ &L_3^{\alpha,t}(x)=x^3-\frac{3\alpha+9}{t}x^2+\frac{3(\alpha^2+5\alpha+6)}{t^2}x-\frac{\alpha^3+6\alpha^2+11\alpha+6}{t^3},\\ &Q_1^{\alpha,t}(x)=\frac{\Gamma(\alpha+1)}{t^{\alpha+1}},\\ &Q_2^{\alpha,t}(x)=\frac{\Gamma(\alpha+1)}{t^{\alpha+1}}\left(x-\frac{\alpha+3}{t}\right),\\ &Q_3^{\alpha,t}(x)=\frac{\Gamma(\alpha+1)}{t^{\alpha+1}}\left(x^2-\frac{2(\alpha+4)}{t}x+\frac{\alpha(\alpha+6)+11}{t^2}\right). \end{aligned}$$

On the other hand, the first Jacobi-type monic polynomials and the corresponding second kind polynomials are

$$\begin{aligned} &J_0^{\alpha,\beta,t}(x)=1,\\ &J_1^{\alpha,\beta,t}(x)=x-\frac{(1+\beta)}{\alpha+\beta+2}\,(1-t),\\ &J_2^{\alpha,\beta,t}(x)=x^2-\frac{2(\beta+2)}{\alpha+\beta+4}\,(1-t)\,x+\left(\frac{\beta-\alpha}{\alpha+\beta+4}+\frac{(\alpha+1)(\alpha+2)}{(\alpha+\beta+3)(\alpha+\beta+4)}\right)(1-t)^2,\\ &Q_1^{\alpha,\beta,t}(x)=(1-t)^{1+\alpha+\beta}\,\mathfrak{B}(\alpha+1,\beta+1),\\ &Q_2^{\alpha,\beta,t}(x)=(1-t)^{1+\alpha+\beta}\,\mathfrak{B}(\alpha+1,\beta+1)\left(x-\frac{(\alpha+\beta)(\beta+3)+4}{(\alpha+\beta+4)(\alpha+\beta+2)}(1-t)\right). \end{aligned}$$

We point out that we recover the Laguerre polynomials in ([6], Example 3.1) when $t=1$ and $\alpha=2$, and the Jacobi polynomials when $t=0$ and $\alpha=\beta=2$.

We choose $F_3^{\alpha}(x,t)$ with $\mathfrak{b}=\mathfrak{b}(\alpha)=0.1+t^{-\alpha}\,\Gamma(\alpha)$ for our first example. Figure 1 illustrates the motion of its zeros when t varies from $t_i=1$ to $t_f=200$ with a step size of 0.5. Only one of the two conjugate complex zeros of $F_3^{\alpha}(x,t)$ was plotted, when α takes the values $1/2$ (circles), $3/2$ (filled triangles), and 3 (squares). For $\alpha = 3/2$, notice that the single real zero tends to $\lim_{t\to\infty} -\mathfrak{b} = -0.1$ and the complex zeros tend to the origin when $t \to \infty$ as described in Corollary 3.

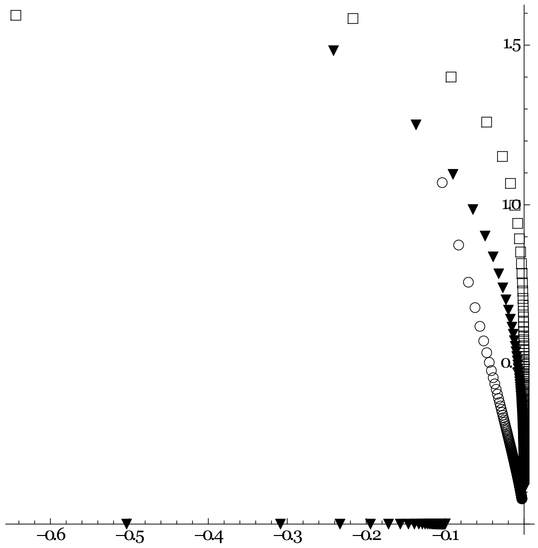
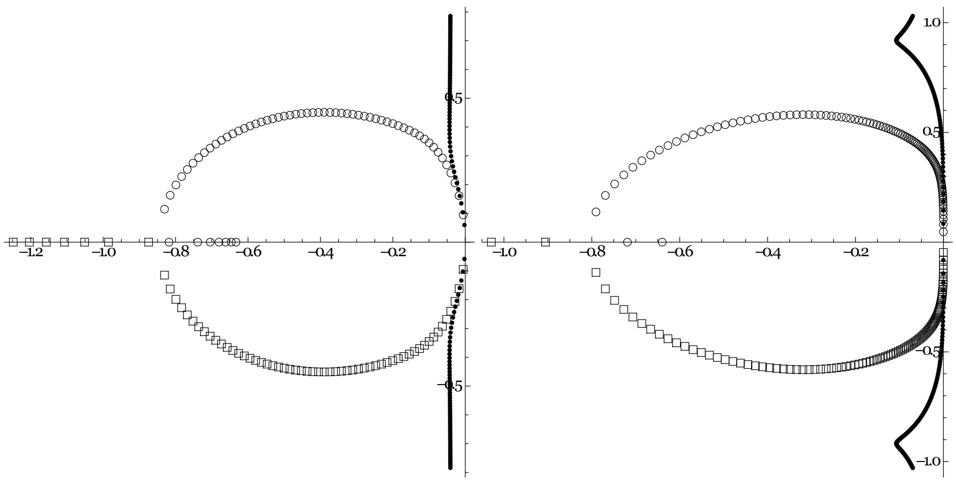


Figure 1. Motion of the complex and real zeros of $F_3^{\alpha}(x,t)$ for several values of t.

Analogously, Figure 2 shows the location of the zeros of $\mathcal{F}_4^{\alpha,1/2}(x,t)$ when t varies from $t_i = -1/4$ to $t_f = 1$ (panel (**a**)) and from $t_i = -1.15$ to $t_f = 1$ (panel (**b**)), both cases with a step size of 0.02. The image on the left corresponds to $\alpha = -1/2$. Notice that the two real zeros (squares and circles) approach a common point ($x \approx -0.85$) just before becoming complex zeros. This occurs at $t \approx -0.1287$. A similar situation (occurring when $t \approx -1.1238$) appears in the image on the right, which corresponds to the case $\alpha = 1$. Furthermore, all zeros tend to the origin when $t \to 1$ as described in Corollary 5.



(**a**) (**b**)

Figure 2. (**a**) The position of the zeros for $\mathcal{F}_4^{-1/2,\beta}(x,t)$ for several values of t and $\beta = 1/2$. (**b**) The position of the zeros for $\mathcal{F}_4^{1,\beta}(x,t)$ for several values of t and $\beta = 1/2$.

To deduce equations for the curves describing the motion of the zeros in terms of t constitutes a very interesting open problem that will be addressed in a future contribution.

6. Conclusions and Future Work

In this paper, we have considered perturbations of well-known families of classical polynomials in terms of a single parameter t. As a consequence, a new t-dependent sequence of Laguerre-type monic orthogonal polynomials is obtained. Several algebraic properties, such as their expressions in terms of hypergeometric functions (see Propositions 5 and 6), three-term recurrence relations (Corollary 1) and equations of motion for zeros with respect to t (Propositions 5), are obtained. Similar results are obtained for Jacobi-type orthogonal polynomials.

On the other hand, we have used such t-dependent sequences to construct sequences of Hurwitz polynomials that are robustly stable with respect to t (see Theorems 1–4). We also show that these polynomials satisfy a recurrence relation (Corollaries 4 and 6), in a similar way as the orthogonal polynomials do.

Finally, we point out that the approach used in this contribution can be extended in at least two directions that constitute open problems for future research:

- Other types of perturbations: Notice that the perturbations considered here are defined in terms of the orthogonality weights, given by (2) and (5). In the literature, other types of perturbations of orthogonal polynomials have been considered. In general, the interest in such perturbations is motivated by the well-known connections of the theory of orthogonal polynomials with the spectral theory. Thus, the focus is placed in algebraic and analytic properties of the perturbed polynomials, expressed in terms of the original non-perturbed polynomials. For instance, in [29] the authors consider a perturbation introduced on the coefficients of the recurrence relation, whereas perturbations on the sequence of moments have been considered in [30]. In both cases, an interesting problem is to construct sequences of Hurwitz polynomials by using the approach considered here, and to determine the structure of the obtained uncertainty.
- Pole-placement design: In control theory, the aim of pole placement is to construct a controller that gives a closed-loop system with a specified characteristic polynomial (see for instance [31,32]). We propose this latter polynomial to be a stable polynomial associated with the Laguerre-type or Jacobi-type weights, and to study desired capabilities of the system such as sensitivity, disturbance rejection, and closed-loop frequency response (see [33]).

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