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# Hermite–Hadamard-Type Inequalities for Convex Functions via the Fractional Integrals with Exponential Kernel

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**Abstract:** In this paper, we establish three fundamental integral identities by the first- and second-order derivatives for a given function via the fractional integrals with exponential kernel. With the help of these new fractional integral identities, we introduce a few interesting Hermite–Hadamard-type inequalities involving left-sided and right-sided fractional integrals with exponential kernels for convex functions. Finally, some applications to special means of real number are presented.

**Keywords:** convex functions, Hermite–Hadamard-type inequalities, fractional integrals, exponential kernel.

**MSC:** 26A33

## 1. Introduction

Let  $h : [a, b] \subset \mathfrak{R} \rightarrow \mathfrak{R}$  be a convex function. Then,  $h$  meets the following classic Hermite–Hadamard inequality (see [1])

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b h(s) ds \leq \frac{h(a)+h(b)}{2}. \quad (1)$$

If  $h$  is a concave function, the inequalities in (1) are presented in the negative direction. The Hermite–Hadamard inequality provides us the estimates for the integral average of a continuous convex function on a compact interval.

For the latest results on generalizing, improving, and extending this classical Hermite–Hadamard inequality, one can see [2–9] and the references therein.

In [10], Dragomir and Agarwal proved the following result connected with the right part of (1). In [11], Alomart also elicited the similar result for functions whose second derivatives absolute values are convex.

**Lemma 1** (see [10], Theorem 2.2). *Assuming  $h : [a, b] \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$  is a differentiable function,  $h' \in L[a, b]$  and  $|h''|$  is convex on  $[a, b]$ . Then, the bellow inequality holds:*

$$\left| \frac{h(a)+h(b)}{2} - \frac{1}{b-a} \int_a^b h(s) ds \right| \leq \frac{(b-a)}{8} (|h'(a)| + |h'(b)|). \quad (2)$$

**Lemma 2** (see [11], Theorem 3). *Assuming  $h : [a, b] \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$  is a twice differentiable function,  $h'' \in L[a, b]$  and  $|h''|$  is convex on  $[a, b]$ . Then, the following inequality holds*

$$\left| \frac{h(a) + h(b)}{2} - \frac{1}{b-a} \int_a^b h(s) ds \right| \leq \frac{(b-a)^2}{24} (|h''(a)| + |h''(b)|). \tag{3}$$

Now, fractional calculus has turned into an enchanting field of mathematics. Many extensive investigations have been carried out in this area. Due to the wide applications of Hermite–Hadamard inequalities and fractional integrals, many researchers have extended their research to Hermite–Hadamard inequalities involving fractional integrals rather than integer integrals, see [12–22]. Sarikaya et al. [12] have deduced an amusing inequality of Hermite–Hadamard-type involving fractional integrals in the place of ordinary integrals. This research fascinates many researchers to consider this respect. As a result, some new integral inequalities by the approach of fractional calculus have been obtained in the literature until now. In addition, Ahmad et al. [16] gave the new fractional integral operators with an exponential kernel and proved similar inequalities.

**Definition 1** (see [16], Definition 2). *Let  $h \in L[a, b]$ . The fractional left-side integral  $\mathcal{J}_{a^+}^\alpha h$  and right-side integral  $\mathcal{J}_b^\alpha h$  of order  $\alpha \in (0, 1)$  are, respectively, defined by*

$$\mathcal{J}_{a^+}^\alpha h(x) = \frac{1}{\alpha} \int_a^x e^{-\frac{1-\alpha}{\alpha}(x-s)} h(s) ds, \quad x > a, \tag{4}$$

and

$$\mathcal{J}_b^\alpha h(x) = \frac{1}{\alpha} \int_x^b e^{-\frac{1-\alpha}{\alpha}(x-s)} h(s) ds, \quad x < b. \tag{5}$$

**Lemma 3** (see [16], Theorem 1). *Let  $h : [a, b] \rightarrow \mathfrak{R}$  be a positive convex function with  $0 \leq a < b$  and  $h \in L[a, b]$ . The following inequality for fractional integrals (4) and (5) holds:*

$$h\left(\frac{a+b}{2}\right) \leq \frac{1-\alpha}{2(1-e^{-\rho})} [\mathcal{J}_{a^+}^\alpha h(b) + \mathcal{J}_b^\alpha h(a)] \leq \frac{h(a) + h(b)}{2}. \tag{6}$$

**Remark 1.** *In (6), note that*

$$\rho = \frac{1-\alpha}{\alpha}(b-a).$$

In addition, Ahmad et al. [16] derived the bound estimate of the difference between the mean value of the endpoints and the average of the fractional integrals.

**Lemma 4** (see [16], Theorem 3). *Assuming  $h : [a, b] \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$  is differentiable,  $h' \in L[a, b]$  and  $|h'|$  are convex on  $[a, b]$ . Then, the following inequality holds:*

$$|Q_{mr}| \leq \frac{(b-a)}{2\rho} \tanh\left(\frac{\rho}{4}\right) (|h'(a)| + |h'(b)|),$$

where

$$Q_{mr} := \frac{h(a) + h(b)}{2} - \frac{1-\alpha}{2(1-e^{-\rho})} [\mathcal{J}_{a^+}^\alpha h(b) + \mathcal{J}_b^\alpha h(a)]$$

denotes the bound estimate of the difference between the mean value of the endpoints and the average of the fractional integrals.

However, the bound for the left of the Hermite–Hadamard inequality (6) has not been studied. It will be interesting to find

$$|Q_{ml}| \leq \text{what?}$$

Here,

$$Q_{ml} := \frac{1 - \alpha}{2(1 - e^{-\rho})} [\mathcal{J}_{a^+}^\alpha h(b) + \mathcal{J}_{b^-}^\alpha h(a)] - h\left(\frac{a + b}{2}\right)$$

denotes the bound estimate of the difference between the value of the midpoint and the average of the fractional integrals.

Furthermore, if  $|h''|$  is convex, it is natural to study the right- and left-type Hermite–Hadamard inequality via the fractional integral with an exponential kernel similar to Lemma 2, i.e., we want to find the constants  $\rho_1$  and  $\rho_2$  satisfying the following inequities:

$$|Q_{mr}| \leq \rho_1 \cdot (|h''(a)| + |h''(b)|),$$

and

$$|Q_{ml}| \leq \rho_2 \cdot (|h''(a)| + |h''(b)|).$$

Motivated by [12,15,16], we will demonstrate three new fractional-type integral identities and set up their corresponding Hermite–Hadamard-type inequalities involving left-sided and right-sided fractional integrals for convex functions, respectively.

## 2. New Fractional Integral Identity and Hermite–Hadamard-Type Inequality for First Order Derivative

We firstly prove the following lemma in order to attest the following result.

**Lemma 5.** *Assuming  $h : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable mapping and  $h' \in L[a, b]$ . Then, the following equality for the fractional integrals (4) and (5) holds:*

$$Q_{ml} = \frac{b - a}{2} \int_0^1 kh'(sa + (1 - s)b)ds - \frac{b - a}{2(1 - e^{-\rho})} \left[ \int_0^1 e^{-\rho s} h'(sa + (1 - s)b)ds - \int_0^1 e^{-\rho(1-s)} h'(sa + (1 - s)b)ds \right], \quad (7)$$

where

$$k = \begin{cases} 1, & 0 \leq s < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq s \leq 1. \end{cases}$$

**Proof.** Define

$$\begin{aligned} V &:= \int_0^1 e^{-\rho s} h'(sa + (1 - s)b)ds - \int_0^1 e^{-\rho(1-s)} h'(sa + (1 - s)b)ds \\ &= V_1 - V_2, \end{aligned} \quad (8)$$

where

$$\begin{aligned} V_1 &= \int_0^1 e^{-\rho s} h'(sa + (1 - s)b)ds, \\ V_2 &= \int_0^1 e^{-\rho(1-s)} h'(sa + (1 - s)b)ds. \end{aligned}$$

Integrating by parts, one has

$$\begin{aligned}
 V_1 &= \int_0^1 e^{-\rho s} h'(sa + (1-s)b) ds \\
 &= \frac{1}{a-b} \int_0^1 e^{-\rho s} d(h(sa + (1-s)b)) \\
 &= \frac{1}{a-b} \left[ e^{-\rho s} h(sa + (1-s)b) \Big|_0^1 - \int_0^1 h(sa + (1-s)b) d(e^{-\rho s}) \right] \\
 &= \frac{1}{a-b} \left[ e^{-\rho} h(a) - h(b) + \rho \int_0^1 h(sa + (1-s)b) e^{-\rho s} ds \right] \\
 &= \frac{e^{-\rho} h(a) - h(b)}{a-b} + \frac{\rho}{(a-b)^2} \int_b^a h(x) e^{-\rho \frac{x-b}{a-b}} dx \\
 &= \frac{e^{-\rho} h(a) - h(b)}{a-b} - \frac{\rho}{(a-b)^2} \int_a^b e^{-\frac{1-\alpha}{\alpha}(b-x)} h(x) dx \\
 &= \frac{e^{-\rho} h(a) - h(b)}{a-b} - \frac{\rho}{(a-b)^2} \alpha \mathcal{J}_{a^+}^\alpha h(b) \\
 &= \frac{e^{-\rho} h(a) - h(b)}{a-b} - \frac{1-\alpha}{(b-a)} \mathcal{J}_{a^+}^\alpha h(b), \tag{9}
 \end{aligned}$$

and

$$\begin{aligned}
 V_2 &= \int_0^1 e^{-\rho(1-s)} h'(sa + (1-s)b) ds \\
 &= \frac{1}{a-b} \int_0^1 e^{-\rho(1-s)} d(h(sa + (1-s)b)) \\
 &= \frac{1}{a-b} \left[ e^{-\rho(1-s)} h(sa + (1-s)b) \Big|_0^1 - \int_0^1 h(sa + (1-s)b) d(e^{-\rho(1-s)}) \right] \\
 &= \frac{1}{a-b} \left[ h(a) - e^{-\rho} h(b) - \rho \int_0^1 h(sa + (1-s)b) e^{-\rho(1-s)} ds \right] \\
 &= \frac{h(a) - e^{-\rho} h(b)}{a-b} - \frac{\rho}{(a-b)^2} \int_b^a h(x) e^{-\rho \frac{a-x}{a-b}} dx \\
 &= \frac{h(a) - e^{-\rho} h(b)}{a-b} + \frac{\rho}{(a-b)^2} \int_a^b e^{-\frac{1-\alpha}{\alpha}(x-a)} h(x) dx \\
 &= \frac{h(a) - e^{-\rho} h(b)}{a-b} + \frac{\rho}{(a-b)^2} \alpha \mathcal{J}_{b^-}^\alpha h(a) \\
 &= \frac{h(a) - e^{-\rho} h(b)}{a-b} + \frac{1-\alpha}{(b-a)} \mathcal{J}_{b^-}^\alpha h(a). \tag{10}
 \end{aligned}$$

Substituting (9) and (10) into (8), we get that

$$\begin{aligned}
 V &= V_1 - V_2 \\
 &= \frac{(1 - e^{-\rho})(h(a) + h(b))}{b-a} - \frac{1-\alpha}{(b-a)} [\mathcal{J}_{a^+}^\alpha h(b) + \mathcal{J}_{b^-}^\alpha h(a)]. \tag{11}
 \end{aligned}$$

Note

$$\begin{aligned}
 & \frac{b-a}{2} \int_0^1 kh'(sa + (1-s)b)dt \\
 = & \frac{b-a}{2} \int_0^{\frac{1}{2}} h'(sa + (1-s)b)ds - \frac{b-a}{2} \int_{\frac{1}{2}}^1 h'(sa + (1-s)b)ds \\
 = & \frac{f(b) - h\left(\frac{a+b}{2}\right)}{2} + \frac{h(a) - h\left(\frac{a+b}{2}\right)}{2} \\
 = & \frac{h(a) + h(b)}{2} - h\left(\frac{a+b}{2}\right). \tag{12}
 \end{aligned}$$

Substituting (12) and (11) into the right-side of (7), we obtain the left of (7). This testifies the proof.  $\square$

Then, we can declare the first theorem including Hermite–Hadamard-type inequality.

**Theorem 1.** *If  $h : [a, b] \subseteq \mathfrak{R} \rightarrow \mathfrak{R}$  is differentiable,  $|h'|$  is convex on  $[a, b]$ , and  $h' \in L[a, b]$ , then the following inequality about the fractional integrals (4) and (5) holds:*

$$|Q_{ml}| \leq \frac{b-a}{2} \left[ \frac{1}{2} - \frac{\tanh\left(\frac{\rho}{4}\right)}{\rho} \right] (|h'(a)| + |h'(b)|). \tag{13}$$

**Proof.** Using Lemma 5, convexity of  $|h'|$ , and  $e^{-\rho s} \geq e^{-\rho}$  and  $e^{-\rho(1-s)} \geq e^{-\rho}$  for any  $s \in [0, 1]$ , we obtain

$$\begin{aligned}
 |Q_{ml}| &= \left| \frac{b-a}{2} \int_0^1 kh'(sa + (1-s)b)ds \right. \\
 &\quad \left. - \frac{b-a}{2(1-e^{-\rho})} \left[ \int_0^1 e^{-\rho s} h'(sa + (1-s)b)ds - \int_0^1 e^{-\rho(1-s)} h'(sa + (1-s)b)ds \right] \right| \\
 &= \frac{b-a}{2(1-e^{-\rho})} \left| \int_0^{\frac{1}{2}} [(1-e^{-\rho}) - e^{-\rho s} + e^{-\rho(1-s)}] h'(sa + (1-s)b)ds \right. \\
 &\quad \left. - \int_{\frac{1}{2}}^1 [(1-e^{-\rho}) - e^{-\rho(1-s)} + e^{-\rho s}] h'(sa + (1-s)b)ds \right| \\
 &\leq \frac{b-a}{2(1-e^{-\rho})} \left[ \int_0^{\frac{1}{2}} (1-e^{-\rho} - e^{-\rho s} + e^{-\rho(1-s)})(s|h'(a)| + (1-s)|h'(b)|)ds \right. \\
 &\quad \left. + \int_{\frac{1}{2}}^1 (1-e^{-\rho} - e^{-\rho(1-s)} + e^{-\rho s})(s|h'(a)| + (1-s)|h'(b)|)ds \right] \\
 &= \frac{b-a}{2(1-e^{-\rho})} \left[ \int_0^{\frac{1}{2}} (1-e^{-\rho} - e^{-\rho s} + e^{-\rho(1-s)})(s|h'(a)| + (1-s)|h'(b)|)ds \right. \\
 &\quad \left. + \int_0^{\frac{1}{2}} (1-e^{-\rho} - e^{-\rho s} + e^{-\rho(1-s)})((1-s)|h'(a)| + s|h'(b)|)ds \right] \\
 &= \frac{b-a}{2(1-e^{-\rho})} \int_0^{\frac{1}{2}} (1-e^{-\rho} - e^{-\rho s} + e^{-\rho(1-s)})(|h'(a)| + |h'(b)|)ds \\
 &= \frac{b-a}{2(1-e^{-\rho})} \left[ \frac{1-e^{-\rho}}{2} - \frac{1}{\rho} (1-e^{-\frac{\rho}{2}})^2 \right] (|h'(a)| + |h'(b)|) \\
 &= \frac{b-a}{2} \left[ \frac{1}{2} - \frac{\tanh\left(\frac{\rho}{4}\right)}{\rho} \right] (|h'(a)| + |h'(b)|).
 \end{aligned}$$

The proof is completed.  $\square$

### 3. New Fractional Integral Identity and Hermite–Hadamard-Type Inequality for Second Order Derivative

In [16] Lemma 4, Ahmad et al. gave the equality

$$Q_{mr} = \frac{(b-a)}{2(1-e^{-\rho})} \left[ \int_0^1 e^{-\rho s} h'(sa + (1-s)b) ds - \int_0^1 e^{-\rho(1-s)} h'(sa + (1-s)b) ds \right]. \tag{14}$$

By (14), we will prove the Hermite–Hadamard-type inequality of the order derivatives via the fractional integrals with an exponential kernel for convex functions. Before we prove our main results in this section, we give the following lemmas.

**Lemma 6.** Assuming  $h : [a, b] \rightarrow \mathfrak{R}$  is a twice differentiable function. If  $h'' \in L[a, b]$ , then the following equality for fractional integrals holds:

$$Q_{mr} = \frac{(b-a)^2}{2\rho(1-e^{-\rho})} \int_0^1 \left( 1 + e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)} \right) h''(sa + (1-s)b) ds. \tag{15}$$

**Proof.** By using equality (14), we note

$$\begin{aligned} K_1 &= \int_0^1 e^{-\rho s} h'(sa + (1-s)b) ds = -\frac{1}{\rho} \int_0^1 h'(sa + (1-s)b) d(e^{-\rho s}) \\ &= \frac{1}{\rho} \left[ h'(b) - e^{-\rho} h'(a) + (a-b) \int_0^1 e^{-\rho s} h''(sa + (1-s)b) ds \right], \end{aligned} \tag{16}$$

and

$$\begin{aligned} K_2 &= \int_0^1 e^{-\rho(1-s)} h'(sa + (1-s)b) ds = \frac{1}{\rho} \int_0^1 h'(sa + (1-s)b) d(e^{-\rho(1-s)}) \\ &= \frac{1}{\rho} \left[ h'(a) - e^{-\rho} h'(b) - (a-b) \int_0^1 e^{-\rho(1-s)} h''(sa + (1-s)b) ds \right]. \end{aligned} \tag{17}$$

Inserting the values of  $K_1$  and  $K_2$  in (14), we obtain

$$\begin{aligned} &\frac{h(a) + h(b)}{2} - \frac{1-\alpha}{2(1-e^{-\rho})} [\mathcal{J}_{a^+}^\alpha h(b) + \mathcal{J}_{b^-}^\alpha h(a)] \\ &= \frac{b-a}{2\rho(1-e^{-\rho})} \left[ (1+e^{-\rho})(h'(b) - h'(a)) \right. \\ &\quad \left. - (b-a) \int_0^1 \left( e^{-\rho s} + e^{-\rho(1-s)} \right) h''(sa + (1-s)b) ds \right] \\ &= \frac{b-a}{2\rho(1-e^{-\rho})} \left[ (1+e^{-\rho})(b-a) \int_0^1 h''(sa + (1-s)b) ds \right. \\ &\quad \left. - (b-a) \int_0^1 \left( e^{-\rho s} + e^{-\rho(1-s)} \right) h''(sa + (1-s)b) ds \right] \\ &= \frac{(b-a)^2}{2\rho(1-e^{-\rho})} \int_0^1 \left( 1 + e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)} \right) h''(sa + (1-s)b) ds. \end{aligned} \tag{18}$$

This completes the proof.  $\square$

**Lemma 7.** Assuming  $h : [a, b] \rightarrow \mathfrak{R}$  is a twice differentiable function. If  $h'' \in L[a, b]$ , then the following equality for fractional integrals holds:

$$Q_{ml} = \frac{(b-a)^2}{2} \int_0^1 m(s) h''(sa + (1-s)b) ds, \tag{19}$$

where

$$m(s) = \begin{cases} s - \frac{1+e^{-\rho}-e^{-\rho s}-e^{-\rho(1-s)}}{\rho(1-e^{-\rho})}, & 0 \leq s < \frac{1}{2}, \\ (1-s) - \frac{1+e^{-\rho}-e^{-\rho s}-e^{-\rho(1-s)}}{\rho(1-e^{-\rho})}, & \frac{1}{2} \leq s < 1. \end{cases}$$

**Proof.** By using the proof of the Lemma 5, we can get

$$Q_{ml} = \frac{b-a}{2} \int_0^1 kh'(sa + (1-s)b)ds - \frac{b-a}{2(1-e^{-\rho})} \left[ \int_0^1 e^{-\rho s} h'(sa + (1-s)b)ds - \int_0^1 e^{-\rho(1-s)} h'(sa + (1-s)b)ds \right]. \tag{20}$$

Thus,

$$\begin{aligned} & \frac{b-a}{2} \int_0^1 kh'(sa + (1-s)b)ds \\ = & \frac{b-a}{2} \int_0^{\frac{1}{2}} h'(sa + (1-s)b)ds - \frac{b-a}{2} \int_{\frac{1}{2}}^1 h'(sa + (1-s)b)ds \\ = & \frac{b-a}{2} \left[ sh'(sa + (1-s)b) \Big|_0^{\frac{1}{2}} - (a-b) \int_0^{\frac{1}{2}} sh''(sa + (1-s)b)ds \right] \\ & - \frac{b-a}{2} \left[ sh'(sa + (1-s)b) \Big|_{\frac{1}{2}}^1 - (a-b) \int_{\frac{1}{2}}^1 sh''(sa + (1-s)b)ds \right] \\ = & \frac{b-a}{2} \left[ \frac{1}{2} h' \left( \frac{a+b}{2} \right) - (a-b) \int_0^{\frac{1}{2}} sh''(sa + (1-s)b)ds \right] \\ & - \frac{b-a}{2} \left[ h'(a) - \frac{1}{2} h' \left( \frac{a+b}{2} \right) - (a-b) \int_{\frac{1}{2}}^1 sh''(sa + (1-s)b)ds \right] \\ = & \frac{b-a}{2} \left[ h' \left( \frac{a+b}{2} \right) - h'(a) \right] \\ & + \frac{(b-a)^2}{2} \int_0^{\frac{1}{2}} sh''(sa + (1-s)b)ds - \frac{(b-a)^2}{2} \int_{\frac{1}{2}}^1 sh''(sa + (1-s)b)ds \\ = & \frac{(b-a)^2}{2} \int_{\frac{1}{2}}^1 h''(sa + (1-s)b)ds \\ & + \frac{(b-a)^2}{2} \int_0^{\frac{1}{2}} sh''(sa + (1-s)b)ds - \frac{(b-a)^2}{2} \int_{\frac{1}{2}}^1 sh''(sa + (1-s)b)ds \\ = & \frac{(b-a)^2}{2} \int_0^{\frac{1}{2}} sh''(sa + (1-s)b)ds + \frac{(b-a)^2}{2} \int_{\frac{1}{2}}^1 (1-s)h''(sa + (1-s)b)ds. \end{aligned} \tag{21}$$

Submitting (16), (17), and (21) to (20), we get (19). This completes the proof.  $\square$

Now, we can prove our Hermite–Hadamard-type inequalities by the second order derivatives.

**Theorem 2.** Assuming  $h : [a, b] \rightarrow \mathfrak{R}$  is a twice differentiable function. If  $h'' \in L[a, b]$  and  $|h''|$  is convex on  $[a, b]$ , the following inequality for fractional integrals with exponential kernel holds:

$$|Q_{mr}| \leq \frac{(b-a)^2}{2\rho(1-e^{-\rho})} \left( \frac{1+e^{-\rho}}{2} - \frac{1-e^{-\rho}}{\rho} \right) (|h''(a)| + |h''(b)|). \tag{22}$$

**Proof.** Note that

$$\begin{aligned}
 & \int_0^1 \left(1 + e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}\right) s ds \\
 &= (1 + e^{-\rho}) \int_0^1 s ds - \int_0^1 s e^{-\rho s} ds - \int_0^1 s e^{-\rho(1-s)} ds \\
 &= \frac{1 + e^{-\rho}}{2} + \frac{1}{\rho} \left(e^{-\rho} - \frac{1 - e^{-\rho}}{\rho}\right) - \frac{1}{\rho} \left(1 - \frac{1 - e^{-\rho}}{\rho}\right) \\
 &= \frac{1 + e^{-\rho}}{2} - \frac{1 - e^{-\rho}}{\rho},
 \end{aligned} \tag{23}$$

and

$$\begin{aligned}
 & \int_0^1 \left(1 + e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}\right) (1 - s) ds \\
 &= \int_0^1 \left(1 + e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}\right) ds - \int_0^1 s \left(1 + e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}\right) ds \\
 &= (1 + e^{-\rho}) \Big|_0^1 + \frac{e^{-\rho s}}{\rho} \Big|_0^1 - \frac{e^{-\rho(1-s)}}{\rho} \Big|_0^1 - \left(\frac{1 + e^{-\rho}}{2} - \frac{1 - e^{-\rho}}{\rho}\right) \\
 &= \frac{1 + e^{-\rho}}{2} - \frac{1 - e^{-\rho}}{\rho}.
 \end{aligned} \tag{24}$$

According to (15), (23), (24), and the convex of  $|h''|$ , we can get

$$\begin{aligned}
 |Q_{mr}| &= \left| \frac{(b-a)^2}{2\rho(1-e^{-\rho})} \int_0^1 \left(1 + e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}\right) h''(sa + (1-s)b) ds \right| \\
 &\leq \frac{(b-a)^2}{2\rho(1-e^{-\rho})} \int_0^1 \left|1 + e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}\right| |h''(sa + (1-s)b)| ds \\
 &\leq \frac{(b-a)^2}{2\rho(1-e^{-\rho})} \int_0^1 \left(1 + e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}\right) (s|h''(a)| + (1-s)|h''(b)|) ds \\
 &= \frac{(b-a)^2}{2\rho(1-e^{-\rho})} \left(\frac{1 + e^{-\rho}}{2} - \frac{1 - e^{-\rho}}{\rho}\right) (|h''(a)| + |h''(b)|).
 \end{aligned}$$

The proof is finished.  $\square$

**Remark 2.**  $\alpha \rightarrow 1$  in (22) of Theorem 2, then  $\rho = \frac{1-\alpha}{\alpha}(b-a) \rightarrow 0$ , one obtains

$$\lim_{\rho \rightarrow 0} \frac{1 - \alpha}{2(1 - e^{-\rho})} = \lim_{x \rightarrow 1} \frac{1 - \frac{b-a}{b-a-\ln x}}{2(1-x)} = \lim_{x \rightarrow 1} \frac{-\ln x}{2(1-x)(b-a-\ln x)} = \frac{1}{2(b-a)}, \tag{25}$$

and

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho(1 - e^{-\rho})} \left(\frac{1 + e^{-\rho}}{2} - \frac{1 - e^{-\rho}}{\rho}\right) = \lim_{\rho \rightarrow 0} \frac{\rho + \rho e^{-\rho} - 2 + 2e^{-\rho}}{2\rho^2(1 - e^{-\rho})} = \frac{1}{12}. \tag{26}$$

So (22) is transformed to

$$\left| \frac{h(a) + h(b)}{2} - \frac{1}{b-a} \int_a^b h(s) ds \right| \leq \frac{(b-a)^2}{24} (|h''(a)| + |h''(b)|).$$

This result coincides the conclusion in [11], Theorem 3.



**Theorem 3.** Assuming  $h : [a, b] \rightarrow \mathfrak{R}$  is a twice differentiable function. If  $h'' \in L[a, b]$  and  $|h''|$  is convex on  $[a, b]$ , then the following inequality for fractional integrals with exponential kernel holds:

$$|Q_{ml}| \leq \frac{(b-a)^2}{2} \left[ \frac{1}{8} + \frac{1+e^{-\rho}}{2\rho(1-e^{-\rho})} - \frac{1}{\rho^2} \right] (|h''(a)| + |h''(b)|). \tag{27}$$

**Proof.** According to Lemma 7 and the convex of  $|h''|$ , we can get

$$\begin{aligned} |Q_{ml}| &= \left| \frac{(b-a)^2}{2} \int_0^1 m(s)h''(sa + (1-s)b)ds \right| \leq \frac{(b-a)^2}{2} \int_0^1 |m(s)| |h''(sa + (1-s)b)| ds \\ &\leq \frac{(b-a)^2}{2} \int_0^{\frac{1}{2}} \left( s + \frac{1+e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}}{\rho(1-e^{-\rho})} \right) (s|h''(a)| + (1-s)|h''(b)|) ds \\ &\quad + \frac{(b-a)^2}{2} \int_{\frac{1}{2}}^1 \left( 1-s + \frac{1+e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}}{\rho(1-e^{-\rho})} \right) (s|h''(a)| + (1-s)|h''(b)|) ds \\ &= \frac{(b-a)^2}{2} \left[ \int_0^{\frac{1}{2}} (s^2|h''(a)| + t(1-t)|h''(b)|) ds + \int_{\frac{1}{2}}^1 (s(1-s)|h''(a)| + (1-s)^2|h''(b)|) ds \right] \\ &\quad + \frac{1}{\rho(1-e^{-\rho})} \int_0^1 (1+e^{-\rho} - e^{-\rho s} - e^{-\rho(1-s)}) (s|h''(a)| + (1-s)|h''(b)|) ds \Big] \\ &= \frac{(b-a)^2}{2} \left[ \frac{1}{24}|h''(a)| + \frac{1}{12}|h''(b)| + \frac{1}{12}|h''(a)| + \frac{1}{24}|h''(b)| \right. \\ &\quad \left. + \frac{1}{\rho(1-e^{-\rho})} \left( \frac{1+e^{-\rho}}{2} - \frac{1-e^{-\rho}}{\rho} \right) (|h''(a)| + |h''(b)|) \right] \\ &= \frac{(b-a)^2}{2} \left[ \frac{1}{8} + \frac{1+e^{-\rho}}{2\rho(1-e^{-\rho})} - \frac{1}{\rho^2} \right] (|h''(a)| + |h''(b)|). \end{aligned}$$

This completes the proof.  $\square$

**Remark 3.** Let  $\alpha \rightarrow 1$  in (27), one has

$$\left| \frac{1}{b-a} \int_a^b h(s)ds - h\left(\frac{a+b}{2}\right) \right| \leq \frac{5(b-a)^2}{48} (|h''(a)| + |h''(b)|).$$

### 4. Application to Special Means

Think on the following particular means [23] for  $\forall p, q \in R, p \neq q$  as follows:

- (i)  $H(p, q) = \frac{2}{\frac{1}{p} + \frac{1}{q}}, p, q \in R \setminus \{0\};$
- (ii)  $A(p, q) = \frac{p+q}{2}, p, q \in R;$
- (iii)  $L(p, q) = \frac{q-p}{\ln|p| - \ln|q|}, |p| \neq |q|, pq \neq 0;$
- (iv)  $L_m(p, q) = \left[ \frac{q^{m+1} - p^{m+1}}{(m+1)(q-p)} \right]^{\frac{1}{m}}, m \in Z \setminus \{-1, 0\}, p, q \in R, p \neq q.$

Next, making use of the acquired results in Section 3, we give some applications to particular means of real number.

**Proposition 1.** Let  $p, q \in \mathfrak{R}, p < q, pq > 0$  and  $m \in \mathbb{Z}, |m| \geq 2$ . Then,

$$|L_m^m(p, q) - A^m(p, q)| \leq \frac{5}{24}(q-p)^2 |m(m-1)| A(|p|^{m-2}, |q|^{m-2}). \tag{28}$$

**Proof.** Applying Remark 3 for  $h(x) = x^m$ , we can get the conclusion immediately.  $\square$

The upper bound is smaller than the result of Proposition 3.1 in [5] when  $|q - p| \leq 1$  and  $|p|, |q| > 1$  obviously.

**Proposition 2.** Let  $p, q \in \mathfrak{R}, p < q, pq > 0$ . Then,

$$\left| L^{-1}(p, q) - A^{-1}(p, q) \right| \leq \frac{5}{12}(q - p)^2 A(|p|^{-3}, |q|^{-3}). \tag{29}$$

**Proof.** The inference follows from Remark 3 used for  $h(x) = \frac{1}{x}$ .  $\square$

**Proposition 3.** Let  $p, q \in \mathfrak{R}, p < q, pq > 0$  and  $m \in \mathbb{Z}, |m| \geq 2$ . Then, we have

$$\left| L_m^m(q^{-1}, p^{-1}) - H^{-m}(q, p) \right| \leq \frac{5}{24} \left( \frac{1}{p} - \frac{1}{q} \right)^2 |m(m - 1)| H^{-1}(|p|^{m-2}, |q|^{m-2}), \tag{30}$$

and

$$\left| L^{-1}(p, q) - H(p, q) \right| \leq \frac{5}{12}(q - p)^2 H^{-1}(|p|^3, |q|^3). \tag{31}$$

**Proof.** Doing the replacement  $q^{-1} \rightarrow p, p^{-1} \rightarrow q$  in the inequalities (28) and (29), we can obtain the required inequalities (30) and (31), respectively. Here, we have observed  $A^{-1}(p^{-1}, q^{-1}) = H(p, q) = 2/(\frac{1}{p} + \frac{1}{q}), q^{-1} < p^{-1}$ .  $\square$

At last, we will present an application to a midpoint formula. In [23], let  $w$  be a division  $p = s_0 < s_1 \cdots < s_{m-1} < s_m = q$  of the interval  $[p, q]$  and inspect the quadrature formula

$$\int_p^q h(s) ds = T(h, w) + E(h, w), \tag{32}$$

where

$$T(h, w) = \sum_{i=0}^{m-1} h\left(\frac{s_i + s_{i+1}}{2}\right)(s_{i+1} - s_i)$$

is the midpoint version and  $E(h, w)$  refers to the approximation error. Here, we deduce the error estimate for the midpoint formula.

**Proposition 4.** Let  $h : [p, q] \rightarrow \mathfrak{R}$  be a twice differentiable mapping on  $(p, q)$  with  $p < q$ . If  $h'' \in L[a, b]$  and  $|h''|$  is convex on  $[p, q]$ , then in (32), for every division  $w$  of  $[p, q]$ , the following inequality holds:

$$|E(f, w)| \leq \frac{5}{48} \sum_{i=0}^{m-1} (s_{i+1} - s_i)^3 (|h''(s_i)| + |h''(s_{i+1})|).$$

**Proof.** Applying Remark 3 on subinterval  $[s_i, s_{i+1}] (i = 0, 1, \dots, m - 1)$  of the division  $w$ , we derive

$$\left| \int_{s_i}^{s_{i+1}} h(s) ds - h\left(\frac{s_i + s_{i+1}}{2}\right)(s_i - s_{i+1}) \right| \leq \frac{5}{48} (s_{i+1} - s_i)^3 (|h''(s_i)| + |h''(s_{i+1})|).$$

Summing over from 0 to  $m - 1$  and making use of the convexity of  $|h''|$ , we infer that

$$\begin{aligned} & \left| \int_p^q h(s) ds - T(h, w) \right| = \left| \sum_{i=0}^{m-1} \left[ \int_{s_i}^{s_{i+1}} h(s) ds - f\left(\frac{s_i + s_{i+1}}{2}\right)(s_i - s_{i+1}) \right] \right| \\ & \leq \sum_{i=0}^{m-1} \left| \int_{s_i}^{s_{i+1}} h(s) ds - h\left(\frac{s_i + s_{i+1}}{2}\right)(s_i - s_{i+1}) \right| \\ & \leq \frac{5}{48} \sum_{i=0}^{m-1} (s_{i+1} - s_i)^3 (|h''(s_i)| + |h''(s_{i+1})|). \end{aligned}$$

The proof is completed.  $\square$

## 5. Conclusions

Based on the above interpretation, we acquire the bound estimates of the difference between the average of the fractional integrals with an exponential kernel and the mean values of the endpoints and the midpoint.

By comparing these bound estimates, we have obtained the following conclusions:

- (i) With the first and second order derivatives of a given function, the Hermite–Hadamard-type inequalities involving left-sided and right-sided, the fractional integrals are different. The Hermite–Hadamard-type inequalities with the second order derivatives of a given function are more accurate.
- (ii) With the same order derivatives of a given function, the Hermite–Hadamard-type inequalities involving different fractional integrals finally tend to be same when  $\alpha \rightarrow 1$ .

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