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Faces of 2-Dimensional Simplex of Order and Chain Polytopes

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Abstract: Each of the descriptions of vertices, edges, and facets of the order and chain polytope of a finite partially ordered set are well known. In this paper, we give an explicit description of faces of 2-dimensional simplex in terms of vertices. Namely, it will be proved that an arbitrary triangle in 1-skeleton of the order or chain polytope forms the face of 2-dimensional simplex of each polytope. These results mean a generalization in the case of 2-faces of the characterization known in the case of edges.

Keywords: order polytope; chain polytope; partially ordered set

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1. Introduction

The combinatorial structure of the order polytope $\mathscr{O}(P)$ and the chain polytope $\mathscr{C}(P)$ of a finite poset (partially ordered set) P is explicitly discussed in [1]. Moreover, in [2], the problem when the order polytope $\mathscr{O}(P)$ and the chain polytope $\mathscr{C}(P)$ are unimodularly equivalent is solved. It is also proved that the number of edges of the order polytope $\mathscr{O}(P)$ is equal to that of the chain polytope $\mathscr{C}(P)$ in [3]. In the present paper we give an explicit description of faces of 2-dimensional simplex of $\mathscr{O}(P)$ and $\mathscr{C}(P)$ in terms of vertices. In other words, we show that triangles in 1-skeleton of $\mathscr{O}(P)$ or $\mathscr{C}(P)$ are in one-to-one correspondence with faces of 2-dimensional simplex of each polytope. These results are a direct generalizations of [4] (Lemma 4, Lemma 5).

2. Definition and Known Results

Let $P = \{x_1, \ldots, x_d\}$ be a finite poset. To each subset $W \subset P$, we associate $\rho(W) = \sum_{i \in W} \mathbf{e}_i \in \mathbb{R}^d$, where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the canonical unit coordinate vectors of $\mathbb{R}^d$. In particular $\rho(\emptyset)$ is the origin of $\mathbb{R}^d$. A *poset ideal* of P is a subset I of P such that, for all x_i and x_j with $x_i \in I$ and $x_j \leqslant x_i$, one has $x_j \in I$. An *antichain* of P is a subset A of P such that x_i and x_j belonging to A with $i \neq j$ are incomparable. The empty set $\emptyset$ is a poset ideal as well as an antichain of P. We say that x_j *covers* x_i if $x_i < x_j$ and $x_i < x_k < x_j$ for no $x_k \in P$. A chain $x_{j_1} < x_{j_2} < \cdots < x_{j_\ell}$ of P is called *saturated* if x_{j_q} covers $x_{j_{q-1}}$ for $1 < q \leqslant \ell$. A *maximal chain* is a saturated chain such that x_{j_1} is a minimal element and x_{j_ℓ} is a maximal element of the poset. The *rank* of P is $\sharp(C) - 1$, where C is a chain with maximum length of P.

The *order polytope* of P is the convex polytope $\mathscr{O}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots a_d) \in \mathbb{R}^d$ such that $0 \leqslant a_i \leqslant 1$ for every $1 \leqslant i \leqslant d$ together with

$$a_i \geqslant a_j$$

if $x_i \leqslant x_j$ in P.

The *chain polytope* of P is the convex polytope $\mathscr{C}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $a_i \geqslant 0$ for every $1 \leqslant i \leqslant d$ together with

$$a_{i_i} + a_{i_2} + \cdots + a_{i_k} \leqslant 1$$

for every maximal chain $x_{i_1} < x_{i_2} < \cdots < x_{i_k}$ of P.

One has $\dim\mathscr{O}(P) = \dim\mathscr{C}(P) = d$. The vertices of $\mathscr{O}(P)$ is those $\rho(I)$ for which I is a poset ideal of P ([1] (Corollary1.3)) and the vertices of $\mathscr{C}(P)$ is those $\rho(A)$ for which A is an antichain of P ([1] (Theorem2.2)). It then follows that the number of vertices of $\mathscr{O}(P)$ is equal to that of $\mathscr{C}(P)$. Moreover, the volume of $\mathscr{O}(P)$ and that of $\mathscr{C}(P)$ are equal to $e(P)/d!$, where $e(P)$ is the number of linear extensions of P ([1] (Corollary4.2)). It also follows from [1] that the facets of $\mathscr{O}(P)$ are the following:

- $x_i = 0$, where $x_i \in P$ is maximal;
- $x_j = 1$, where $x_j \in P$ is minimal;
- $x_i = x_j$, where x_j covers x_i,

and that the facets of $\mathscr{C}(P)$ are the following:

- $x_i = 0$, for all $x_i \in P$;
- $x_{i_1} + \cdots + x_{i_k} = 1$, where $x_{i_1} < \cdots < x_{i_k}$ is a maximal chain of P.

In [4] a characterization of edges of $\mathscr{O}(P)$ and those of $\mathscr{C}(P)$ is obtained. Recall that a subposet Q of finite poset P is said to be *connected* in P if, for each x and y belonging to Q, there exists a sequence $x = x_0, x_1, \ldots, x_s = y$ with each $x_i \in Q$ for which x_{i-1} and x_i are comparable in P for each $1 \leqslant i \leqslant s$.

Lemma 1 ([4] (Lemma 4, Lemma 5))**.** *Let P be a finite poset.*

1. *Let I and J be poset ideals of P with $I \neq J$. Then the convex hull of $\{\rho(I), \rho(J)\}$ forms an edge of $\mathscr{O}(P)$ if and only if $I \subset J$ and $J \backslash I$ is connected in P.*
2. *Let A and B be antichains of P with $A \neq B$. Then the convex hull of $\{\rho(A), \rho(B)\}$ forms an edge of $\mathscr{C}(P)$ if and only if $(A \backslash B) \cup (B \backslash A)$ is connected in P.*

3. Faces of 2-Dimensional Simplex

Using Lemma 1, we show the following description of faces of 2-dimensional simplex.

Theorem 1. *Let P be a finite poset. Let I, J, and K be pairwise distinct poset ideals of P. Then the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ forms a 2-face of $\mathscr{O}(P)$ if and only if $I \subset J \subset K$ and $K \backslash I$ is connected in P.*

Proof. **("Only if")** If the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ forms a 2-face of $\mathscr{O}(P)$, then the convex hulls of $\{\rho(I), \rho(J)\}$, $\{\rho(J), \rho(K)\}$, and $\{\rho(I), \rho(K)\}$ form edges of $\mathscr{O}(P)$. It then follows from Lemma 1 that $I \subset J \subset K$ and $K \backslash I$ is connected in P.

("If") Suppose that the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ has dimension 1. Then there exists a line passing through the lattice points $\rho(I), \rho(J)$, and $\rho(K)$. Hence $\rho(I), \rho(J)$, and $\rho(K)$ cannot be vertices of $\mathscr{O}(P)$. Thus the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ has dimension 2.

Let $P = \{x_1, \ldots, x_d\}$. If there exists a maximal element x_i of P not belonging to $I \cup J \cup K$, then the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ lies in the facet $x_i = 0$. If there exists a minimal element x_j of P belonging to $I \cap J \cap K$, then the convex hull of $\{\rho(I), \rho(J), \rho(K)\}$ lies in the facet $x_j = 1$. Hence, working with induction on $d (\geqslant 2)$, we may assume that $I \cup J \cup K = P$ and $I \cap J \cap K = \varnothing$. Suppose that $\varnothing = I \subset J \subset K = P$ and $K \backslash I = P$ is connected.

Case 1. $\sharp(J) = 1$.

Let $J = \{x_i\}$ and $P' = P \backslash \{x_i\}$. Then P' is a connected poset. Let $x_{i_1}, \ldots, x_{i_q}$ be the maximal elements of P and $\mathcal{A}_{ij} = \{y \in P' \mid y < x_{i_j}\}$, where $1 \leqslant j \leqslant q$. Then we write

$$b_k = \begin{cases} \sharp(\{i_j \mid x_k \in \mathcal{A}_{ij}\}) & \text{if } k \notin \{i_1, \ldots, i_q, i\} \\ 0 & \text{if } k = i \\ -\sharp(\mathcal{A}_{ij}) & \text{if } k \in \{i_1, \ldots, i_q\} \end{cases}.$$

We then claim that the hyperplane $\mathscr{H}$ of $\mathbb{R}^d$ defined by the equation $h(\mathbf{x}) = \sum_{k=1}^{d} b_k x_k = 0$ is a supporting hyperplane of $\mathscr{O}(P)$ and that $\mathscr{H} \cap \mathscr{O}(P)$ coincides with the convex hull of $\{\rho(\varnothing), \rho(J), \rho(P)\}$. Clearly $h(\rho(\varnothing)) = h(\rho(P)) = 0$ and $h(\rho(J)) = b_i = 0$. Let I be a poset ideal of P with $I \neq \varnothing$, $I \neq P$ and $I \neq J$. We have to prove that $h(\rho(I)) > 0$. To simplify the notation, suppose that $I \cap \{x_{i_1}, \ldots, x_{i_q}\} = \{x_{i_1}, \ldots, x_{i_r}\}$, where $0 \leqslant r < q$. If $r = 0$, then $h(\rho(J)) > 0$. Let $1 \leqslant r < q$, $I' = I \backslash \{x_i\}$, and $K = \bigcup_{j=1}^{r} (\mathcal{A}_{i_j} \cup \{x_{i_j}\})$. Then I' and K are poset ideals of P and $h(\rho(K)) \leqslant h(\rho(I')) = h(\rho(I))$. We claim $h(\rho(K)) > 0$. One has $h(\rho(K)) \geqslant 0$. Moreover, $h(\rho(K)) = 0$ if and only if no $z \in K$ belongs to $\mathcal{A}_{i_{r+1}} \cup \cdots \cup \mathcal{A}_{i_q}$. Now, since P' is connected, it follows that there exists $z \in K$ with $z \in \mathcal{A}_{i_{r+1}} \cup \cdots \cup \mathcal{A}_{i_q}$. Hence $h(\rho(K)) > 0$. Thus $h(\rho(I)) > 0$.

Case 2. $\sharp(J) = d - 1$.

Let $P \backslash J = \{x_i\}$ and $P' = P \backslash \{x_i\}$. Then P' is a connected poset. Thus we can show the existence of a supporting hyperplane of $\mathscr{O}(P)$ which contains the convex hull of $\{\rho(\varnothing), \rho(J), \rho(P)\}$ by the same argument in Case 1.

Case 3. $2 \leqslant \sharp(J) \leqslant d - 2$.

To simplify the notation, suppose that $J = \{x_1, \ldots, x_\ell\}$. Then $P \backslash J = \{x_{\ell+1}, \ldots, x_d\}$. Since J and $P \backslash J$ are subposets of P, these posets are connected. Let $x_{i_1}, \ldots, x_{i_q}$ be the maximal elements of J and $x_{i_{q+1}}, \ldots, x_{i_{q+r}}$ the maximal elements of $P \backslash J$. Then we write

$$\mathcal{A}_{ij} = \begin{cases} \{y \in J \mid y < x_{i_j}\} & \text{if } 1 \leqslant j \leqslant q \\ \{y \in P \backslash J \mid y < x_{i_j}\} & \text{if } q+1 \leqslant j \leqslant r \end{cases}$$

and

$$b_k = \begin{cases} \sharp(\{i_j \mid x_i \in \mathcal{A}_{ij}\}) & \text{if } k \notin \{i_1, \ldots, i_q, i_{q+1}, \ldots, i_{q+r}\} \\ -\sharp(\mathcal{A}_{ij}) & \text{if } k \in \{i_1, \ldots, i_q, i_{q+1}, \ldots, i_{q+r}\} \end{cases}.$$

We then claim that the hyperplane $\mathscr{H}$ of $\mathbb{R}^d$ defined by the equation $h(\mathbf{x}) = \sum_{k=1}^{d} b_k x_k = 0$ is a supporting hyperplane of $\mathscr{O}(P)$ and $\mathscr{H} \cap \mathscr{O}(P)$ coincides with the convex hull of $\{\rho(\varnothing), \rho(J), \rho(P)\}$. Clearly $h(\rho(\varnothing)) = h(\rho(J)) = h(\rho(P \backslash J)) = 0$, then $h(\rho(P)) = h(\rho(J)) + h(\rho(P \backslash J)) = 0$. Let I be a poset ideal of P with $I \neq \varnothing$, $I \neq P$ and $I \neq J$. What we must prove is $h(\rho(I)) > 0$.

If $I \subset J$, then I is a poset ideal of J. To simplify the notation, suppose that $I \cap \{x_{i_1}, \ldots, x_{i_q}\} = \{x_{i_1}, \ldots, x_{i_s}\}$, where $0 \leqslant s < q$. If $s = 0$, then $h(\rho(I)) > 0$. Let $1 \leqslant s < q$, $K = \bigcup_{j=1}^{s} (\mathcal{A}_{i_j} \cup \{x_{i_j}\})$. Then K is a poset ideal of J and $h(\rho(K)) \leqslant h(\rho(I))$. Thus we can show $h(\rho(K)) > 0$ by the same argument in Case 1 (Replace r with s and P' with J).

If $J \subset I$, then $I \backslash J$ is a poset ideal of $P \backslash J$. To simplify the notation, suppose that $(I \backslash J) \cap \{x_{i_{q+1}}, \ldots, x_{i_{q+r}}\} = \{x_{i_{q+1}}, \ldots, x_{i_{q+t}}\}$, where $0 \leqslant t < r$. If $t = 0$, then $h(\rho(I)) = h(\rho(J)) + h(\rho(I \backslash J)) = h(\rho(I \backslash J)) > 0$. Let $1 \leqslant t < r$, $K = \bigcup_{j=q+1}^{q+t} (\mathcal{A}_{i_j} \cup \{x_{i_j}\})$. Then K is a poset ideal of $P \backslash J$ and $h(\rho(K)) \leqslant h(\rho(I \backslash J)) = h(\rho(I))$. Thus we can show $h(\rho(K)) > 0$ by the same argument in Case 1 (Replace r with $q + t$, q with $q + r$ and P' with $P \backslash J$). Consequently, $h(\rho(I)) > 0$, as desired. □

Let $A \triangle B$ denote the symmetric difference of the sets A and B, that is $A \triangle B = (A \backslash B) \cup (B \backslash A)$.

Theorem 2. *Let P be a finite poset. Let A, B, and C be pairwise distinct antichains of P. Then the convex hull of $\{\rho(A), \rho(B), \rho(C)\}$ forms a 2-face of $\mathscr{C}(P)$ if and only if $A \triangle B$, $B \triangle C$ and $C \triangle A$ are connected in P.*

Proof. **("Only if")** If the convex hull of $\{\rho(A), \rho(B), \rho(C)\}$ forms a 2-face of $\mathscr{C}(P)$, then the convex hulls of $\{\rho(A), \rho(B)\}$, $\{\rho(B), \rho(C)\}$, and $\{\rho(A), \rho(C)\}$ form edges of $\mathscr{C}(P)$. It then follows from Lemma 1 that $A \triangle B$, $B \triangle C$ and $C \triangle A$ are connected in P.

("If") Suppose that the convex hull of $\{\rho(A), \rho(B), \rho(C)\}$ has dimension 1. Then there exists a line passing through the lattice points $\rho(A)$, $\rho(B)$, and $\rho(C)$. Hence $\rho(A)$, $\rho(B)$, and $\rho(C)$ cannot be vertices of $\mathscr{C}(P)$. Thus the convex hull of $\{\rho(A), \rho(B),$
$\rho(C)\}$ has dimension 2.

Let $P = \{x_1, \ldots, x_d\}$. If $A \cup B \cup C \neq P$ and $x_i \notin A \cup B \cup C$, then the convex hull of $\{\rho(A), \rho(B), \rho(C)\}$ lies in the facet $x_i = 0$. Furthermore, if $A \cup B \cup C = P$ and $A \cap B \cap C \neq \emptyset$, then $x_j \in A \cap B \cap C$ is isolated in P and x_j itself is a maximal chain of P. Thus the convex hull of $\{\rho(A), \rho(B), \rho(C)\}$ lies in the facet $x_j = 1$. Hence, working with induction on $d (\geqslant 2)$, we may assume that $A \cup B \cup C = P$ and $A \cap B \cap C = \emptyset$. As stated in the proof of [3] ([Theorem 2.1]), if $A \triangle B$ is connected in P, then A and B satisfy either (i) $B \subset A$ or (ii) $y < x$ whenever $x \in A$ and $y \in B$ are comparable. Hence, we consider the following three cases:

(a) If $B \subset A$, then $A \triangle B = A \backslash B$ is connected in P, and thus $\sharp(A \backslash B) = 1$. Let $A \backslash B = \{x_k\}$. If $C \cap A \neq \emptyset$, then $C \cap A = \{x_k\}$, since $A \cap B \cap C = C \cap B = \emptyset$. Namely x_k is isolated in P. Hence $B \triangle C = B \cup C = A \cup B \cup C = P$ cannot be connected. Thus $C \cap A = \emptyset$. In this case, we may assume $z < x$ if $x \in A$ and $z \in C$ are comparable. Furthermore, P has rank 1.

(b) If $B \not\subset A$ and $B \cap A \neq \emptyset$, then we may assume $y < x$ if $x \in A$ and $y \in B$ are comparable. If $C \subset B$ with $C \cap A \cap B = \emptyset$, then as stated in (a), $C \triangle A$ cannot be connected. Since $C \not\subset B$, we may assume $z < y$ if $y \in B$ and $z \in C$ are comparable. If $C \cap B \neq \emptyset$, then $C \cap A = \emptyset$ and P has rank 1 or 2. Similarly, if $C \cap B = \emptyset$, then $C \cap A = \emptyset$ and P has rank 2.

(c) Let $B \not\subset A$ and $B \cap A = \emptyset$. We may assume that if $x \in A$ and $y \in B$ are comparable, then $y < x$. If $C \subset B$, then we regard this case as equivalent to (a). Let $C \not\subset B$. We may assume $z < y$ if $y \in B$ and $z \in C$ are comparable. Moreover, if $C \cap B \neq \emptyset$, then we regard this case as equivalent to (b). If $C \cap B = \emptyset$, then $C \cap A = \emptyset$ and P has rank 2.

Consequently, there are five cases as regards antichains for $\mathscr{C}(P)$.

Case 1. $B \subset A$, $C \cap A = \emptyset$, and $C \cap B = \emptyset$.

For each $x_i \in B$ we write b_i for the number of elements $z \in C$ with $z < x_i$. For each $x_j \in C$ we write c_j for the number of elements $y \in B$ with $x_j < y$. Let $a_k = 0$ for $A \backslash B = \{x_k\}$. Clearly $\sum_{x_i \in B} b_i = \sum_{x_j \in C} c_j = q$, where q is the number of pairs (y, z) with $y \in B$, $z \in C$ and $z < y$. Let $h(\mathbf{x}) = \sum_{x_i \in B} b_i x_i + \sum_{x_j \in C} c_j x_j + a_k x_k$ and let $\mathscr{H}$ be the hyperplane of $\mathbb{R}^d$ defined by $h(\mathbf{x}) = q$. Then $h(\rho(A)) = h(\rho(B)) = h(\rho(C)) = q$. We claim that, for any antichain D of P with $D \neq A$, $D \neq B$, and $D \neq C$, one has $h(\rho(D)) < q$. Let $D = B_1 \cup C_1$ or $D = \{x_k\} \cup C_1$ with $B_1 \subsetneq B$ and $C_1 \subsetneq C$. Suppose $D = B_1 \cup C_1$. Since $B \triangle C$ is connected and since D is an antichain of P, it follows that $\sum_{x_i \in B_1} b_i + \sum_{x_j \in C_1} c_j < q$. Thus $h(\rho(D)) < q$. Suppose that $D = \{x_k\} \cup C_1$. It follows that $\sum_{x_j \in C_1} c_j + a_k = \sum_{x_j \in C_1} c_j < \sum_{x_j \in C} c_j = q$. Thus $h(\rho(D)) < q$.

Case 2. $B \not\subset A$, $B \cap A \neq \emptyset$, $C \not\subset B$, $C \cap B \neq \emptyset$, $C \cap A = \emptyset$, and P has rank 1.

We define four numbers as follows:

$$\alpha_i = \sharp(\{y \in B \backslash A \mid y < x_i \ , \ x_i \in A \backslash B\});$$
$$\gamma_j = \sharp(\{x \in A \backslash B \mid x_j < x \ , \ x_j \in B \backslash A\});$$
$$\alpha_k = \sharp(\{z \in C \backslash B \mid z < x_k \ , \ x_k \in B \backslash C\});$$
$$\gamma_\ell = \sharp(\{y \in B \backslash C \mid x_\ell < y \ , \ x_\ell \in C \backslash B\}).$$

Since P has rank 1, $B \subset A \cup C = P$. It follows that $A = (A \backslash B) \cup (B \backslash C)$, $C = (B \backslash A) \cup (C \backslash B)$. Then

$$\sum_{x_s \in A} \alpha_s = \sum_{x_i \in A \backslash B} \alpha_i + \sum_{x_k \in B \backslash C} \alpha_k = q;$$
$$\sum_{x_j \in B \backslash A} \gamma_j + \sum_{x_k \in B \backslash C} \alpha_k = q;$$
$$\sum_{x_u \in C} \gamma_u = \sum_{x_j \in B \backslash A} \gamma_j + \sum_{x_\ell \in C \backslash B} \gamma_\ell = q,$$

where q_1 is the number of pairs (x, y) with $x \in A \backslash B$, $y \in B \backslash A$ and $y < x$, q_2 is the number of pairs (y, z) with $y \in B \backslash C$, $z \in C \backslash B$ and $z < y$, and $q = q_1 + q_2$. Let

$$\begin{aligned} h(\mathbf{x}) &= \sum_{x_s \in A} \alpha_s x_s + \sum_{x_u \in C} \gamma_u x_u \\ &= \sum_{x_i \in A \backslash B} \alpha_i x_i + \left(\sum_{x_j \in B \backslash A} \gamma_j x_j + \sum_{x_k \in B \backslash C} \alpha_k x_k \right) + \sum_{x_\ell \in C \backslash B} \gamma_\ell x_\ell \end{aligned}$$

and $\mathscr{H}$ the hyperplane of $\mathbb{R}^d$ defined by $h(\mathbf{x}) = q$. Then $h(\rho(A)) = h(\rho(B)) = h(\rho(C)) = q$. We claim that, for any antichain D of P with $D \neq A$, $D \neq B$ and $D \neq C$, one has $h(\rho(D)) < q$. Let $D = D_1 \cup D_2$ with D_1 is an antichain of $A \triangle B$ and D_2 is an antichain of $B \triangle C$. Since $A \triangle B$, $B \triangle C$ are connected, it follows that $h(\rho(D_1)) < q_1$ and $h(\rho(D_2)) < q_2$. Thus $h(\rho(D)) = h(\rho(D_1)) + h(\rho(D_2)) < q_1 + q_2 = q$.

Case 3. $B \not\subset A$, $B \cap A \neq \emptyset$, $C \not\subset B$, $C \cap B \neq \emptyset$, $C \cap A = \emptyset$, and P has rank 2.

For each $x_i \in P$ we write $c(i)$ for the number of maximal chains, which contain x_i. Let q be the number of maximal chains in P. Since each $x_i \in A$ is maximal element and each $x_k \in C$ is minimal element, $\sum_{x_i \in A} c(i) = \sum_{x_k \in C} c(k) = q$. Then

$$\begin{aligned} \sum_{x_j \in B} c(j) &= \sum_{x_s \in B \cap A} c(s) + \sum_{x_t \in B \cap C} c(t) + \sum_{x_u \in B \backslash (A \cup C)} c(u) \\ &= \sum_{x_s \in B \cap A} c(s) + \sum_{x_t \in B \cap C} c(t) + \left(\sum_{x_v \in A \backslash B} c(v) - \sum_{x_t \in B \cap C} c(t) \right) \\ &= \sum_{x_i \in A} c(i) = q. \end{aligned}$$

Let $h(\mathbf{x}) = \sum_{x_i \in P} c(i) x_i$ and $\mathscr{H}$ the hyperplane of $\mathbb{R}^d$ defined by $h(\mathbf{x}) = q$. Then $h(\rho(A)) = h(\rho(B)) = h(\rho(C)) = q$. We claim that, for any antichain D of P with $D \neq A$, $D \neq B$ and $D \neq C$, one has $h(\rho(D)) < q$. $D = A_1 \cup B_1 \cup C_1$ with $A_1 \subset A \backslash B$, $B_1 \subsetneq B$, and $C_1 \subsetneq C \backslash B$. Now, we define two subsets of B:

$$B_2 = \{x_j \in B \mid x_j < x_i,\ x_i \in A_1\};$$
$$B_3 = \{x_j \in B \mid x_k < x_j,\ x_k \in C_1\}.$$

Then $B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \emptyset$ and $B_1 \cup B_2 \cup B_3 \subset B_3$. Let $\sum_{x_i \in A} c(i) = q_1$, $\sum_{x_j \in B_1} c(j) = q_2$, $\sum_{x_k \in C_1} c(k) = q_3$, $\sum_{x_j \in B_2} c(j) = q_1'$, and $\sum_{x_j \in B_3} c(j) = q_3'$. Since $A \triangle B$, $B \triangle C$ are connected, it follows that $q_1 < q_1'$ and $q_3 < q_3'$. Hence

$$\begin{aligned} h(\rho(D)) &= \sum_{x_i \in A_1} c(i) + \sum_{x_j \in B_1} c(j) + \sum_{x_k \in C_1} c(k) \\ &= q_1 + q_2 + q_3 < q_1' + q_2 + q_3' \\ &= \sum_{x_j \in B_2} c(j) + \sum_{x_j \in B_1} c(j) + \sum_{x_j \in B_3} c(j) \leqslant \sum_{x_j \in B} c(j) = q. \end{aligned}$$

Thus $h(\rho(D)) < q$.

Case 4. $B \not\subset A$, $B \cap A \neq \emptyset$, $C \cap B = \emptyset$, and $C \cap A = \emptyset$.

Since P has rank 2, we can show $h(\rho(D)) < q$ by the same argument in Case 3 (Suppose $C \cap B = \emptyset$).

Case 5. $B \not\subset A$, $B \cap A = \emptyset$, $C \cap B = \emptyset$ and $C \cap A = \emptyset$.

Since P has rank 2, we can show $h(\rho(D)) < q$ by the same argument in Case 3 (Suppose $B \cap A = C \cap B = \emptyset$).

In conclusion, each $\mathscr{H}$ is a supporting hyperplane of $\mathscr{C}(P)$ and $\mathscr{H} \cap \mathscr{C}(P)$ coincides with the convex hull of $\{\rho(A), \rho(B), \rho(C)\}$, as desired. □

Corollary 1. *Triangles in 1-skeleton of $\mathscr{O}(P)$ or $\mathscr{C}(P)$ are in one-to-one correspondence with faces of 2-dimensional simplex of each polytope.*

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