Best Proximity Results with Applications to Nonlinear Dynamical Systems

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Abstract: Best proximity point theorem furnishes sufficient conditions for the existence and computation of an approximate solution \( \omega \) that is optimal in the sense that the error \( \sigma(\omega, J\omega) \) assumes the global minimum value \( \sigma(\theta, \vartheta) \). The aim of this paper is to define the notion of Suzuki \( \alpha-\Theta \)-proximal multivalued contraction and prove the existence of best proximity points \( \omega \) satisfying \( \sigma(\omega, J\omega) = \sigma(\theta, \vartheta) \), where \( J \) is assumed to be continuous or the space \( M \) is regular. We derive some best proximity results on a metric space with graphs and ordered metric spaces as consequences. We also provide a non trivial example to support our main results. As applications of our main results, we discuss some variational inequality problems and dynamical programming problems.

Keywords: nonlinear dynamical systems; best proximity point; \( \alpha \)-proximal contraction; multi-valued mappings; graphs

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1. Introduction and Preliminaries

In 1969, Fan [1] initiated and obtained a classical best approximation result, that is, if \( \theta \) is a nonempty compact convex subset of a Hausdorff locally convex topological vector space \( \vartheta \) and \( J : \theta \to \vartheta \) is a continuous mapping, then there exists \( \omega \) in \( \theta \) such that \( \sigma(\omega, J\omega) = \sigma(J\omega, \theta) \). In 2010, Basha [2] introduced the notion of best proximity point of a non-self mapping. Additionally he gave a generalization of the Banach fixed point theorems by a best proximity theorem. Let \( \theta \) and \( \vartheta \) be nonempty subsets of a metric space \( (M, \sigma) \). A point \( \omega \) is called a best proximity point of mapping \( J : \theta \to \vartheta \) if \( \sigma(\omega, J\omega) = \sigma(\theta, \vartheta) \), where \( \sigma(\theta, \vartheta) = \inf\{\sigma(\omega, \omega) : \omega \in \theta, \omega \in \vartheta\} \). Sankar Raj [3] and Zhang et al. [4] defined the notion of \( P \)-property and weak \( P \)-property respectively. Jleli et al. [5] defined the concept of \( P \)-property admissible for non self mapping \( J : \theta \to \vartheta \). Hussain et al. [6] utilized the concept of \( \alpha \)-proximal admissible and introduced Suzuki type \( \alpha^+ - \psi \) proximal contraction to generalize several best proximity results. Chen et al. [7] defined \( \alpha \)-admissible Meir-Keeler-type set contractions which have KKM\(^* \) property on almost convex sets and established some generalized fixed point theorems. In 2014, Ali et al. [8] gave the conception of \( \alpha \)-proximal admissible for multivalued mapping and obtained some best proximity point theorems for multivalued mappings. The goal of this article is to define Suzuki \( \alpha-\Theta \)-proximal multivalued contraction and establish some generalized best proximity point results.

2. Preliminaries

In this section, we give some preliminaries.
Definition 1. Let $\theta$ and $\vartheta$ be nonempty subsets of a metric space $(\mathcal{M}, \sigma)$, and define $\theta_0$ and $\vartheta_0$ by

$$
\theta_0 = \{ \omega \in \theta : \text{there exists some } \varkappa \in \vartheta \text{ such that } \sigma(\omega, \varkappa) = \sigma(\theta, \vartheta) \}
$$

$$
\vartheta_0 = \{ \varkappa \in \vartheta : \text{there exists some } \omega \in \theta \text{ such that } \sigma(\omega, \varkappa) = \sigma(\theta, \vartheta) \}.
$$

- (Sankar Raj [3]) The pair $(\theta, \vartheta)$ is said to satisfy the $P$-property if $\theta_0 \neq \emptyset$ and the following condition is satisfied:

$$
\omega, \vartheta \in \theta_0, \varkappa, \rho \in \vartheta_0, \quad \sigma(\omega, \varkappa) = \sigma(\theta, \vartheta) = \sigma(\rho, \omega) \implies \sigma(\omega, \vartheta) = \sigma(\varkappa, \rho).
$$

- (Zhang et al. [4]) The pair $(\theta, \vartheta)$ is said to have the weak $P$-property if $\theta_0 \neq \emptyset$ and the following condition is satisfied: $\omega, \vartheta \in \theta_0, \varkappa, \rho \in \vartheta_0,$

$$
\sigma(\omega, \varkappa) = \sigma(\rho, \omega) = \sigma(\theta, \vartheta) \implies \sigma(\omega, \vartheta) \leq \sigma(\varkappa, \rho).
$$

Jleli et al. [5] defined the concept of $\alpha$-proximal admissible for non self mapping $\mathcal{J} : \theta \to \theta$ as follows:

Definition 2. Let $\theta$ and $\vartheta$ be two nonempty subsets of a metric space $(\mathcal{M}, \sigma)$. A mapping $\mathcal{J} : \theta \to \theta$ is called $\alpha$-proximal admissible if there exists a mapping $\alpha : \theta \times \theta \to [0, \infty)$ such that

$$
\alpha(\omega, \vartheta) \quad \sigma(\varkappa, \mathcal{J} \omega) = \sigma(\theta, \mathcal{J} \omega) \quad \sigma(\rho, \mathcal{J} \omega) = \sigma(\theta, \mathcal{J} \omega)
$$

where $\omega, \vartheta, \varkappa, \rho \in \theta$.

Later on, Ali et al. [8] extended the notion of $\alpha$-proximal admissible for multivalued mapping in this way.

Definition 3 ([8]). Let $\theta$ and $\vartheta$ be two nonempty subsets of a metric space $(\mathcal{M}, \sigma)$. A mapping $\mathcal{J} : \theta \to 2^\theta$ is called $\alpha$-proximal admissible if there exists a mapping $\alpha : \theta \times \theta \to [0, \infty)$ such that

$$
\alpha(\omega, \vartheta) \quad \sigma(\varkappa, w_1) = \sigma(\theta, \vartheta) \quad \sigma(\rho, w_2) = \sigma(\theta, \vartheta)
$$

where $\omega, \vartheta, \varkappa, \rho \in \theta, w_1 \in \mathcal{J} \omega$ and $w_2 \in \mathcal{J} \omega$.

For more details in the direction of best proximity, we refere the following [9–11].

On the other hand, Jleli et al. [12] introduced a new type of contraction called $\Theta$-contraction and established some new fixed point theorems for such a contraction in the context of generalized metric spaces.

Definition 4. Let $\Theta : (0, \infty) \to (1, \infty)$ be a mapping such that:

- $(\Theta_1)$ $\Theta$ is nondecreasing;
- $(\Theta_2)$ $\forall \{ \omega_n \} \subseteq \mathbb{R}^+, \lim_{n \to \infty} \Theta(\omega_n) = 1 \iff \lim_{n \to \infty} (\omega_n) = 0$;
- $(\Theta_3)$ $\exists 0 < k < 1$ and $l \in (0, \infty)$ such that $\lim_{\omega \to 0^+} \frac{\Theta(\omega) - 1}{\omega^k} = l$. 


A mapping \( \mathcal{J} : \mathcal{M} \to \mathcal{M} \) is called a \( \Theta \)-contraction if there exist some mapping \( \Theta \) satisfying (\( \Theta_1 \))-\( \Theta_3 \)) and a constant \( k \in (0, 1) \) such that

\[
\sigma(\mathcal{J} \omega, \mathcal{J} \omega) \neq 0 \implies \Theta(\sigma(\mathcal{J} \omega, \mathcal{J} \omega)) \leq [\Theta(\sigma(\omega, \omega))]^k
\]

\( \forall \omega, \omega \in \mathcal{M} \). Following Jleli et al. [12], the set of all continuous functions \( \Theta : \mathbb{R}^+ \to (1, \infty) \) satisfying (\( \Theta_1 \)) \( - \) (\( \Theta_3 \)) conditions is represented by \( \mathcal{F} \). For more details in the direction of \( \Theta \)-contractions, we refer the readers to [13–15].

Motivated by [6,8,12], we introduce the notion of Suzuki \( a \)-\( \Theta \)-proximal multivalued contraction by using the concept of \( a \)-proximal admissibility for multivalued mappings and \( \Theta \)-contraction to prove some new results. Our results extend some best proximity results of literature.

3. Results and Discussions

Throughout this paper, \((\mathcal{M}, \sigma)\) is a complete metric space and \( C(\mathcal{M}), CB(\mathcal{M}) \) and \( K(\mathcal{M}) \) denote the families of all nonempty closed subsets, nonempty closed and bounded subsets and compact subsets of \((\mathcal{M}, \sigma)\) respectively. For any \( \theta, \omega \in C(\mathcal{M}) \), let the mapping \( H(\cdot, \cdot) \) be the generalized Hausdorff metric with respect to \( \sigma \) defined by

\[
H(\theta, \omega) = \left\{ \begin{array}{ll}
\max \left\{ \sup_{\omega \in \theta} \sigma(\omega, \theta), \sup_{\omega \in \theta} \sigma(\omega, \omega) \right\}, & \text{if it exists} \\
\infty, & \text{otherwise}
\end{array} \right.
\]

**Definition 5.** Let \((\mathcal{M}, \sigma)\) be a metric space and \( \theta, \omega \) be a non-empty subsets of \( \mathcal{M} \). A multi-valued mapping \( \mathcal{J} : \theta \to 2^\theta \) is said to be Suzuki \( a \)-\( \Theta \)-proximal multivalued contraction if there exist functions \( \Theta \in \mathcal{F} \), \( a : \theta \times \theta \to [0, \infty) \) and some constant \( k \in (0, 1) \) such that

\[
\sigma^a(\mathcal{J} \omega, \mathcal{J} \omega) \leq a(\omega, \omega) \sigma(\omega, \omega) \implies \Theta(H(\mathcal{J} \omega, \mathcal{J} \omega)) \leq [\Theta(\sigma(\omega, \omega))]^k
\]

\( \forall \omega, \omega \in \theta \), where \( \sigma^a(\omega, \omega) = \sigma(\omega, \omega) - \sigma(\theta, \theta) \) satisfying \( H(\mathcal{J} \omega, \mathcal{J} \omega) > 0 \).

Note that, if \( C \) be a compact subset of a metric space \((\mathcal{M}, \sigma)\) and \( \omega \in \mathcal{M} \), then there exists \( \omega \in C \) such that \( \sigma(\omega, C) = \sigma(\omega, \omega) \).

**Theorem 1.** Let \( \theta, \omega \in C(\mathcal{M}) \) such that \( \theta_0 \neq \emptyset \) and \( \mathcal{J} : \theta \to K(\theta) \) be Suzuki \( a \)-\( \Theta \)-proximal multivalued contraction and \( a \)-proximal admissible. Suppose that

(i) \( \forall \omega \in \theta_0 \), we have \( \mathcal{J} \omega \subseteq \theta_0 \) and the pair \((\theta, \theta)\) satisfies the weak P-Property;

(ii) \( \exists \omega_0, \omega_1 \in \theta_0 \) and \( \omega_1 \in \mathcal{J} \omega_0 \) such that

\[
\sigma(\omega_1, \omega_1) = \sigma(\theta, \theta), \text{ and } a(\omega_0, \omega_1) \geq 1.
\]

(iii) \( \mathcal{J} \) is continuous.

Then \( \mathcal{J} \) has a best proximity point.

**Proof.** By supposition (ii), \( \exists \omega_0, \omega_1 \in \theta_0 \) and \( \omega_1 \in \mathcal{J} \omega_0 \) such that

\[
\sigma(\omega_1, \omega_1) = \sigma(\theta, \theta), \text{ and } a(\omega_0, \omega_1) \geq 1.
\]
If \( \omega_1 \in J \omega_1 \), then we obtain

\[
\sigma(\theta, \theta) \leq \sigma(\omega_1, J \omega_1) \leq \sigma(\omega_1, \omega_1) = \sigma(\theta, \theta)
\]

and so \( \omega_1 \) is the required point.

Now, let \( \omega_1 \notin J \omega_1 \). Since \( \omega_1 \in J \omega_0 \), we have

\[
\sigma(\omega_0, J \omega_0) \leq \sigma(\omega_0, \omega_1) \leq \sigma(\omega_0, \omega_1) + \sigma(\omega_1, \omega_1).
\]

From \( \sigma(\omega_1, \omega_1) = \sigma(\theta, \theta) \), we obtain

\[
\sigma(\omega_0, J \omega_0) \leq \sigma(\omega_0, \omega_1) + \sigma(\theta, \theta)
\]

and so

\[
\sigma(\omega_0, J \omega_0) - \sigma(\theta, \theta) \leq \sigma(\omega_0, \omega_1).
\]

By Definition, we have

\[
\sigma^*(\omega_0, J \omega_0) \leq \sigma(\omega_0, \omega_1) \leq \alpha(\omega_0, \omega_1) \sigma(\omega_0, \omega_1).
\]

This pursues that \( H(J \omega_0, J \omega_1) > 0 \). By (1), we have

\[
\Theta(H(J \omega_0, J \omega_1)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k.
\]

Otherwise, as \( 0 < \sigma(\omega_1, J \omega_1) \leq H(J \omega_0, J \omega_1) \), so by (\( \Theta_1 \)), we have

\[
\Theta(\sigma(\omega_1, J \omega_1)) \leq \Theta(H(J \omega_0, J \omega_1)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k.
\]

Since \( J \omega_1 \) is compact, there exists \( \omega_2 \in J \omega_1 \) such that \( \sigma(\omega_1, J \omega_1) = \sigma(\omega_1, \omega_2) \) and so

\[
\Theta(\sigma(\omega_1, \omega_2)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k.
\]

By supposition (i), we get \( J \omega_1 \subseteq \theta_0 \) and so \( \exists \omega_2 \in \theta_0 \) such that

\[
\sigma(\omega_2, \omega_2) = \sigma(\theta, \theta).
\]

Since \( J \) is an \( \alpha \)-proximal admissible, so it follows from (2) and (5) that

\[
\alpha(\omega_1, \omega_2) \geq 1.
\]

Since \( (\theta, \theta) \) satisfies the weak \( P \)-Property, so by (i) we get

\[
\sigma(\omega_1, \omega_2) \leq \sigma(\omega_1, \omega_2).
\]

If \( \omega_1 = \omega_2 \), then we obtain \( \omega_1 \) as the required point. Suppose that \( \omega_1 \neq \omega_2 \). From (4), (7) and (\( \Theta_1 \)), it follows that

\[
1 < \Theta(\sigma(\omega_1, \omega_2)) \leq \Theta(\sigma(\omega_1, \omega_2)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k.
\]

If \( \omega_2 \in J \omega_2 \), then \( \omega_2 \) is the required point. Now, assume that \( \omega_2 \notin J \omega_2 \). Since \( \omega_2 \in J \omega_1 \), we have

\[
\sigma(\omega_1, J \omega_1) \leq \sigma(\omega_1, \omega_2) \leq \sigma(\omega_1, \omega_2) + \sigma(\omega_2, \omega_2).
\]
From $\sigma(\omega_2, \omega_2) = \sigma(\theta, \theta)$, we obtain
\[
\sigma(\omega_1, J\omega_1) \leq \sigma(\omega_1, \omega_2) + \sigma(\theta, \theta)
\]
and so
\[
\sigma(\omega_1, J\omega_1) - \sigma(\theta, \theta) \leq \sigma(\omega_1, \omega_2).
\]
By Definition, we have
\[
\sigma^* (\omega_1, J\omega_1) \leq \sigma(\omega_1, \omega_2) \leq \alpha(\omega_1, \omega_2) \sigma(\omega_1, \omega_2).
\]
It pursues that $H(J\omega_1, J\omega_2) > 0$. From (1), we have
\[
\Theta(H(J\omega_1, J\omega_2)) \leq \Theta(\sigma(\omega_1, \omega_2)) \leq |\Theta(\sigma(\omega_1, \omega_2))|^k.
\]
Otherwise, as $0 < \sigma(\omega_2, J\omega_2) \leq H(J\omega_1, J\omega_2)$, so by $(\Theta_1)$, we have
\[
\Theta(\sigma(\omega_2, J\omega_2)) \leq \Theta(H(J\omega_1, J\omega_2)) \leq |\Theta(\sigma(\omega_1, \omega_2))|^k.
\]
Since $J\omega_2$ is compact, so $\exists \omega_3 \in J\omega_2$ such that $\sigma(\omega_2, J\omega_2) = \sigma(\omega_2, \omega_3)$ and so
\[
\Theta(\sigma(\omega_2, \omega_3)) \leq |\Theta(\sigma(\omega_1, \omega_2))|^k.
\]
By supposition (i), we have $J\omega_2 \subseteq \theta_0$ and so $\exists \omega_3 \in \theta_0$ such that
\[
\sigma(\omega_3, \omega_3) = \sigma(\theta, \theta).
\]
Since $J$ is an $\alpha$-proximal admissible, so it follows from (5) and (11) that
\[
\alpha(\omega_2, \omega_3) \geq 1.
\]
Since $(\theta, \theta)$ satisfies the weak $P$-Property, so by supposition (i), we have
\[
\sigma(\omega_2, \omega_3) \leq \sigma(\omega_2, \omega_3).
\]
If $\omega_2 = \omega_3$, then $\omega_2$ is the required best proximity point of $J$. Assume that $\omega_2 \neq \omega_3$. From (10), (12) and $(\Theta_1)$, it follows that
\[
\Theta(\sigma(\omega_2, \omega_3)) \leq \Theta(\sigma(\omega_2, \omega_3)) \leq |\Theta(\sigma(\omega_1, \omega_2))|^k.
\]
Hence, by induction, we have $\{\omega_n\} \subseteq \theta_0$ and $\{\omega_n\} \subseteq \theta_0$ such that
\begin{enumerate}[(a)]
\item $\alpha(\omega_n, \omega_{n+1}) \geq 1$ and $\omega_n \neq \omega_{n+1}$;
\item $\omega_n \in J\omega_{n-1}$ and $\omega_n \notin J\omega_n$;
\item $\sigma(\omega_n, \omega_n) = \sigma(\theta, \theta)$ and
\end{enumerate}
\[
1 < \Theta(\sigma(\omega_n, \omega_{n+1})) \leq \Theta(\sigma(\omega_n, \omega_{n+1})) \leq |\Theta(\sigma(\omega_{n-1}, \omega_n))|^k
\]
for all $n \geq 1$. Which further implies that
\[
1 < \Theta(\sigma(\omega_n, \omega_{n+1})) \leq |\Theta(\sigma(\omega_{n-1}, \omega_n))|^k \leq |\Theta(\sigma(\omega_{n-2}, \omega_{n-1}))|^2 \leq \ldots \leq |\Theta(\sigma(\omega_0, \omega_1))|^k.
\]
From (16), we obtain
\[
\lim_{n \to \infty} \Theta(\sigma(\omega_n, \omega_{n+1})) = 1.
\]
Then from (Θ₂), we get
\[
\lim_{n \to \infty} \sigma(\omega_n, \omega_{n+1}) = 0.
\] (18)

By (Θ₃), \( \exists 0 < k < 1 \) and \( l \in (0, \infty) \) such that
\[
\lim_{n \to \infty} \frac{\Theta(\sigma(\omega_n, \omega_{n+1})) - 1}{\sigma(\omega_n, \omega_{n+1})^k} = l.
\]

Assume that \( l < \infty \). In this instance, let \( \gamma = \frac{l}{2} > 0 \). By definition of the limit, \( \exists n₁ \in \mathbb{N} \) such that
\[
\forall n > n₁. \text{ It implies that } \frac{\Theta(\sigma(\omega_n, \omega_{n+1})) - 1}{\sigma(\omega_n, \omega_{n+1})^k} \geq l - \gamma = \frac{l}{2} = \gamma
\]
\[
\forall n > n₁. \text{ Then } n(\sigma(\omega_n, \omega_{n+1}))^k \leq an[\Theta(\sigma(\omega_n, \omega_{n+1})) - 1]
\]
\[
\forall n > n₁ \text{, where } \eta = \frac{1}{\gamma}. \text{ Presently we assume that } l = \infty. \text{ Let } \gamma > 0. \text{ By the definition of the limit, } \exists n₁ \in \mathbb{N} \text{ such that }
\gamma \leq \frac{\Theta(\sigma(\omega_n, \omega_{n+1})) - 1}{\sigma(\omega_n, \omega_{n+1})^k}
\]
\[
\forall n > n₁. \text{ This implies that } n(\sigma(\omega_n, \omega_{n+1}))^k \leq \eta n[\Theta(\sigma(\omega_n, \omega_{n+1})) - 1]
\]
\[
\forall n > n₁, \text{ where } \eta = \frac{1}{\gamma}. \text{ Thus, in all cases, there exist } \eta > 0 \text{ and } n₁ \in \mathbb{N} \text{ such that }
\eta(\sigma(\omega_n, \omega_{n+1}))^k \leq \eta n[\Theta(\sigma(\omega_n, \omega_{n+1})) - 1]
\]
\[
\forall n > n₁. \text{ Hence by (16), we get }
\eta(\sigma(\omega_n, \omega_{n+1}))^k \leq \eta n[\Theta(\sigma(\omega_n, \omega_{n+1})) - 1].
\]

Taking \( n \to \infty \), we get
\[
\lim_{n \to \infty} n(\sigma(\omega_n, \omega_{n+1}))^k = 0.
\]

Thus, there exists \( n₂ \in \mathbb{N} \) such that
\[
\sigma(\omega_n, \omega_{n+1}) \leq \frac{1}{n₁²k}
\]
(19)

for all \( n > n₂ \). Now we prove that \( \{ \omega_n \} \) is a Cauchy sequence in \( \theta \). For \( m > n > n₂ \) we have,
\[
\sigma(\omega_n, \omega_m) \leq \sum_{i=n}^{m-1} \sigma(\omega_i, \omega_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{1/k}
\]
(20)

Since, \( 0 < k < 1 \), then \( \sum_{i=1}^{\infty} \frac{1}{1/k} \) converges. Therefore, \( \sigma(\omega_n, \omega_m) \to 0 \) as \( m, n \to \infty \). Hence \( \{ \omega_n \} \) is Cauchy in \( \theta \). By (15) and (Θ₁), we have
\[
\sigma(\omega_n, \omega_{n+1}) < \sigma(\omega_{n-1}, \omega_n).
\]
Then, likewise, we can prove that \( \{ \omega_n \} \) is a Cauchy sequence in \( \theta \). As \( \theta, \vartheta \in C(M) \), so \( \exists \omega^* \in \theta \) and \( \omega^* \in \vartheta \) such that \( \omega_n \to \omega^* \) and \( \omega_n \to \omega^* \) as \( n \to \infty \), respectively. As \( \sigma(\omega_n, \omega_n) \to \sigma(\theta, \vartheta) \) for all \( n \to \infty \), we conclude that

\[
\lim_{n \to \infty} \sigma(\omega_n, \omega_n) = \sigma(\omega^*, \omega^*) = \sigma(\theta, \vartheta).
\]

Since \( \mathcal{J} \) is continuous, we have \( \lim_{n \to \infty} H(\mathcal{J} \omega_n, \mathcal{J} \omega^*) = 0 \). On the other hand, since \( \omega_{n+1} \in \mathcal{J} \omega_n \), we have

\[
\sigma(\omega^*, \mathcal{J} \omega^*) \leq \sigma(\omega^*, \omega_{n+1}) + \sigma(\omega_{n+1}, \mathcal{J} \omega^*) \leq \sigma(\omega^*, \omega_{n+1}) + H(\mathcal{J} \omega_n, \mathcal{J} \omega^*).
\]

Letting \( n \to \infty \), we obtain

\[
\sigma(\omega^*, \mathcal{J} \omega^*) \leq 0
\]

which leads to \( \omega^* \in \mathcal{J} \omega^* \). Furthermore, one has

\[
\sigma(\theta, \vartheta) \leq \sigma(\omega^*, \mathcal{J} \omega^*) \leq \sigma(\omega^*, \omega^*) = \sigma(\theta, \vartheta).
\]

Therefore, \( \omega^* \) is the required best proximity point of \( \mathcal{J} \).

\( \square \)

If \( K(\theta) \) is replaced with \( CB(\theta) \) in Theorem 1, then we get this result.

**Theorem 2.** Let \( \theta, \vartheta \in C(M) \) such that \( \theta_0 \neq \emptyset \) and \( \mathcal{J} : \theta \to CB(\theta) \) be Suzuki \( \alpha\Theta \)-proximal multivalued contraction and \( \alpha \)-proximal admissible. Assume that

(i) \( \forall \omega \in \theta_0 \), we have \( \mathcal{J} \omega \subseteq \theta_0 \) and \( (\theta, \vartheta) \) satisfies the weak P-Property;

(ii) \( \exists \omega_0, \omega_1 \in \theta_0 \) and \( \omega_1 \in \mathcal{J} \omega_0 \) such that

\[
\sigma(\omega_1, \omega_1) = \sigma(\theta, \vartheta), \quad \alpha(\omega_0, \omega_1) \geq 1.
\]

(iii) \( \mathcal{J} \) is continuous

(iv) \( (\Theta_4) \) holds.

Then \( \mathcal{J} \) has a best proximity point.

**Proof.** By supposition (ii), \( \exists \omega_0, \omega_1 \in \theta_0 \) and \( \omega_1 \in \mathcal{J} \omega_0 \) such that

\[
\sigma(\omega_1, \omega_1) = \sigma(\theta, \vartheta), \quad \alpha(\omega_0, \omega_1) \geq 1. \tag{21}
\]

Next, suppose that \( \omega_1 \notin \mathcal{J} \omega_1 \). Since \( \omega_1 \in \mathcal{J} \omega_0 \), we have

\[
\sigma(\omega_0, \mathcal{J} \omega_0) \leq \sigma(\omega_0, \omega_1) \leq \sigma(\omega_0, \omega_1) + \sigma(\omega_1, \omega_1).
\]

From \( \sigma(\omega_1, \omega_1) = \sigma(\theta, \vartheta) \), we obtain

\[
\sigma(\omega_0, \mathcal{J} \omega_0) \leq \sigma(\omega_0, \omega_1) + \sigma(\theta, \vartheta)
\]

and so

\[
\sigma(\omega_0, \mathcal{J} \omega_0) - \sigma(\theta, \vartheta) \leq \sigma(\omega_0, \omega_1).
\]

By Definition, we have

\[
\sigma^*(\omega_0, \mathcal{J} \omega_0) \leq \sigma(\omega_0, \omega_1) \leq \alpha(\omega_0, \omega_1) \sigma(\omega_0, \omega_1).
\]
This pursues that \( H(\mathcal{J} \omega_0, \mathcal{J} \omega_1) > 0 \). By (1), we have
\[
\Theta(H(\mathcal{J} \omega_0, \mathcal{J} \omega_1)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k. \tag{22}
\]
Otherwise, as \( 0 < \sigma(\omega_1, \mathcal{J} \omega_1) \leq H(\mathcal{J} \omega_0, \mathcal{J} \omega_1) \), so by \((\Theta_1)\) we have
\[
\Theta(\sigma(\omega_1, \mathcal{J} \omega_1)) \leq \Theta(H(\mathcal{J} \omega_0, \mathcal{J} \omega_1)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k.
\]
From \((\Theta_4)\), we can write
\[
\Theta(\sigma(\omega_1, \mathcal{J} \omega_1)) = \inf\{\Theta(\sigma(\omega_1, z)) : z \in \mathcal{J} \omega_1\}.
\]
Hence there exists \( \omega_2 \in \mathcal{J} \omega_1 \) such that
\[
\Theta(\sigma(\omega_1, \omega_2)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k.
\]

Doing the same as we have done in Theorem 1, we get \( \{\omega_n\} \subseteq \theta_0 \) and \( \{\omega_n\} \subseteq \theta_0 \) such that
\[\begin{align*}
(a) & \; \alpha(\omega_n, \omega_{n+1}) \geq 1, \; \omega_n \neq \omega_{n+1}; \\
(b) & \; \omega_n \in \mathcal{J} \omega_{n-1} \text{ and } \omega_n \notin \mathcal{J} \omega_n; \\
(c) & \; \sigma(\omega_n, \omega_n) = \sigma(\theta, \theta) \text{ and }
\end{align*}\]
\[
1 < \Theta(\sigma(\omega_n, \omega_{n+1})) \leq \Theta(\sigma(\omega_{n-1}, \omega_n)) \leq |\Theta(\sigma(\omega_{n-1}, \omega_n))|^k \tag{23}
\]
\( \forall n \geq 1 \). Furthermore, we obtain \( \{\omega_n\} \) in \( \theta \) and \( \{\omega_n\} \) in \( \theta \) as Cauchy sequences. As \( \theta, \theta \in C(\mathcal{M}) \), so \( \exists \omega^* \in \theta \) and \( \omega^* \in \theta \) such that \( \omega_n \to \omega^* \) and \( \omega_n \to \omega^* \) as \( n \to \infty \), respectively. By the proof of Theorem 1, we can get \( \omega^* \) as best proximity point of \( \mathcal{J} \).

\[\square\]

The next result can given by replacing the continuity of the mapping \( \mathcal{J} \) with the property \( H \).

(H) If \{\( \omega_n \)\} is a sequence in \( \theta \) with \( \alpha(\omega_n, \omega_{n+1}) \geq 1 \), \( \forall n \in \mathbb{N} \) and \( \omega_n \to \omega \in \theta \) as \( n \to \infty \), then \( \exists \{\omega_{n_k}\} \) of \{\( \omega_n \)\} such that \( \alpha(\omega_{n_k}, \omega) \geq 1 \) for all \( k \geq 1 \).

**Theorem 3.** Let \( \theta, \theta \in C(\mathcal{M}) \) such that \( \theta_0 \neq \emptyset \) and \( \mathcal{J} : \theta \to K(\theta) \) be Suzuki \( \alpha-\Theta \)-proximal multivalued contraction and \( \alpha \)-proximal admissible. Assume that
(i) for each \( \omega \in \theta_0 \), we have \( \mathcal{J} \omega \subseteq \theta_0 \) and the pair \((\theta, \theta)\) satisfies the weak P-Property;
(ii) there exists \( \omega_{0, \omega_1} \in \theta_0 \) and \( \omega_1 \in \mathcal{J} \omega_0 \) such that
\[
\sigma(\omega_1, \omega_1) = \sigma(\theta, \theta), \quad \text{and} \quad \alpha(\omega_0, \omega_1) \geq 1.
\]
(iii) Property \( (H) \) holds.
Then \( \mathcal{J} \) has a best proximity point.

**Proof.** By Theorem 1, we get \( \{\omega_n\} \subseteq \theta_0 \) and \( \{\omega_n\} \subseteq \theta_0 \) such that
\[\begin{align*}
(a) & \; \alpha(\omega_n, \omega_{n+1}) \geq 1, \; \omega_n \neq \omega_{n+1}; \\
(b) & \; \omega_n \in \mathcal{J} \omega_{n-1} \text{ and } \omega_n \notin \mathcal{J} \omega_n; \\
(c) & \; \sigma(\omega_n, \omega_n) = \sigma(\theta, \theta) \text{ and }
\end{align*}\]
\[
1 < \Theta(\sigma(\omega_n, \omega_{n+1})) \leq \Theta(\sigma(\omega_{n-1}, \omega_n)) \leq |\Theta(\sigma(\omega_{n-1}, \omega_n))|^k \tag{24}
\]
\[\square\]
for all \( n \geq 1 \). Also, there exist \( \omega^* \in \vartheta \) and \( \alpha^* \in \vartheta \) such that \( \omega_n \to \omega^* \) and \( \alpha_n \to \alpha^* \) as \( n \to \infty \), respectively, and \( \sigma(\omega^*, \alpha^*) = \sigma(\vartheta, \vartheta) \). We prove that \( \omega^* \) is a best proximity point of \( \mathcal{J} \). If \( \exists \{ \omega_{n_k} \} \) of \( \{ \omega_n \} \) such that \( \mathcal{J} \omega_{n_k} = \mathcal{J} \omega^*, \forall k \geq 1 \), then we have

\[
\sigma(\vartheta, \vartheta) \leq \sigma(\omega_{n_k+1}, \mathcal{J} \omega_{n_k}) \leq \sigma(\omega_{n_k+1}, \omega_{n_k+1}) = \sigma(\vartheta, \vartheta)
\]

which yields that,

\[
\sigma(\vartheta, \vartheta) \leq \sigma(\omega_{n_k+1}, \mathcal{J} \omega^*) \leq \sigma(\vartheta, \vartheta)
\]

for all \( k \geq 1 \). Letting \( k \to \infty \), we obtain

\[
\sigma(\vartheta, \vartheta) \leq \sigma(\omega^*, \mathcal{J} \omega^*) \leq \sigma(\vartheta, \vartheta).
\]

Hence \( \omega^* \) is a best proximity point of \( \mathcal{J} \). Without any loss, we assume that \( \mathcal{J} \omega_n \neq \mathcal{J} \omega^*, \forall n \in \mathbb{N} \). By (H), \( \exists \{ \omega_{n_k} \} \) of \( \{ \omega_n \} \) such that \( a(\omega_{n_k}, \omega) \geq 1, \forall k \geq 1 \). By supposition (ii), we get \( \omega_{n_k+1} \in \mathcal{J} \omega_{n_k} \) such that

\[
\sigma(\omega_{n_k}, \mathcal{J} \omega_{n_k}) \leq \sigma(\omega_{n_k}, \omega_{n_k+1}) \leq \sigma(\omega_{n_k}, \omega_{n_k+1}) + \sigma(\omega_{n_k+1}, \omega_{n_k+1}).
\]

Since \( \sigma(\omega_{n_k+1}, \omega_{n_k+1}) = \sigma(\vartheta, \vartheta) \), we obtain

\[
\sigma(\omega_{n_k+1}, \mathcal{J} \omega_{n_k}) \leq \sigma(\omega_{n_k}, \omega_{n_k+1}) + \sigma(\vartheta, \vartheta)
\]

and so

\[
\sigma(\omega_{n_k}, \mathcal{J} \omega_{n_k}) - \sigma(\vartheta, \vartheta) \leq \sigma(\omega_{n_k}, \omega_{n_k+1}).
\]

By Definition, we have

\[
\sigma^*(\omega_{n_k}, \mathcal{J} \omega_{n_k}) \leq \sigma(\omega_{n_k}, \omega_{n_k+1}) \leq a(\omega_{n_k}, \omega^*) \sigma(\omega_{n_k}, \omega_{n_k+1}).
\]

Form (1), we have

\[
\Theta(H(\mathcal{J} \omega_{n_k}, \mathcal{J} \omega^*)) \leq [\Theta(\sigma(\omega_{n_k}, \omega^*))]^k < \Theta(\sigma(\omega_{n_k}, \omega^*)).
\]

From (O1), we obtain

\[
H(\mathcal{J} \omega_{n_k}, \mathcal{J} \omega^*) < \sigma(\omega_{n_k}, \omega^*).
\]

On the other hand, we have

\[
\sigma(\omega^*, \mathcal{J} \omega^*) \leq \sigma(\omega^*, \omega_{n_k+1}) + \sigma(\omega_{n_k+1}, \mathcal{J} \omega^*) \leq \sigma(\omega^*, \omega_{n_k+1}) + H(\mathcal{J} \omega_{n_k}, \mathcal{J} \omega^*).
\]

Since \( H(\mathcal{J} \omega_{n_k}, \mathcal{J} \omega^*) < \sigma(\omega_{n_k}, \omega^*) \), we obtain

\[
\sigma(\omega^*, \mathcal{J} \omega^*) \leq \sigma(\omega^*, \omega_{n_k+1}) + \sigma(\omega_{n_k}, \omega^*).
\]

Letting \( k \to \infty \), we obtain \( \sigma(\omega^*, \mathcal{J} \omega^*) = 0 \). Hence, we have

\[
\sigma(\vartheta, \vartheta) \leq \sigma(\omega^*, \mathcal{J} \omega^*) \leq \sigma(\omega^*, \omega^*) + \sigma(\omega^*, \mathcal{J} \omega^*).
\]

Moreover, since \( \sigma(\omega^*, \mathcal{J} \omega^*) = 0 \) and \( \sigma(\omega^*, \omega^*) = \sigma(\vartheta, \vartheta) \), we have

\[
\sigma(\vartheta, \vartheta) \leq \sigma(\omega^*, \mathcal{J} \omega^*) \leq \sigma(\vartheta, \vartheta).
\]
Therefore, $\omega^*$ is a best proximity point of $J$. 

\[ \square \]

**Theorem 4.** Let $\theta, \vartheta \in C(M)$ such that $\theta_0 \neq \emptyset$ and $J : \theta \to CB(\vartheta)$ be Suzuki $\delta$-$\Theta$-proximal multivalued contraction and $\alpha$-proximal admissible. Suppose that

(i) for each $\omega \in \theta_0$, we have $J(\omega) \subseteq \theta_0$ and the pair $(\theta, \vartheta)$ satisfies the weak P-Property;

(ii) there exists $\omega_0, \omega_1 \in \theta_0$ and $\vartheta_1 \in J(\omega_0)$ such that

$$\sigma(\omega_1, \omega_1) = \sigma(\theta, \vartheta), \quad \text{and} \quad \alpha(\omega_0, \omega_1) \geq 1.$$

(iii) Property (H) holds.

(iv) $(\Theta_4)$ holds.

Then $J$ has a best proximity point.

**Proof.** The proof of this Theorem can easily be done like Theorem 3 and so we omit the proof here. \[ \square \]

The following result if a direct consequence of Theorem 1 for non self mapping.

**Corollary 1.** Let $\theta, \vartheta \in C(M)$ such that $\theta_0 \neq \emptyset$ and $J : \theta \to \vartheta$ be a non self mapping such that

$$\sigma^*(\omega, J(\omega)) \leq \alpha(\omega, \vartheta) \sigma(\omega, \vartheta) \implies \Theta(\sigma(\omega, J(\omega))) \leq \Theta(\sigma(\omega, \vartheta))^k.$$

Suppose that the following conditions hold:

(i) $J$ is $\alpha$-proximal admissible,

(ii) $J(\theta_0) \subseteq \theta_0$ and the pair $(\theta, \vartheta)$ satisfies the weak P-Property;

(iii) there exists $\omega_0, \omega_1 \in \theta_0$ such that

$$\sigma(\omega_1, J(\omega_0)) = \sigma(\theta, \vartheta), \quad \text{and} \quad \alpha(\omega_0, \omega_1) \geq 1.$$

(iv) $J$ is continuous or Property (H) holds.

Then $J$ has a best proximity point.

**Example 1.** Let $M = [0, \infty) \times [0, \infty)$ be endowed with the usual metric $\sigma$, $\theta = \{\frac{1}{3}\} \times [0, \infty)$ and $\vartheta = \{0\} \times [0, \infty)$. Define $J : \theta \to CB(\vartheta)$ by

$$J\left(\frac{1}{3}, \theta\right) = \begin{cases} \{(0, \frac{a}{3}) : 0 \leq \omega \leq a\} & \text{if } a \leq 1 \\ \{(0, \omega^2) : 0 \leq \omega \leq a\} & \text{if } a > 1 \end{cases}$$

and a function $\alpha : \theta \times \theta \to [0, \infty)$ as follows:

$$\alpha(\omega, \vartheta) = \begin{cases} 1 & \text{if } \omega, \vartheta \in \{(\frac{1}{3}, a) : 0 \leq a \leq 1\} \\ 0 & \text{otherwise} \end{cases}$$

Take $\Theta : (0, \infty) \to (1, \infty)$ by $\Theta(t) = e^{t^2}$ for $t > 0$ and $\sigma(\theta, \vartheta) = \frac{1}{3}$. Note that $\vartheta_0 = \theta_0 = \emptyset$ and $J(\omega) \subseteq \theta_0$ for all $\omega$ in $\theta_0$. Also $(\theta, \vartheta)$ satisfies the weak P-property. Let $\omega_0, \omega_1 \in \{(\frac{1}{3}, \omega) : 0 \leq \omega \leq 1\}$. Then we have $J(\omega_0, J(\omega_1) \subseteq \{(0, \frac{\omega_0}{3}) : 0 \leq \omega \leq 1\}.$

Consider $\omega_1 \in J(\omega_0, \omega_2) \in J(\omega_1)$ and $x_1, x_2 \in \vartheta$ such that $\sigma(x_1, \omega_1) = \sigma(\theta, \vartheta)$ and $\sigma(x_2, \omega_2) = \sigma(\theta, \vartheta)$. Then we have $x_1, x_2 \in \{(\frac{1}{3}, \omega) : 0 \leq \omega \leq \frac{1}{3}\}$. Hence $\alpha(x_1, x_2) = 1$ implies that $J$ is an
\( \alpha \)-proximal admissible. For \( \omega_0 = (\frac{1}{3}, 1) \) and \( \omega_1 = (0, \frac{1}{3}) \in J\omega_0 \subseteq \theta_0 \), we have \( \omega_1 = (\frac{1}{3}, \frac{1}{3}) \in \theta_0 \) such that \( \sigma(\omega_1, \omega_1) = \sigma(\theta, \theta) \) and \( \alpha(\omega_0, \omega_1) = 1 \). Further, we have
\[
\sigma(\omega_0, J\omega_0) = \sigma((\frac{1}{3}, 1), (0, \frac{1}{3})) \\
\leq \sigma((\frac{1}{3}, 1), (\frac{1}{3}, \frac{1}{3})) + \sigma((\frac{1}{3}, \frac{1}{3}), (0, \frac{1}{3})) \\
= \sigma(\omega_0, \omega_1) + \sigma(\omega_1, \omega_1).
\]

Since \( \sigma(\omega_1, \omega_1) = \sigma(\theta, \theta) \), we obtain
\[
\sigma(\omega_0, J\omega_0) \leq \sigma(\omega_0, \omega_1) + \sigma(\theta, \theta)
\]
and so
\[
\sigma(\omega_0, J\omega_0) - \sigma(\theta, \theta) \leq \sigma(\omega_0, \omega_1).
\]

By definition, we have
\[
\sigma^*(\omega_0, J\omega_0) \leq \sigma(\omega_0, \omega_1).
\]

Since \( \alpha(\omega_0, \omega_1) \geq 1 \), we have
\[
\sigma^*(\omega_0, J\omega_0) \leq \sigma(\omega_0, \omega_1) \leq \alpha(\omega_0, \omega_1)\sigma(\omega_0, \omega_1).
\]

Since \( \delta(J\omega, J\omega) = \delta(J\omega, J\omega) \) for all \( \omega, \varpi \in \theta_0 \). Assume that \( \omega_0 = (\frac{1}{3}, 1) \) and \( \omega_1 = (\frac{1}{3}, \frac{1}{3}) \) and consider
\[
H(J\omega_0, J\omega_1) = \max \{\delta(J\omega_0, J\omega_1), \delta(J\omega_1, J\omega_0)\}.
\]

Then, we have
\[
\delta(J\omega_0, J\omega_1) = \delta(J(\frac{1}{3}, 1), J(\frac{1}{3}, \frac{1}{3})) \\
= \max \{\sigma((0, \frac{1}{3}), J(\frac{1}{3}, \frac{1}{3}))\}
\]
and so
\[
\sigma((0, \frac{1}{3}), J(\frac{1}{3}, \frac{1}{3})) = \min \{\sigma(0, \frac{1}{3}), \sigma(0, \frac{1}{3})\} \\
= \frac{2}{9} \\
= \frac{1}{3} \sigma((\frac{1}{3}, 1), (\frac{1}{3}, \frac{1}{3})) \\
= \frac{1}{3} \sigma(\omega_0, \omega_1).
\]

Similarly, we have
\[
\delta(J\omega_1, J\omega_0) = \frac{1}{3} \sigma(\omega_0, \omega_1).
\]

This yields that
\[
H(J\omega_0, J\omega_1) = \frac{1}{3} \sigma(\omega_0, \omega_1).
\]
Since $\Theta$ is increasing, we have
\[ \Theta(H(J\omega_0, J\omega_1)) = e^{\sqrt{H(J\omega_0, J\omega_1)}} = e^{\sqrt{2\sigma(\omega_0, \omega_1)}} \leq e^{\sqrt{3\sigma(\omega_0, \omega_1)}} = [\Theta(\sigma(\omega_0, \omega_1))]^k \]
with $k = \sqrt{\frac{3}{2}}$. Hence (1) is satisfied. Also, $J$ is continuous and supposition (ii) of Theorem 4 is verified. Indeed, for $\omega_0 = (\frac{1}{3}, 1), \omega_1 = (\frac{1}{3}, 0)$ and $\omega_1 = (0, 0)$, we obtain
\[ \sigma(\omega_1, \omega_1) = \sigma((\frac{1}{3}, 0), (0, 0)) = \frac{1}{3} = \sigma(\theta, \theta) \]
and $\alpha(\omega_0, \omega_1) = 1$. Thus all the supposition of Theorem 4 are satisfied and $\omega^* = (\frac{1}{3}, 0)$ is the required proximity point.

4. Consequences

4.1. Fixed Point Results in Complete Metric Space

In this section, we deduce some fixed point results for multi-valued and single-valued mappings from our main results.

Theorem 5. Let $J : \mathcal{M} \to K(\mathcal{M}) \ (CB(\mathcal{M}))$ be such that
\[ \sigma(\omega, J\omega) \leq \alpha(\omega, \omega)\sigma(\omega, \omega) \Rightarrow \Theta(H(J\omega, J\omega)) \leq [\Theta(\sigma(\omega, \omega))]^k. \]

Suppose that the following conditions hold:
(i) there exist $\omega_0 \in \mathcal{M}$ and $\omega_1 \in J\omega_0$ such that $\alpha(\omega_0, \omega_1) \geq 1$,
(ii) $J$ is $\alpha$-admissible,
(iii) $J$ is continuous or Property (H) holds ($\Theta_4$ holds).

Then $J$ has a fixed point.

Proof. By supposition (i), $\exists \omega_0 \in \mathcal{M}$ and $\omega_1 \in J\omega_0$ such that $\alpha(\omega_0, \omega_1) \geq 1$. If $\omega_0 = \omega_1$, then $\omega_0$ is fixed point of $J$. Now
\[ \sigma(\omega_0, J\omega_0) \leq \sigma(\omega_0, \omega_1) \leq \alpha(\omega_0, \omega_1)\sigma(\omega_0, \omega_1). \]

If $\omega_1 \in J\omega_1$, then $\omega_1$ is the required fixed point and we have nothing to prove. So we suppose that $\omega_1 \notin J\omega_1$. By assumption, we have
\[ \Theta(\sigma(\omega_1, J\omega_1)) \leq \Theta(H(J\omega_0, J\omega_1)) \leq [\Theta(\sigma(\omega_0, \omega_1))]^k. \]

Since $J\omega_1$ is compact, there exists $\omega_2 \in J\omega_1$ such that $\sigma(\omega_1, J\omega_1) = \sigma(\omega_1, \omega_2)$. Thus we have
\[ \Theta(\sigma(\omega_1, \omega_2)) \leq \Theta(H(J\omega_0, J\omega_1)) \leq [\Theta(\sigma(\omega_0, \omega_1))]^k. \]

Since $\alpha(\omega_0, \omega_1) \geq 1$ and $J$ is $\alpha$-admissible, so we have $\alpha(J\omega_0, J\omega_1) \geq 1$. Thus we have
\[ \alpha(\omega_1, \omega_2) \geq 1. \]

Now
\[ \sigma(\omega_1, J\omega_1) \leq \sigma(\omega_1, \omega_2) \leq \alpha(\omega_1, \omega_2)\sigma(\omega_1, \omega_2). \]
If \( \omega_2 \in \mathcal{J} \omega_2 \), then \( \omega_2 \) is the requited fixed point and we have nothing to prove. So we suppose that \( \omega_2 \notin \mathcal{J} \omega_2 \). By assumption, we have
\[
\Theta(\sigma(\omega_2, \mathcal{J} \omega_2)) \leq \Theta(H(\mathcal{J} \omega_1, \mathcal{J} \omega_2)) \leq [\Theta(\sigma(\omega_1, \omega_2))]^k.
\]
Since \( \mathcal{J} \omega_2 \) is compact, there exists \( \omega_3 \in \mathcal{J} \omega_2 \) such that \( \sigma(\omega_2, \mathcal{J} \omega_2) = \sigma(\omega_2, \omega_3) \). Thus we have
\[
\Theta(\sigma(\omega_2, \omega_3)) \leq \Theta(H(\mathcal{J} \omega_1, \mathcal{J} \omega_2)) \leq [\Theta(\sigma(\omega_1, \omega_2))]^k.
\]
Thus by induction, we have \( a(\omega_n, \omega_{n+1}) \geq 1 \) and
\[
\Theta(\sigma(\omega_n, \omega_{n+1})) \leq [\Theta(\sigma(\omega_{n-1}, \omega_n))]^k
\]
for all \( n \in \mathbb{N} \). Now is easy to satisfy all the assertions of Theorems 1 and 3 and thus we get the conclusion. \( \square \)

**Corollary 2.** Let \( \mathcal{J} : \mathcal{M} \to \mathcal{M} \) be such that
\[
\sigma(\omega, \mathcal{J} \omega) \leq a(\omega, \omega)\sigma(\omega, \omega) \implies \Theta(\sigma(\mathcal{J} \omega, \mathcal{J} \omega)) \leq [\Theta(\sigma(\omega, \omega))]^k.
\]
Suppose that the following conditions hold:
(i) there exist \( \omega_0 \in \mathcal{M} \) and such that \( a(\omega_0, \mathcal{J} \omega_0) \geq 1 \),
(ii) \( \mathcal{J} \) is \( a \)-admissible,
(iii) \( \mathcal{J} \) is continuous or Property (H) holds.
Then \( \mathcal{J} \) has a fixed point.

4.2. Some Results in Partially Ordered Metric Spaces

In this section, we derive following new results in partially ordered metric spaces \((\mathcal{M}, \sigma, \preceq)\) from our main results.

**Definition 6** ([17]). Let \( \theta \) and \( \theta \) be two non empty subsets of a partially ordered metric space \((\mathcal{M}, \sigma, \preceq)\). A mapping \( \mathcal{J} : \theta \to 2^\theta \) is said to be proximal nondreasing if
\[
\begin{align*}
\omega_1 & \preceq \omega_2 \\
\sigma(\varphi_1, \omega_1) & = \sigma(\theta, \theta) \\
\sigma(\varphi_2, \omega_2) & = \sigma(\theta, \theta)
\end{align*}
\]
\( \implies \varphi_1 \preceq \varphi_2 \)

where \( \omega_0, \omega_1, \varphi_1, \varphi_2 \in \theta \) and \( \omega_1, \omega_2 \in \mathcal{J} \omega_1, \omega_2 \in \mathcal{J} \omega_2 \).

The property \((H_p)\) will be need in the next result.

\((H_p)\) If \{\( \omega_n \)\} is a sequence in \( \theta \) such that \( \omega_n \leq \omega_{n+1} \), \( \forall \ n \in \mathbb{N} \) and \( \omega_n \to \omega \in \theta \) as \( n \to \infty \), then \( \exists \ \{\omega_{n_k}\} \) of \{\( \omega_n \)\} such that \( \omega_{n_k} \leq \omega, \forall k \geq 1 \).

**Theorem 6.** Let \( \theta, \theta \in \mathcal{C}(\mathcal{M}) \) such that \( \theta_0 \neq \emptyset \) and \( \mathcal{J} : \theta \to CB(\theta), \Theta \in \Omega, \alpha : \theta \times \theta \to [0, \infty) \) and some constant \( k \in (0, 1) \) such that
\[
\sigma^*(\omega, \mathcal{J} \omega) \leq \sigma(\omega, \omega) \implies \Theta(H(\mathcal{J} \omega, \mathcal{J} \omega)) \leq [\Theta(\sigma(\omega, \omega))]^k
\]
for all \( \omega, \omega \in \theta \) with \( \omega \preceq \omega \), where \( \sigma^*(\omega, \omega) = \sigma(\omega, \omega) - \sigma(\theta, \theta) \) satisfying \( H(\mathcal{J} \omega, \mathcal{J} \omega) > 0 \). Suppose that
(i) for each \( \omega \in \theta_0 \), we have \( \mathcal{J} \omega \subseteq \theta_0 \) and the pair \( (\theta, \theta) \) satisfies the weak P-Property;
(ii) there exists $\omega_0, \omega_1 \in \theta_0$ and $\omega_1 \in \mathcal{J}\omega_0$ such that

$$\sigma(\omega_1, \omega_1) = \sigma(\theta, \theta), \text{ and } \omega_0 \leq \omega_1.$$ 

(iii) $\mathcal{J}$ is proximal nondecreasing;

(iv) $\mathcal{J}$ is continuous or Property (H$_p$) holds.

Then $\mathcal{J}$ has a best proximity point.

**Proof.** Consider $\alpha : \theta \times \theta \to [0, \infty)$ such that

$$\alpha(\omega, \omega) = \begin{cases} 1 & \text{if } \omega \leq \omega \\ 0 & \text{Otherwise.} \end{cases}$$

By supposition (ii), $\exists \omega_0, \omega_1 \in \theta$ and $\omega_1 \in \mathcal{J}\omega_0$ such that

$$\sigma(\omega_1, \omega_1) = \sigma(\theta, \theta) \text{ and } \omega_0 \leq \omega_1.$$ 

Since $\omega_0 \leq \omega_1$ we obtain $\alpha(\omega_0, \omega_1) = 1$. Next, suppose that $\omega_1 \not\in \mathcal{J}\omega_1$. Since $\omega_1 \in \mathcal{J}\omega_0$, we have

$$\sigma(\omega_0, \mathcal{J}\omega_0) \leq \sigma(\omega_0, \omega_1) \leq \sigma(\omega_0, \omega_1) + \sigma(\omega_1, \omega_1).$$ 

Since $\sigma(\omega_1, \omega_1) = \sigma(\theta, \theta)$, we obtain

$$\sigma(\omega_0, \mathcal{J}\omega_0) \leq \sigma(\omega_0, \omega_1) + \sigma(\theta, \theta)$$

and so

$$\sigma(\omega_0, \mathcal{J}\omega_0) - \sigma(\theta, \theta) \leq \sigma(\omega_0, \omega_1).$$

By Definition, we have

$$\sigma^*(\omega_0, \mathcal{J}\omega_0) \leq \sigma(\omega_0, \omega_1) = \alpha(\omega_0, \omega_1)\sigma(\omega_0, \omega_1).$$

It pursues that $H(\mathcal{J}\omega_0, \mathcal{J}\omega_1) > 0$ and

$$\Theta(H(\mathcal{J}\omega_0, \mathcal{J}\omega_1)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k.$$ 

Otherwise, as $0 < \sigma(\omega_1, \mathcal{J}\omega_1) \leq H(\mathcal{J}\omega_0, \mathcal{J}\omega_1)$, so by (H$_p$), we have

$$\Theta(\sigma(\omega_1, \mathcal{J}\omega_1)) \leq \Theta(H(\mathcal{J}\omega_0, \mathcal{J}\omega_1)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k$$

From (H$_p$), we can write $\Theta(\sigma(\omega_1, \mathcal{J}\omega_1)) = \inf\{\Theta(\sigma(\omega_1, z)) : z \in \mathcal{J}\omega_1\}$. Hence there exists $\omega_2 \in \mathcal{J}\omega_1$ such that

$$\Theta(\sigma(\omega_1, \omega_2)) \leq |\Theta(\sigma(\omega_0, \omega_1))|^k.$$ 

From the proof of Theorem 2, we get $\{\omega_n\} \subseteq \theta_0$ and $\{\omega_n\} \subseteq \theta_0$ such that

(a) $\omega_n \leq \omega_{n+1}$ and $\omega_n \neq \omega_{n+1}$;
(b) $\omega_n \in \mathcal{J}\omega_{n-1}$ and $\omega_n \not\in \mathcal{J}\omega_n$;
(c) $\sigma(\omega_n, \omega_n) = \sigma(\theta, \theta)$ and

$$1 < \Theta(\sigma(\omega_n, \omega_{n+1})) \leq \Theta(\sigma(\omega_{n-1}, \omega_{n+1})) \leq |\Theta(\sigma(\omega_{n-1}, \omega_n))|^k$$

$\forall n \geq 1$. Hence by the proof of Theorem 2, we get the conclusion when $\mathcal{J}$ is continuous. Similarly by the proof of Theorem 4 if (H$_p$) is satisfied.

$\surd$
4.3. Some Results on Graphic Contraction

Let \((M, \sigma)\) be a metric space and \(G = (V(G), E(G))\) be the direct graph such that \(V(G) = M\) and \(E(G)\) contains all loops, i.e., \(\Delta = \{(\omega, \omega) : \omega \in M\} \subseteq E(G)\).

**Definition 7** ([18]). A mapping \(J : M \to M\) is called \(G\)-continuous if for \(\omega \in M\) and sequence \(\{\omega_n\}\) in \(M\) such that \(\omega_n \to \omega\) as \(n \to \infty\) and \((\omega_n, \omega_{n+1}) \in E(G)\) for all \(n \in \mathbb{N}\) implies \(J\omega_n \to J\omega\).

**Definition 8** ([8]). Let \(\theta\) and \(\vartheta\) be nonempty subsets of a metric space \((M, \sigma)\) endowed with a graph \(G\). A mapping \(J : \theta \to 2^\theta\) is said to be \(G\)-proximal if

\[
\begin{align*}
\sigma(\omega_1, \omega_2) & \in E(G) \\
\sigma(\vartheta_1, \vartheta_1) & = \sigma(\theta, \theta) \\
\sigma(\vartheta_2, \vartheta_2) & = \sigma(\theta, \theta)
\end{align*}
\]

implies \(J\omega_1 \in J\vartheta_1, J\vartheta_2 \in J\vartheta_2\). If \(\sigma(\theta, \theta) = 0\), then we say that \(J\) preserves the edges of \(G\).

Assume the following:

\(\text{(H}_G\text{)}\) If \(\{\omega_n\}\) is a sequence in \(\theta\) such that \((\omega_n, \omega_{n+1}) \in E(G), \forall n \in \mathbb{N}\) and \(\omega_n \to \omega \in \theta\) as \(n \to \infty\), then \(\exists \{\omega_{n_k}\} \subseteq \{\omega_n\}\) such that \((\omega_{n_k}, \omega) \in E(G)\) for all \(k \geq 1\).

**Theorem 7.** Let \(\theta, \vartheta \in C(M)\) such that \(\text{dist}(\theta) \neq 0\) and \(J : \theta \to (K(\vartheta), \Theta \subseteq \Omega, \kappa : \theta \times \theta \to [0, \infty)\) and some constant \(k \in (0, 1)\) such that

\[
\sigma^*(\omega, J\omega) \leq \sigma(\omega, \omega) \implies \Theta(H(J\omega, J\omega)) \leq [\Theta(\sigma(\omega, \omega))]^k
\]

for all \(\omega, \omega \in \theta\) with \((\omega, \omega) \in E(G)\), where \(\sigma^*(\omega, \omega) = \sigma(\omega, \omega) - \sigma(\theta, \theta)\) satisfying \(H(J\omega, J\omega) > 0\).

Assume that

(i) for each \(\omega \in \theta_0\), we have \(J\omega \subseteq \theta_0\) and the pair \((\theta, \theta)\) satisfies the weak P-Property;

(ii) there exists \(\omega_0, \omega_1 \in \theta_0\) and \(\omega_1 \in J\omega_0\) such that

\[
\sigma(\omega_1, \omega_1) = \sigma(\theta, \theta), \text{ and } (\omega_0, \omega_1) \in E(G).
\]

(iii) \(J\) is \(G\)-proximal

(iv) \(J\) is \(G\)-continuous or Property \((\text{H}_G)\) holds.

Then \(J\) has a best proximity point.

**Proof.** Taking \(\kappa : \theta \times \theta \to [0, \infty)\) such that

\[
\kappa(\omega, \omega) = \begin{cases} 
1 & \text{if } (\omega, \omega) \in E(G) \\
0 & \text{Otherwise.}
\end{cases}
\]

By supposition (ii), \(\exists \omega_0, \omega_1 \in \theta\) and \(\omega_1 \in J\omega_0\) such that

\[
\sigma(\omega_1, \omega_1) = \sigma(\theta, \theta) \text{ and } (\omega_0, \omega_1) \in E(G).
\]

Since \((\omega_0, \omega_1) \in E(G)\), we obtain \(\kappa(\omega_0, \omega_1) = 1\). Next, suppose that \(\omega_1 \notin J\omega_1\). Since \(\omega_1 \in J\omega_0\), we have

\[
\sigma(\omega_0, J\omega_0) \leq \sigma(\omega_0, \omega_1) \leq \sigma(\omega_0, \omega_1) + \sigma(\omega_1, \omega_1).
\]
Since $\sigma(\omega_1, \omega_1) = \sigma(\theta, \theta)$, we obtain
\[
\sigma(\omega_0, J\omega_0) \leq \sigma(\omega_0, \omega_1) + \sigma(\theta, \theta)
\]
and so
\[
\sigma(\omega_0, J\omega_0) - \sigma(\theta, \theta) \leq \sigma(\omega_0, \omega_1).
\]
By Definition, we have
\[
\sigma^*(\omega_0, J\omega_0) \leq \sigma(\omega_0, \omega_1) = \sigma(\omega_0, \omega_1).
\]
It pursues that $H(J\omega_0, J\omega_1) > 0$, so from (1), we have
\[
\Theta(H(J\omega_0, J\omega_1)) \leq [\Theta(\sigma(\omega_0, \omega_1))]^k.
\]
Otherwise, as $0 < \sigma(\omega_1, J\omega_1) \leq H(J\omega_0, J\omega_1)$, so from $(\Theta_1)$, we have
\[
\Theta(\sigma(\omega_1, J\omega_1)) \leq \Theta(H(J\omega_0, J\omega_1)) \leq [\Theta(\sigma(\omega_0, \omega_1))]^k.
\]
Hence there exists $\omega_2 \in J\omega_1$ such that
\[
\Theta(\sigma(\omega_1, \omega_2)) \leq [\Theta(\sigma(\omega_0, \omega_1))]^k.
\]
Proceeding the same as we have done in Theorem 2, we get \{\omega_n\} $\subseteq \theta_0$ and \{\omega_n\} $\subseteq \theta_0$ such that
\begin{enumerate}
  \item[(a)] $(\omega_n, \omega_{n+1}) \in E(G)$ and $\omega_n \neq \omega_{n+1};$
  \item[(b)] $\omega_n \in J\omega_{n-1}$ and $\omega_n \notin J\omega_n;$
  \item[(c)] $\sigma(\omega_n, \omega_n) = \sigma(\theta, \theta)$ and
\end{enumerate}
\[
1 < \Theta(\sigma(\omega_n, \omega_{n+1})) \leq \Theta(\sigma(\omega_n, \omega_{n+1})) \leq [\Theta(\sigma(\omega_{n-1}, \omega_n))]^k
\]
\[
\forall \ n \geq 1.
\]
Hence from the Theorem 2, we get the deduction when $J$ is $G$-continuous or also if $(H_G)$ is satisfied then we get conclusion from the proof of Theorem 4. \qed

5. Some Applications

In this section we present the applications of our results for variational inequality problems and dynamical programming.

5.1. Application to Variational Inequality Problem

Let $C$ be nonempty, closed and convex subset of real Hilbert space $H$ with inner product $(,)$ and induced norm $\|\cdot\|$. Recall that an operator $S : H \to H$ is called monotone if $(Sx - Sq, q - x) \geq 0$. We consider a monotone variational inequality problem as follows:

**Problem 1.** Find $x \in C$ such that $(Sx, q - x) \geq 0$ for all $q \in C$, where $S : H \to H$ is a monotone operator.

The interest for variational inequalities theory is due to the fact that a wide class of equilibrium problems, arising in pure and applied sciences, can be treated in an unified framework [19]. Now, we recall the metric projection, say $P_C : H \to C$, which is a powerful tool for solving a variational inequality problem. Referring to classical books on approximation theory in inner product spaces, (see [20]), we recall that for each $x \in H$, there exists a unique nearest point $P_Cx \in C$ such that
\[
\|x - P_Cx\| \leq \|x - q\| \quad \text{for all} \quad q \in C.
\]
We need the following crucial lemmas.

**Lemma 1.** Let \( z \in H \). Then \( \varphi = P_{C}(z) \) if and only if \( \langle \varphi - z, \omega - \varphi \rangle \leq 0 \) for all \( \omega \in C \).

**Lemma 2.** Let \( S : C \to C \) be monotone. Then \( \varphi \in C \) is a solution of \( (S\varphi, \varphi - \varphi) \geq 0 \) for all \( \varphi \in C \) if and only if \( \varphi = P_{C}(\varphi - \lambda S\varphi) \), \( \lambda > 0 \).

Now we prove the results for the solution of Problem 1.

**Theorem 8.** Let \( C \) be a non-empty, closed and convex subset of a real Hilbert space \( H \). Suppose that \( S : C \to C \) is monotone. Then \( \varphi \in C \) is a solution of \( (S\varphi, \varphi - \varphi) \geq 0 \) for all \( \varphi \in C \) if and only if \( \varphi = P_{C}(\varphi - \lambda S\varphi) \), \( \lambda > 0 \).

**Proof.** Define \( J : C \to C \) by \( J\omega = P_{C}(\omega - \lambda S\omega) \) for all \( \omega \in C \), then \( J \) satisfies all the hypothesis of Corollary 2 and so \( J \) has a unique fixed point \( \varphi \). Hence by Lemma 2, \( \varphi \in C \) is solution of \( (S\varphi, \varphi - \varphi) \geq 0 \) for all \( \varphi \in C \) if and only if \( \varphi \) is a fixed point of \( J \). This completes the proof.

**Corollary 3.** Let \( C \) be a non-empty, closed and convex subset of a real Hilbert space \( H \). Assume that there exists \( \lambda \in (0, 1) \) and \( \Theta \in \mathcal{F} \) such that for all \( \omega, \omega \in C \),

\[
\|\omega - P_{C}^{\omega}\| \leq \Theta(\|\omega - \omega\|) \quad \text{implies} \quad \Theta(\|P_{C}^{\omega} - P_{C}^{\omega}\|) \leq [\Theta(\|\omega - \omega\|)]^{k},
\]

where \( I_{C} \) is the identity operator on \( C \). Then there exists a unique element \( \varphi \in C \) such that \( (S\varphi, \varphi - \varphi) \geq 0 \) for all \( \varphi \in C \).

**5.2. Application to Nonlinear Dynamical System**

Here, we apply our results in order to prove the existence of a solution of the following functional equation:

\[
q(\omega) = \sup_{\omega \in D} \{f(\omega, \omega) + G(\omega, \omega, \beta(\omega, \omega))\}, \quad \omega \in W,
\]

(25)

where \( f : W \times D \to \mathbb{R} \) and \( G : W \times D \times \mathbb{R} \to \mathbb{R} \) are bounded, \( \beta : W \times D \to W \), \( W \) and \( D \) are Banach spaces. These types of equations have their application in computer programming, mathematical optimization and dynamic programming, which allow instruments for answering boundary value problems emanating in physical sciences and engineering.

Let \( B(W) \) denotes the set of bounded real-valued functions on \( W \). The pair \( (B(W), ||.||) \), where

\[
||h|| = \sup_{\omega \in W}|h(\omega)|, \quad h \in B(W),
\]

is a Banach space with \( \sigma(\omega, \omega) = ||\omega - \omega|| \), a distance associated to the norm.

In order to prove the existence of a solution of equation (25), we take \( F : B(W) \to B(W) \) of the type

\[
(Fh)(\omega) = \sup_{\omega \in D} \{f(\omega, \omega) + G(\omega, \omega, h(\omega, \omega))\}
\]

(26)
We establish the following result:

**Theorem 9.** Let $F : B(W) \to B(W)$ be an operator defined by (26) and assume that the following conditions are satisfied:

(A) $G$ and $\beta$ are bounded;
(B) for $(\omega, \omega) \in W \times D, p, q \in B(W)$ and $t \in W$

\[
|p(\omega) - Fp(\omega)| \leq |p(\omega) - q(\omega)| \Rightarrow |G(\omega, \omega, P(t)) - G(\omega, \omega, q(t))| \leq \sqrt{|p(\omega) - q(\omega)|} - 2\sqrt{|Fp(\omega) - Fq(\omega)|}.
\]

Then, the functional equation 5.2 has a unique and bounded solution.

**Proof.** Define $\alpha : B(W) \times B(W) \to [0, \infty)$ and $\Theta : (0, \infty) \to (1, \infty)$ by $\alpha(p, q) = 1$ and $\Theta(t) = e^{\sqrt{t}}$ for all $p, q \in B(W)$, $t \in (0, \infty)$. Let $\delta > 0$ be any positive number and $p_1, p_2 \in B(W)$. Pick $\omega \in W$ arbitrarily and choose $\omega_1, \omega_2 \in W$ such that

\[
Fp_1 < f(\omega, \omega_1) + G(\omega, \omega, p_1(\omega)) + \delta \tag{27}
\]

where $\omega_i = \beta(\omega, \omega), i = 1, 2$.

By definition of $F$, we have

\[
Fp_1(\omega) \geq f(\omega, \omega_2) + G(\omega, \omega_2, p_1(\omega_2)) \tag{28}
\]

\[
Fp_2(\omega) \geq f(\omega, \omega_1) + G(\omega, \omega_1, p_2(\omega_1)). \tag{29}
\]

Now, from (27)–(29), we have

\[
Fp_1(\omega) - Fp_2(\omega) \leq |G(\omega, \omega_1, p_1(\omega_1)) - G(\omega, \omega_1, p_2(\omega_1))| + \delta. \tag{30}
\]

Similarly,

\[
Fp_2(\omega) - Fp_1(\omega) \leq |G(\omega, \omega_1, p_2(\omega_1)) - G(\omega, \omega_1, p_1(\omega_1))| + \delta. \tag{31}
\]

Assume that $\|p - Fp\| \leq \|p - q\|$, then from (30) and (31) with (B), we have

\[
0 \leq |Fp_1(\omega) - Fp_2(\omega)| \leq |G(\omega, \omega_1, p_1(\omega_1)) - G(\omega, \omega_1, p_2(\omega_1))| + \delta \leq \sqrt{|p(\omega) - q(\omega)|} - 2\sqrt{|Fp(\omega) - Fq(\omega)|} + \delta,
\]

which implies

\[
1 \leq e^{\sqrt{|p(\omega) - q(\omega)|} - 2\sqrt{|Fp(\omega) - Fq(\omega)|} + \delta} = \frac{e^{\sqrt{|p(\omega) - q(\omega)|} - 2\sqrt{|Fp(\omega) - Fq(\omega)|}}}{e^{2\sqrt{|Fp(\omega) - Fq(\omega)|}}} = \left(\frac{1}{e^{2\sqrt{|Fp(\omega) - Fq(\omega)|}}}ight)^{\frac{1}{2}} e^{\frac{\delta}{2}}.
\]

Which is equivalent to

\[
e^{\sqrt{|p - Fq|}} \leq [e^{\sqrt{|p - q|}}]^{\frac{1}{2}} e^{\frac{\delta}{2}}.
\]

Since $\delta > 0$ is arbitrary, we get

\[
e^{\sqrt{|p - Fq|}} \leq [e^{\sqrt{|p - q|}}]^{\frac{1}{2}}.
\]
Thus we have
\[ \Theta(\|Fp - Fq\|) \leq \left[ \Theta(\|p - q\|) \right]^{1/2}. \]

Thus, all the suppositions of Corollary 2 are satisfied for \( k = \frac{1}{2} \). Therefore, there exists \( p \), such that \( Fp = p \), which is the bounded solution of the functional Equation (25). □

6. Conclusions

In the present paper, we defined the notion of Suzuki \( \alpha-\Theta \)-proximal multivalued contractions to discuss the existence of best proximity points in the context of complete metric spaces. We derived some best proximity theorems on a metric space with graphs and ordered metric spaces as consequences. We discussed some variational inequality problems and dynamical programming problems as applications of our main results. We also gave a significant example to support our main results. We hope that the results contained in this article will build new connections for those who are working in \( \Theta \)-proximal contractions (or its generalizations) and its applications to variational inequality problems and dynamical programming problems.

Similar generalizations of such contractions for the fuzzy mappings \( \mathcal{F} : \mathcal{M} \rightarrow \mathcal{F}(\mathcal{M}) \) would be a special topic for future study. Another direction of future work would be to apply our results in the solution of fractional differential inclusions.

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