On Mann Viscosity Subgradient Extragradient Algorithms for Fixed Point Problems of Finitely Many Strict Pseudocontractions and Variational Inequalities

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Abstract: In a real Hilbert space, we denote CFPP and VIP as common fixed point problem of finitely many strict pseudocontractions and a variational inequality problem for Lipschitzian, pseudomonotone operator, respectively. This paper is devoted to explore how to find a common solution of the CFPP and VIP. To this end, we propose Mann viscosity algorithms with line-search process by virtue of subgradient extragradient techniques. The designed algorithms fully assimilate Mann approximation approach, viscosity iteration algorithm and inertial subgradient extragradient technique with line-search process. Under suitable assumptions, it is proven that the sequences generated by the designed algorithms converge strongly to a common solution of the CFPP and VIP, which is the unique solution to a hierarchical variational inequality (HVI).

Keywords: method with line-search process; pseudomonotone variational inequality; strictly pseudocontractive mappings; common fixed point; sequentially weak continuity

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1. Introduction and Preliminaries

Throughout this article, we suppose that the real vector space $H$ is a Hilbert one and the nonempty subset $C$ of $H$ is a convex and closed one. An operator $S : C \to H$ is called:

(i) $L$-Lipschitzian if there exists $L > 0$ such that \( \|Su - Sv\| \leq L\|u - v\| \ \forall u, v \in C \);

(ii) sequentially weakly continuous if for any \( \{w_n\} \subset C \), the following implication holds: \( w_n \rightharpoonup w \Rightarrow Sw_n \rightharpoonup Sw \);

(iii) pseudomonotone if \( \langle Su, u - v \rangle \leq 0 \Rightarrow \langle Sv, u - v \rangle \leq 0 \ \forall u, v \in C \);

(iv) monotone if \( \langle Su - Sv, v - u \rangle \leq 0 \ \forall u, v \in C \);

(v) $\gamma$-strongly monotone if \( \exists \gamma > 0 \ s.t. \ \langle Su - Sw, u - w \rangle \geq \gamma \|u - w\|^2 \ \forall u, w \in C \).

It is not difficult to observe that monotonicity ensures the pseudomonotonicity. A self-mapping $S : C \to C$ is called a $\eta$-strict pseudocontraction if the relation holds: \( \langle Su - Sv, u - v \rangle \leq \|u - v\|^2 - \frac{L^2}{2}\| (I - S)u - (I - S)v \|^2 \ \forall u, v \in C \) for some $\eta \in [0, 1)$. By [1] we know that, in the case where $S$ is $\eta$-strictly pseudocontractive, $S$ is Lipschitzian, i.e., \( \|Su - Sv\| \leq \frac{1 + \eta}{1 - \eta} \|u - v\| \ \forall u, v \in C \). It is clear that the class of strict pseudocontractions includes the class of nonexpansive operators, i.e., \( \|Su - Sv\| \leq \|u - v\| \ \forall u, v \in C \). Both classes of nonlinear operators received much attention and many numerical algorithms were designed for calculating their fixed points in Hilbert or Banach spaces; see e.g., [2–11].
Let $A$ be a self-mapping on $H$. The classical variational inequality problem (VIP) is to find $z \in C$ such that $\langle Az, y - z \rangle \geq 0 \ \forall y \in C$. The solution set of such a VIP is indicated by $\text{VI}(C, A)$. To the best of our knowledge, one of the most effective methods for solving the VIP is the gradient-projection method. Recently, many authors numerically investigated the VIP in finite dimensional spaces, Hilbert spaces or Banach spaces; see e.g., [12–20].

In 2014, Kraikaew and Saejung [21] suggested a Halpern-type gradient-like algorithm to deal with the VIP

$$
\begin{align*}
\ell
\end{align*}
$$

with $\ell \in (0, \frac{1}{1})$. Under mild assumptions, they proved that $\{u_k\}$ converge weakly to a point of $\text{VI}(C, A)$.

Very recently, Thong and Hieu [23] suggested two inertial algorithms with linear-search process, to solve the VIP for Lipschitzian, monotone operator $A$ and the FPP for a quasi-nonexpansive operator $S$ satisfying a demiclosedness property in $H$. Under appropriate assumptions, they proved that the sequences constructed by the suggested algorithms converge weakly to a point of $\text{Fix}(S) \cap \text{VI}(C, A)$. Further research on common solutions problems, we refer the readers to [24–38].

In this paper, we first introduce Mann viscosity algorithms via subgradient extragradient techniques, and then establish some strong convergence theorems in Hilbert spaces. It is remarkable that our algorithms involve line-search process.

The following lemmas are useful for the convergence analysis of our algorithms in the sequel.

**Lemma 1.** [39] Let the operator $A$ be pseudomonotone and continuous on $C$. Given a point $w \in C$. Then the relation holds: $\langle Aw, w - y \rangle \leq 0 \ \forall y \in C \Leftrightarrow \langle Ay, w - y \rangle \leq 0 \ \forall y \in C$.

**Lemma 2.** [40] Suppose that $\{s_k\}$ is a sequence in $[0, +\infty)$ such that $s_{k+1} \leq t_k b_k + (1 - t_k)s_k \ \forall k \geq 1$, where $\{t_k\}$ and $\{b_k\}$ lie in real line $\mathbb{R} := (-\infty, \infty)$, such that:

(a) $\{t_k\} \subset [0, 1]$ and $\sum_{k=1}^{\infty} t_k = \infty$;

(b) $\lim \sup_{k \to \infty} b_k \leq 0 \text{ or } \sum_{k=1}^{\infty} |t_k b_k| < \infty$. Then $s_k \to 0$ as $k \to \infty$.

From Ceng et al. [2] it is not difficult to find that the following lemmas hold.

**Lemma 3.** Let $\Gamma$ be an $\eta$-strictly pseudocontractive self-mapping on $C$. Then $I - \Gamma$ is demiclosed at zero.

**Lemma 4.** For $l = 1, \ldots, N$, let $\Gamma_l$ be an $\eta_l$-strictly pseudocontractive self-mapping on $C$. Then for $l = 1, \ldots, N$, the mapping $\Gamma_1$ is an $\eta$-strictly pseudocontraction with $\eta = \max\{\eta_l : 1 \leq l \leq N\}$, such that

$$
\|\Gamma_1 u - \Gamma_1 v\| \leq \frac{1 + \eta}{1 - \eta} \|u - v\| \ \forall u, v \in C.
$$

**Lemma 5.** Let $\Gamma$ be an $\eta$-strictly pseudocontractive self-mapping on $C$. Given two reals $\gamma, \beta \in [0, +\infty)$. If $(\gamma + \beta) \eta \leq \gamma$, then $\|\gamma(u - v) + \beta(\Gamma u - \Gamma v)\| \leq (\gamma + \beta)\|u - v\| \ \forall u, v \in C$. 
2. Main Results

Our first algorithm is specified below:

Algorithm 1

Initial Step: Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0$, $\ell \in (0, 1)$, $\mu \in (0, 1)$.

Iteration Steps: Compute $x_{n+1}$ below:

Step 1. Put $v_n = x_n - \sigma_n(x_n - x_{n-1})$ and calculate $u_n = P_C(v_n - \ell_n Au_n)$, where $\ell_n$ is picked to be the largest $\ell \in \{\gamma, \gamma \ell, \gamma^2 \ell, \ldots\}$ s.t. $\ell \|Au_n - Au_n\| \leq \mu \|v_n - u_n\|$. 

\begin{equation}
\ell \|Au_n - Au_n\| \leq \mu \|v_n - u_n\|.
\end{equation}

Step 2. Calculate $z_n = (1 - \alpha_n)P_{C_n}(v_n - \ell_n Au_n) + \alpha_n f(x_n)$ with $C_n := \{v \in H : \langle v - \ell_n Au_n - u_n, u_n - v \rangle \geq 0\}$.

Step 3. Calculate $x_{n+1} = \gamma_n P_{C_n}(v_n - \ell_n Au_n) + \delta_n T_n z_n + \beta_n x_n$.

Update $n := n + 1$ and return to Step 1.

In this section, we always suppose that the following hypotheses hold:

$T_k$ is a $\zeta_k$-strictly pseudocontractive mapping on $H$ for $k = 1, \ldots, N$ s.t. $\zeta \in [0, 1)$ with $\zeta = \max \{\xi_k : 1 \leq k \leq N\}$.

A is $L$-Lipschitzian, pseudomonotone self-mapping on $H$, and sequentially weakly continuous on $C$, such that $\Omega := \bigcap_{k=1}^N Fix(T_k) \cap VI(C, A) \neq \emptyset$.

\[ f : H \rightarrow C \text{ is a } \delta\text{-contraction with } \delta \in [0, \frac{1}{4}) \text{.} \]

\[ \{\sigma_n\} \subset \{0, 1\} \text{ and } \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1) \text{ are such that} : \]

(i) $\beta_n + \gamma_n + \delta_n = 1$ and $\sup_{n \geq 1} \frac{\delta_n}{\gamma_n} < \infty$;

(ii) $|1 - 2\delta_n| \gamma_n \geq \gamma_n + \delta_n$, $\forall n \geq 1$ and $\liminf_{n \rightarrow \infty} |1 - 2\delta_n| \gamma_n > 0$;

(iii) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sup_{n \geq 1} \beta_n = \infty$;

(iv) $\liminf_{n \rightarrow \infty} \beta_n > 0$, $\liminf_{n \rightarrow \infty} \delta_n > 0$ and $\limsup_{n \rightarrow \infty} \beta_n < 1$.

Following Xu and Kim [40], we denote $T_n := T_{n \text{mod} N}$, $\forall n \geq 1$, where the mod function takes values in $\{1, 2, \ldots, N\}$, i.e., whenever $n = jN + q$ for some $j \geq 0$ and $0 \leq q < N$, we obtain that $T_n = T_q$ in the case of $q = 0$ and $T_n = T_j$ in the case of $0 < q < N$.

Lemma 6. The Armijo-like search rule (1) is well defined, and $\min \{\gamma, \frac{\mu}{4}\} \leq \ell_n \leq \gamma$.

Proof. Obviously, (1) holds for all $\gamma^m \leq \frac{\mu}{4}$. So, $\ell_n$ is well defined and $\ell_n \leq \gamma$. In the case of $\ell_n = \gamma$, the inequality is true. In the case of $\ell_n < \gamma$, (1) ensures $\|Av_n - AP_C(v_n - \frac{\mu}{4} Au_n)\| > \frac{\mu}{4} \|v_n - P_C(v_n - \frac{\mu}{4} Au_n)\|$. The L-Lipschitzian property of $A$ yields $\ell_n > \frac{\mu}{4}$. \hfill \Box

Lemma 7. Let $\{v_n\}, \{u_n\}$ and $\{z_n\}$ be the sequences constructed by Algorithm 1. Then

\[ \|z_n - \omega\|^2 \leq (1 - \alpha_n)\|v_n - \omega\|^2 + \alpha_n \delta \|x_n - \omega\|^2 - (1 - \alpha_n)(1 - \mu)\|v_n - u_n\|^2 \]

\[ + \|h_n - u_n\|^2 + 2\alpha_n< f\omega - \omega, z_n - \omega > \forall \omega \in \Omega. \]

where $h_n := P_{C_n}(v_n - \ell_n Au_n) \forall n \geq 1$.

Proof. First, taking an arbitrary $p \in \Omega \subset C \subset C_n$, we observe that

\[ 2\|h_n - p\|^2 \leq 2< h_n - p, v_n - \ell_n Au_n - p > \]

\[ = \|h_n - p\|^2 + \|v_n - p\|^2 - \|h_n - v_n\|^2 - 2< \ell_n Au_n, h_n - p >. \]

So, it follows that $\|v_n - p\|^2 - 2< h_n - p, \ell_n Au_n > - \|h_n - v_n\|^2 \geq \|h_n - p\|^2$, which together with (1), we deduce that $0 \geq \langle p - u_n, Au_n \rangle$ and

\[ \|h_n - p\|^2 \leq \|v_n - p\|^2 - \|h_n - v_n\|^2 + 2\ell_n(\langle Au_n, p - u_n \rangle + \langle Au_n, u_n - h_n \rangle) \]

\[ \leq \|v_n - p\|^2 - \|Au_n - h_n\|^2 - \|v_n - u_n\|^2 + 2\langle u_n - v_n + \ell_n Au_n, u_n - h_n \rangle. \]
Since \( h_n = P_{C_n}(v_n - \ell_n Au_n) \) with \( C_n := \{ v \in H : \langle u_n - v + \ell_n Av_n, u_n - v \rangle \leq 0 \} \), we have \( \langle u_n - v_n + \ell_n Av_n, u_n - h_n \rangle \leq 0 \), which together with (1), implies that

\[
2\langle u_n - v_n + \ell_n Au_n, u_n - h_n \rangle = 2\langle u_n - v_n + \ell_n Av_n, u_n - h_n \rangle + 2\ell_n \langle Av_n - Au_n, h_n - u_n \rangle \leq 2\mu\|u_n - v_n\|\|u_n - h_n\| \leq \mu(\|v_n - u_n\|^2 + \|h_n - u_n\|^2).
\]

Therefore, substituting the last inequality for (4), we infer that

\[
\|h_n - p\|^2 \leq \|v_n - p\|^2 - (1 - \mu)\|v_n - u_n\|^2 - (1 - \mu)\|h_n - u_n\|^2 \quad \forall p \in \Omega.
\]

In addition, we have

\[
z_n - p = (1 - \alpha_n)(h_n - p) + \alpha_n(f - I)p + \alpha_n(f(x_n) - f(p)).
\]

Using the convexity of the function \( h(t) = t^2 \forall t \in \mathbb{R} \), from (5) we get

\[
\|z_n - p\|^2 \leq [\alpha_n\delta\|x_n - p\| + (1 - \alpha_n)\|h_n - p\|]^2 + 2\alpha_n\langle (f - I)p, z_n - p \rangle \\
 \leq \alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n)\|h_n - p\|^2 + 2\alpha_n\langle (f - I)p, z_n - p \rangle \\
 \leq \alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n)(\|v_n - p\|^2 - (1 - \alpha_n)(1 - \mu)\|v_n - u_n\|^2 \\
 + \|h_n - u_n\|^2) + 2\alpha_n\langle (f - I)p, z_n - p \rangle.
\]

\[
\square
\]

**Lemma 8.** Let \( \{x_n\}, \{u_n\}, \) and \( \{v_n\} \) be bounded sequences constructed by Algorithm 1. If \( x_n - x_{n+1} \to 0, \ v_n - u_n \to 0, \ v_n - z_n \to 0 \) and \( \exists \{v_n\} \subset \{v_n\} \) s.t. \( v_{n_i} \to z \in H, \) then \( z \in \Omega. \)

**Proof.** According to Algorithm 1, we get \( \sigma_n(x_n - x_{n-1}) = v_n - x_n \forall n \geq 1, \) and hence \( \|v_n - x_{n-1}\| \geq \|v_n - x_n\|. \) Using the assumption \( x_n - x_{n+1} \to 0, \) we have

\[
\lim_{n \to \infty} \|v_n - x_n\| = 0.
\]

So,

\[
\|z_n - x_n\| \leq \|v_n - z_n\| + \|v_n - x_n\| \to 0.
\]

Since \( \{x_n\} \) is bounded, from \( v_n = x_n - \sigma_n(x_{n-1} - x_n) \) we know that \( \{v_n\} \) is a bounded vector sequence. According to (5), we obtain that \( h_n := P_{C_n}(v_n - \ell_n Au_n) \) is a bounded vector sequence. Also, by Algorithm 1 we get \( \alpha_n f(x_n) + h_n - x_n - \alpha_n h_n = z_n - x_n. \) So, the boundedness of \( \{x_n\}, \{h_n\} \) guarantees that as \( n \to \infty, \)

\[
\|h_n - x_n\| = \|z_n - x_n - \alpha_n f(x_n) + \alpha_n h_n\| \leq \|z_n - x_n\| + \alpha_n(\|f(x_n)\| + \|h_n\|) \to 0.
\]

It follows that

\[
x_{n+1} - z_n = \gamma_n(h_n - x_n) + \delta_n(T_n z_n - z_n) + (1 - \delta_n)(x_n - z_n),
\]

which immediately yields

\[
\delta_n\|T_n z_n - z_n\| = \|x_{n+1} - x_n + x_n - z_n - (1 - \delta_n)(x_n - z_n) - \gamma_n(h_n - x_n)\| \\
\leq \|x_{n+1} - x_n + \delta_n(x_n - z_n) - \gamma_n(h_n - x_n)\| \\
\leq \|x_{n+1} - x_n\| + \|x_n - z_n\| + \|h_n - x_n\|.
\]
Since $x_n - x_{n+1} \to 0$, $z_n - x_n \to 0$, $h_n - x_n \to 0$ and $\liminf_{n \to \infty} \delta_n > 0$, we obtain $\|z_n - T_n z_n\| \to 0$ as $n \to \infty$. This further implies that

$$
\|x_n - T_n x_n\| \leq \|x_n - z_n\| + \|z_n - T_n z_n\| + \frac{1 + \epsilon}{1 - \epsilon} \|z_n - x_n\| \leq \frac{2}{1 - \epsilon} \|x_n - z_n\| + \|z_n - T_n z_n\| \to 0 \quad (n \to \infty).
$$

We have $(v_n - \ell_n Av_n - u_n, v - u_n) \leq 0 \quad \forall v \in C,$ and

$$
\langle v_n - u_n, v - u_n \rangle + \ell_n \langle Av_n, u_n - v_n \rangle \leq \ell_n \langle Av_n, v - v_n \rangle \quad \forall v \in C.
$$

Note that $\ell_n \geq \min \{\gamma, \frac{\|u\|^2}{T}\}.$ So, $\liminf_{n \to \infty} \langle Av_n, v - v_n \rangle \geq 0 \quad \forall v \in C.$ This yields $\liminf_{n \to \infty} \langle Av_n, v - u_n \rangle \geq 0 \quad \forall v \in C.$ Since $v_n - x_n \to 0$ and $v_n \to z$, we get $x_n \to z$. We may assume $k = n \mod N$ for all $i$. By the assumption $x_n - x_{n+k} \to 0$, we have $x_{n+j} \to z$ for all $j \geq 1$. Hence, $\|x_{n+j} - T_{i+k} x_{n+j}\| = \|x_{n+j} - T_{i+k} x_{n+j}\| \to 0.$ Then the demiclosedness principle implies that $z \in \text{Fix}(T_{k+i})$ for all $j$. This ensures that $z \in \bigcap_{k=1}^{N} \text{Fix}(T_k)$.

We now take a sequence $\{\xi_i\} \subset (0, 1)$ satisfying $\xi_i \downarrow 0$ as $i \to \infty$. For all $i \geq 1$, we denote by $m_i$ the smallest natural number satisfying

$$
\langle Au_{n_i}, v - u_{n_i} \rangle + \xi_i \geq 0 \quad \forall i \geq m_i.
$$

Since $\{\xi_i\}$ is decreasing, it is clear that $\{m_i\}$ is increasing. Noticing that $\{u_{n_i}\} \subset C$ ensures $Au_{m_i} \neq 0 \quad \forall i \geq 1$, we set $e_{m_i} = \frac{Au_{m_i}}{\|Au_{m_i}\|}$, we get $\langle Au_{m_i}, e_{m_i} \rangle = 1 \quad \forall i \geq 1.$ So, from (10) we get $\langle Au_{m_i}, v + \xi_i e_{m_i} - u_{m_i} \rangle \geq 0 \quad \forall i \geq 1.$ Also, the pseudomonotonicity of $A$ implies $\langle A(v + \xi_i e_{m_i}), v + \xi_i e_{m_i} - u_{m_i} \rangle \geq 0 \quad \forall i \geq 1.$

This immediately leads to

$$
\langle Av - A(v + \xi_i e_{m_i}), v + \xi_i e_{m_i} - u_{m_i} \rangle - \xi_i \langle Av, e_{m_i} \rangle \leq \langle Av, v - u_{m_i} \rangle \quad \forall i \geq 1.
$$

We claim $\lim_{i \to \infty} \xi_i e_{m_i} = 0$. Indeed, from $v_{n_i} \to z$ and $v_n - u_n \to 0$, we obtain $u_{n_i} \to z$. So, $\{u_{n_i}\} \subset C$ ensures $z \in C$. Also, the sequentially weak continuity of $A$ guarantees that $Au_{n_i} \to Az$. Thus, we have $Az \neq 0$ (otherwise, $z$ is a solution). Moreover, the sequentially weak lower semicontinuity of $\| \cdot \|$ ensures $0 < \|Az\| \leq \liminf_{i \to \infty} \|Au_{n_i}\|$. Since $\{u_{n_i}\} \subset \{u_n\}$ and $\xi_i \downarrow 0$ as $i \to \infty$, we deduce that $0 \leq \limsup_{i \to \infty} \|\xi_i e_{m_i}\| = \limsup_{i \to \infty} \|\xi_i \| = 0$. Hence we get $\xi_i e_{m_i} \to 0$.

Finally we claim $z \in \Omega$. In fact, letting $i \to \infty$, we conclude that the right hand side of (11) tends to zero by the Lipschitzian property of $A$, the boundedness of $\{u_{n_i}\}, \{h_{n_i}\}$ and the limit $\lim_{i \to \infty} \xi_i e_{m_i} = 0$. Thus, we get $\langle Av, v - z \rangle = \liminf_{i \to \infty} \langle Av, v - u_{n_i} \rangle \geq 0 \quad \forall v \in C$. So, $z \in VI(C, A)$. Therefore, from (9) we have $z \in \bigcap_{k=1}^{N} \text{Fix}(T_k) \cap VI(C, A) = \Omega.$

**Theorem 1.** Assume $A(C)$ is bounded. Let $\{x_n\}$ be constructed by Algorithm 1. Then

$$
x_n \to x^* \in \Omega \iff \begin{cases} x_n - x_{n+1} \to 0, \\ \sup_{n \geq 1} \|x_n - f x_n\| < \infty \end{cases}
$$

where $x^* \in \Omega$ is the unique solution to the hierarchical variational inequality (HVI):

$$
\langle (I - f)x^*, x^* - \omega \rangle \leq 0, \quad \forall \omega \in \Omega.
$$
Proof. Taking into account condition (iv) on \( \{ \gamma_n \} \), we may suppose that \( \{ \beta_n \} \subset [a, b] \subset (0, 1) \). Applying Banach’s Contraction Principle, we obtain existence and uniqueness of a fixed point \( x^* \in H \) for the mapping \( P_{\Omega} \circ f \), which means that \( x^* = P_{\Omega}f(x^*) \). Hence, the HVI

\[
\langle (I - f)x^* - x^* - \omega \rangle \leq 0, \quad \forall \omega \in \Omega
\]

(12)

has a unique solution \( x^* \in \Omega := \cap_{k=1}^{N} \text{Fix}(T_k) \cap VI(C, A) \).

It is now obvious that the necessity of the theorem is true. In fact, if \( x_n \to x^* \in \Omega \), then we get

\[
\sup_{n \geq 1} \| x_n - f(x_n) \| \leq \sup_{n \geq 1} (\| x_n - x^* \| + \| x^* - f(x^*) \| + \| f(x^*) - f(x_n) \|) < \infty \quad \text{and}
\]

\[
\| x_n - x_{n+1} \| \leq \| x_n - x^* \| + \| x_{n+1} - x^* \| \to 0 \quad (n \to \infty).
\]

For the sufficient condition, let us suppose \( x_n - x_{n+1} \to 0 \) and \( \sup_{n \geq 1} \| (I - f)x_n \| < \infty \). The sufficiency of our conclusion is proved in the following steps.

\[\square\]

Step 1. We show the boundedness of \( \{ x_n \} \). In fact, let \( p \) be an arbitrary point in \( \Omega \). Then \( T_np = p \forall n \geq 1 \), and

\[
\| v_n - p \|^2 - (1 - \mu)\| h_n - u_n \|^2 - (1 - \mu)\| v_n - u_n \| \geq \| h_n - p \|^2,
\]

(13)

which hence leads to

\[
\| v_n - p \| \geq \| h_n - p \| \quad \forall n \geq 1.
\]

(14)

By the definition of \( v_n \), we have

\[
\| v_n - p \| \leq \| x_n - p \| + \sigma_n \| x_n - x_{n-1} \| \leq \| x_n - p \| + \alpha_n \cdot \frac{\sigma_n}{\alpha_n} \| x_n - x_{n-1} \|.\]

(15)

Noticing \( \sup_{n \geq 1} \| x_n - x_{n-1} \| < \infty \) and \( \sup_{n \geq 1} \| x_n - x_{n-1} \| < \infty \), we obtain that \( \sup_{n \geq 1} \| x_n - x_{n-1} \| < \infty \).

This ensures that \( \exists M_1 > 0 \text{ s.t.} \)

\[
\frac{\sigma_n}{\alpha_n} \| x_n - x_{n-1} \| \leq M_1 \quad \forall n \geq 1.
\]

(16)

Combining (14)–(16), we get

\[
\| h_n - p \| \leq \| v_n - p \| \leq \| x_n - p \| + \alpha_n M_1 \quad \forall n \geq 1.
\]

(17)

Note that \( A(C) \) is bounded, \( u_n = P_C(v_n - \epsilon_n Au_n) \), \( f(H) \subset C \subset C_n \) and \( h_n = P_{C_n}(v_n - \epsilon_n Au_n) \). Hence we know that \( \{ Au_n \} \) is bounded. So, from \( \sup_{n \geq 1} \| (I - f)x_n \| < \infty \), it follows that

\[
\| h_n - f(x_n) \| \leq \| v_n - \epsilon_n Au_n - f(x_n) \|
\]

\[
\leq \| x_n - x_{n-1} \| + \| x_n - f(x_n) \| + \gamma \| Au_n \| \leq M_0,
\]

where \( \exists M_0 > 0 \text{ s.t.} \)

\[
M_0 \geq \sup_{n \geq 1} (\| x_n - x_{n-1} \| + \| x_n - f(x_n) \| + \gamma \| Au_n \|) \quad (\text{due to the assumption } x_n - x_{n+1} \to 0).
\]

Consequently,

\[
\| z_n - p \| \leq \alpha_n \delta \| x_n - p \| + (1 - \alpha_n) \| h_n - p \| + \alpha_n (f - I) p \|
\]

\[
\leq (1 - \alpha_n (1 - \delta)) \| x_n - p \| + \alpha_n (M_1 + \| (f - I) p \|),
\]

which together with \( (\gamma_n + \delta_n) \zeta \leq \gamma_n \), yields

\[
\| x_{n+1} - p \|
\]

\[
\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| \frac{1}{1 - \beta_n} [\gamma_n (z_n - p) + \delta_n (T_n z_n - p)] \| + \gamma_n \| h_n - z_n \|
\]

\[
\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| (1 - \alpha_n (1 - \delta)) \| x_n - p \| + \alpha_n (M_0 + M_1 + \| (f - I) p \|)
\]

\[
= [1 - \alpha_n (1 - \beta_n) (1 - \delta)] \| x_n - p \| + \alpha_n (1 - \beta_n) (1 - \delta) \frac{M_0 + M_1 + \| (f - I) p \|}{1 - \delta}.
\]
This shows that \( \|x_n - p\| \leq \max \{ \|x_1 - p\|, \frac{M_0 + M_1 + \|f(h)p\|}{1 - \nu} \} \) \( \forall n \geq 1 \). Thus, \( \{x_n\} \) is bounded, and so are the sequences \( \{h_n\}, \{v_n\}, \{u_n\}, \{z_n\}, \{T_n z_n\} \).

**Step 2.** We show that \( \exists M_4 > 0 \) s.t.

\[
(1 - \alpha_n)(1 - \beta_n)(1 - \mu) \|\|v_n - y_n\|^2 + \|u_n - y_n\|^2\| \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4.
\]

In fact, using Lemma 7 and the convexity of \( \cdot \|\cdot\|^2 \), we get

\[
\|x_{n+1} - p\|^2 \leq \|\beta_n(x_n - p)\|^2 + \gamma_n\|z_n - p\|^2 + \delta_n\|h_n - f(x_n), x_{n+1} - p\| + 2\gamma_n\alpha_n\langle h_n - f(x_n), x_{n+1} - p\rangle
\]

\[
\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 + 2(1 - \beta_n)\alpha_n\|h_n - f(x_n)\|\|x_{n+1} - p\|
\]

\[
\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)(\alpha_n\|x_n - p\| + (1 - \alpha_n)\|v_n - p\|)^2
\]

\[
- (1 - \alpha_n)(1 - \mu)\|\|v_n - u_n\|^2 + \|h_n - u_n\|^2\| + \alpha_n M_2, \tag{18}
\]

where \( \exists M_2 > 0 \) s.t. \( M_2 \geq \sup_{n \geq 1} 2(\|f - p\|\|z_n - p\| + \|u_n - f(x_n)\|\|x_{n+1} - p\|) \). Also,

\[
\|v_n - p\|^2 \leq \|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_3)
\]

\[
\leq \|x_n - p\|^2 + \alpha_n M_3, \tag{19}
\]

where \( \exists M_3 > 0 \) s.t. \( M_3 \geq \sup_{n \geq 1}(2M_1\|x_n - p\| + \alpha_n M_2^2) \). Substituting (19) for (18), we have

\[
\|x_{n+1} - p\|^2 \leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)(1 - \alpha_n(1 - \delta))\|x_n - p\|^2 + (1 - \alpha_n)\|x_n - M_5
\]

\[
- (1 - \alpha_n)(1 - \mu)\|\|v_n - u_n\|^2 + \|h_n - u_n\|^2\| + \alpha_n M_2
\]

\[
\leq \|x_n - p\|^2 - (1 - \alpha_n)(1 - \beta_n)(1 - \mu)\|\|v_n - u_n\|^2 + \|h_n - u_n\|^2\| + \alpha_n M_4, \tag{20}
\]

where \( M_4 := M_2 + M_3 \). This immediately implies that

\[
(1 - \alpha_n)(1 - \beta_n)(1 - \mu)\|\|v_n - u_n\|^2 + \|h_n - u_n\|^2\| \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4. \tag{21}
\]

**Step 3.** We show that \( \exists M > 0 \) s.t.

\[
\|x_{n+1} - p\|^2 \leq \left[ 1 - \frac{1 - 2\beta_n}{1 - \alpha_n(1 - \delta)} \right] \|x_n - p\|^2 + \frac{1 - 2\beta_n}{1 - \alpha_n(1 - \delta)} \|x_n - p\| f(x_n) - p \|z_n - x_{n+1}\|
\]

\[
+ \frac{2\delta_n}{1 - 2\beta_n} \|z_n - p\| f(x_n) - p \|z_n - x_n\| + \frac{2\delta_n}{1 - 2\beta_n} \|f(p) - p, x_n - p\|
\]

\[
+ \frac{2\delta_n}{\alpha_n(1 - \delta)} \|x_n - x_{n-1}\| M_3. \tag{22}
\]

In fact, we get

\[
\|v_n - p\|^2 \leq \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|\|2|x_n - p|| + \sigma_n\|x_n - x_{n-1}\|
\]

\[
\leq \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\| M_3, \tag{22}
\]

where \( \exists M > 0 \) s.t. \( M \geq \sup_{n \geq 1}\{\|x_n - p\|, \sigma_n\|x_n - x_{n-1}\|\} \). By Algorithm 1 and the convexity of \( \cdot \|\cdot\|^2 \), we have

\[
\|x_{n+1} - p\|^2 \leq \|\beta_n(x_n - p)\|^2 + \gamma_n\|z_n - p\|^2 + \delta_n\|h_n - f(x_n), x_{n+1} - p\| + 2\gamma_n\alpha_n\langle h_n - f(x_n), x_{n+1} - p\rangle
\]

\[
\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 + 2\gamma_n\alpha_n\|h_n - f(x_n)\|\|x_{n+1} - p\|
\]

\[
+ 2\gamma_n\alpha_n\langle h_n - f(x_n), x_{n+1} - p\rangle
\]

which leads to

\[
\|x_{n+1} - p\|^2 \leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)(1 - \alpha_n)\|h_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, z_n - p\rangle
\]

\[
+ \gamma_n\alpha_n\|h_n - p\|^2 + \|x_{n+1} - p\|^2 + 2\gamma_n\alpha_n\langle f(x_n) - p, x_{n+1} - p\rangle.
\]
Using (17) and (22) we obtain that \( \| h_n - p \|^2 \leq \| x_n - p \|^2 + \sigma_n \| x_n - x_{n-1} \| 3M. \) Hence,

\[
\| x_{n+1} - p \|^2 \leq \left[ 1 - \alpha_n (1 - \beta_n) \right] \| x_n - p \|^2 + (1 - \beta_n) (1 - \alpha_n) \sigma_n \| x_n - x_{n-1} \| 3M + 2 \alpha_n \beta_n \| f(x_n) - p \| \| z_n - x_{n+1} \| + \gamma_n \alpha_n \| f(x_n) - p \| \| z_n - x_{n-1} \| 3M + \gamma_n \alpha_n \| f(x_n) - p \| \| z_n - x_{n+1} \| + 2 \alpha_n \beta_n \| f(x_n) - p \| \| z_n - x_{n+1} \| + 2 \alpha_n \beta_n \| f(x_n) - p \| \| z_n - x_{n-1} \| 3M + 2 \gamma_n \alpha_n \| f(x_n) - p \| \| z_n - x_{n+1} \| + 2 \alpha_n \beta_n \| f(x_n) - p \| \| z_n - x_{n-1} \| 3M,
\]

which immediately yields

\[
\| x_{n+1} - p \|^2 \leq \left[ 1 - \left( 1 - \frac{2\beta_n}{\gamma_n} \right) \alpha_n \right] \| x_n - p \|^2 + \left( 1 - \frac{2\beta_n}{\gamma_n} \right) \alpha_n \| f(x_n) - p \| \| z_n - x_{n+1} \| + 2 \alpha_n \beta_n \| f(x_n) - p \| \| z_n - x_{n-1} \| 3M.
\] (23)

**Step 4.** We show that \( x_n \to x^* \in \Omega \), where \( x^* \) is the unique solution of (12). Indeed, putting \( p = x^* \), we infer from (23) that

\[
\| x_{n+1} - x^* \|^2 \leq \left[ 1 - \left( 1 - \frac{2\beta_n}{\gamma_n} \right) \alpha_n \right] \| x_n - x^* \|^2 + \left( 1 - \frac{2\beta_n}{\gamma_n} \right) \alpha_n \| f(x_n) - x^* \| \| z_n - x_{n+1} \| + 2 \alpha_n \beta_n \| f(x_n) - x^* \| \| z_n - x_{n-1} \| 3M.
\] (24)

It is sufficient to show that \( \limsup_{n \to \infty} (f(I)x^*, x_n - x^*) \leq 0 \). From (21), \( x_n - x_{n+1} \to 0, \alpha_n \to 0 \) and \( \{\beta_n\} \subset [a, b] \subset (0, 1) \), we get

\[
\limsup_{n \to \infty} (f(I)x^*, x_n - x^*) \leq \left( f(I)x^* - x^* \right), \| x_n - x^* \| + \| x_{n+1} - x^* \| + \| x_{n-1} - x^* \| \leq 0.
\]

This ensures that

\[
\lim_{n \to \infty} \| x_n - u_n \| = 0 \quad \text{and} \quad \lim_{n \to \infty} \| h_n - u_n \| = 0.
\] (25)

Consequently,

\[
\| x_n - u_n \| \leq \| x_n - v_n \| + \| v_n - u_n \| \to 0 \quad (n \to \infty).
\]

Since \( z_n = \alpha_n f(x_n) + (1 - \alpha_n) h_n \) with \( h_n := \rho v_n - (\ell_n A) u_n \), we get

\[
\| z_n - u_n \| = \| \alpha_n f(x_n) - \alpha_n h_n + h_n - u_n \| \leq \alpha_n (\| f(x_n) \| + \| h_n \|) + \| h_n - u_n \| \to 0 \quad (n \to \infty),
\] (26)

and hence

\[
\| z_n - u_n \| \leq \| z_n - u_n \| + \| u_n - x_n \| \to 0 \quad (n \to \infty).
\] (27)

Obviously, combining (25) and (26), guarantees that

\[
\| v_n - z_n \| \leq \| v_n - u_n \| + \| u_n - z_n \| \to 0 \quad (n \to \infty).
\]
From the boundedness of \( \{x_n\} \), it follows that \( \exists x_{n_1} \subset x_n \) s.t.
\[
\lim_{n \to \infty} \sup (f(I)x^*, x_n - x^*) = \lim_{i \to \infty} \sup (f(I)x^*, x_{n_i} - x^*).
\] (28)

Since \( \{x_n\} \) is bounded, we may suppose that \( x_{n_i} \to \hat{x} \). Hence from (28) we get
\[
\lim_{n \to \infty} \sup (f(I)x^*, x_n - x^*) = \lim_{i \to \infty} \sup (f(I)x^*, x_{n_i} - x^*) = \lim_{i \to \infty} \sup (f(I)x^*, \hat{x} - x^*).
\] (29)

It is easy to see from \( v_n - x_n \to 0 \) and \( x_{n_i} \to \hat{x} \) that \( v_{n_i} \to \hat{x} \). Since \( x_{n_i} - x_{n_i+1} \to 0 \), \( v_n - u_n \to 0 \), \( v_n - z_n \to 0 \) and \( v_{n_i} \to \hat{x} \), by Lemma 8 we infer that \( \hat{x} \in \Omega \). Therefore, from (12) and (29) we conclude that
\[
\lim_{n \to \infty} \sup (f(I)x^*, x_n - x^*) = \lim_{i \to \infty} \sup (f(I)x^*, \hat{x} - x^*) \leq 0.
\] (30)

Note that \( \liminf_{n \to \infty} \frac{(1-2\delta_n - \gamma_n)}{1 - \alpha_n} > 0 \). It follows that \( \sum_{n=0}^{\infty} \frac{(1-2\delta_n - \gamma_n)}{1 - \alpha_n} = \infty \). It is clear that
\[
\limsup_{n \to \infty} \left\{ \frac{2\gamma_n}{(1-2\delta_n - \gamma_n)} \|f(x_n) - x^*\| \|z_n - x_{n-1}\| + \frac{2\delta_n}{(1-2\delta_n - \gamma_n)} \|f(x_n) - x^*\| \|z_n - x_n\| + \frac{2\delta_n}{(1-2\delta_n - \gamma_n)} \|f(x_n) - x^*\| \|x_n - x_{n-1}\| \|3M\| \right\} \leq 0.
\] (31)

Therefore, by Lemma 2 we immediately deduce that \( x_n \to x^* \).

Next, we introduce another Mann viscosity algorithm with line-search process by the subgradient extragradient technique.

**Algorithm 2**

**Initial Step:** Given \( x_0, x_1 \in H \) arbitrarily. Let \( \gamma > 0 \), \( I \in (0, 1) \), \( \mu \in (0, 1) \).

**Iteration Steps:** Compute \( x_{n+1} \) below:

Step 1. Put \( v_n = x_n - \sigma_n(x_n - x_n) \) and calculate \( u_n = P_C(v_n - \ell_n Av_n) \), where \( \ell_n \) is picked to be the largest \( \ell \in \{\gamma, \gamma I, \gamma I^2, \ldots\} \) s.t.
\[
\ell \|Av_n - Au_n\| \leq \mu \|v_n - u_n\|.
\] (32)

Step 2. Calculate \( z_n = (1 - \alpha_n)P_C(v_n - \ell_n Av_n) + \alpha_n f(x_n) \) with \( C_n := \{v \in H : \langle v_n - \ell_n Av_n - u_n, u_n - v\rangle \geq 0\} \).

Step 3. Calculate
\[
x_{n+1} = \gamma_n P_C(v_n - \ell_n Av_n) + \delta_n T_n z_n + \beta_n v_n.
\] (33)

Update \( n := n + 1 \) and return to Step 1.

It is remarkable that Lemmas 6, 7 and 8 remain true for Algorithm 2.

**Theorem 2.** Assume \( A(C) \) is bounded. Let \( \{x_n\} \) be constructed by Algorithm 2. Then
\[
x_n \to x^* \in \Omega \iff \begin{cases} x_n - x_{n+1} \to 0, \\ \sup_{n \geq 1} \|I - f\|x_n\| < \infty \end{cases}
\]
where \( x^* \in \Omega \) is the unique solution of the HVI: \( \langle I - f\rangle x^*, x^* - \omega \leq 0, \forall \omega \in \Omega \).

**Proof.** For the necessity of our proof, we can observe that, by a similar approach to that in the proof of Theorem 1, we obtain that there is a unique solution \( x^* \in \Omega \) of (12).

We show the sufficiency below. To this aim, we suppose \( x_n - x_{n+1} \to 0 \) and \( \sup_{n \geq 1} \|I - f\|x_n\| < \infty \), and prove the sufficiency by the following steps. □
**Step 1.** We show the boundedness of \( \{ x_n \} \). In fact, by the similar inference to that in Step 1 for the proof of Theorem 1, we obtain that (13)–(17) hold. So, using Algorithm 2 and (17) we obtain
\[
\| z_n - p \| \leq (1 - \alpha_n (1 - \delta)) \| x_n - p \| + \alpha_n (M_1 + \| (f - I)p \|),
\]
which together with \((\gamma_n + \delta_n)z_0 \leq \gamma_n\), yields
\[
\begin{align*}
\| x_{n+1} - p \| & \leq \beta_n \| v_n - p \| + (1 - \beta_n) \| \frac{1}{1 - \beta_n} (\gamma_n (z_n - p) + \delta_n (T_n z_n - p)) \| + \gamma_n \| h_n - z_n \|
& \leq \beta_n \| x_n - p \| + \alpha_n M_1 + (1 - \beta_n) \| (1 - \alpha_n (1 - \delta)) \| x_n - p \|
& \quad + \alpha_n (M_0 + M_1 + \| (f - I)p \|)]
& = [1 - \alpha_n (1 - \beta_n) (1 - \delta)] \| x_n - p \| + \alpha_n (1 - \beta_n) (1 - \delta) \frac{M_0 + \frac{1}{1 - \beta_n} M_1 + \| (f - I)p \|}{1 - \delta}.
\end{align*}
\]
Therefore, we get the boundedness of \( \{ x_n \} \) and hence the one of sequences \( \{ h_n \}, \{ v_n \}, \{ u_n \}, \{ z_n \}, \{ T_n z_n \} \).

**Step 2.** We show that \( \exists M_4 > 0 \) s.t.
\[
(1 - \alpha_n)(1 - \beta_n)(1 - \mu) \| |w_n - y_n| \|^2 + \| u_n - y_n \|^2 \leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + \alpha_n M_4.
\]
In fact, by Lemma 7 and the convexity of \( \| \cdot \| \), we get
\[
\begin{align*}
\| x_{n+1} - p \|^2 & \leq \beta_n \| v_n - p \|^2 + \gamma_n \| z_n - p \| + \delta_n \| T_n z_n - p \| + 2 \gamma_n \alpha_n \| h_n - f(x_n) \| \| x_n - p \| + 2(1 - \beta_n) \alpha_n \| h_n - f(x_n) \| \| x_{n+1} - p \|
& \leq \beta_n \| v_n - p \|^2 + (1 - \beta_n) \| z_n - p \|^2 + 2(1 - \beta_n) \alpha_n \| h_n - f(x_n) \| \| x_{n+1} - p \|
& \quad - (1 - \alpha_n)(1 - \mu) \| \| v_n - u_n \|^2 + \| h_n - u_n \|^2 \| + \| x_{n+1} - p \|^2 \| x_{n+1} - p \| + \alpha_n M_2), \tag{34}
\end{align*}
\]
where \( \exists M_2 > 0 \) s.t. \( M_2 \geq \sup_{n \geq 2} (2 \| (f - I)p \| \| z_n - p \| + \| u_n - f(x_n) \| \| x_{n+1} - p \|) \). Also,
\[
\begin{align*}
\| v_n - p \|^2 & \leq \| x_n - p \|^2 + \alpha_n (2M_1 \| x_n - p \| + \alpha_n M_2^3)
& \leq \| x_n - p \|^2 + \alpha_n M_3, \tag{35}
\end{align*}
\]
where \( \exists M_3 > 0 \) s.t. \( M_3 \geq \sup_{n \geq 1} (2M_1 \| x_n - p \| + \beta_n M_2^3) \). Substituting (35) for (34), we have
\[
\begin{align*}
\| x_{n+1} - p \|^2 & \leq \beta_n \| x_n - p \|^2 + (1 - \beta_n) \{ (1 - \alpha_n (1 - \delta)) \| x_n - p \|^2 + (1 - \alpha_n) \alpha_n M_5
& \quad - (1 - \alpha_n)(1 - \mu) \| \| v_n - u_n \|^2 + \| h_n - u_n \|^2 \| + \alpha_n M_2 \} + \beta_n \alpha_n M_3
& = \| x_n - p \|^2 - (1 - \alpha_n)(1 - \beta_n)(1 - \mu) \| \| v_n - u_n \|^2 + \| h_n - u_n \|^2 \| + \| x_{n+1} - p \| + \alpha_n M_4, \tag{36}
\end{align*}
\]
where \( M_4 := M_2 + M_3 \). This ensures that
\[
(1 - \alpha_n)(1 - \beta_n)(1 - \mu) \| |w_n - y_n| \|^2 + \| h_n - u_n \|^2 \leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + \alpha_n M_4. \tag{37}
\]

**Step 3.** We show that \( \exists M > 0 \) s.t.
\[
\begin{align*}
\| x_{n+1} - p \|^2 & \leq \| x_n - p \|^2 + \| z_n - x_n - p \| + \| |z_n - x_n| - \| x_n - p \| \| p \| \| x_n - x_{n+1} \|
& \quad + \| f(x_n) - p \| \| z_n - x_n \| + \frac{2\beta_n}{1 - \beta_n} \| f(p) - p, x_n - p \| + \frac{2\delta_n}{1 - \alpha_n \gamma_n} \| x_n - x_{n-1} \| 3M.
\end{align*}
\]
In fact, we get
\[
\begin{align*}
\| v_n - p \|^2 & \leq \| x_n - p \|^2 + \sigma_n \| x_n - x_{n-1} \| (2 \| x_n - p \| + \sigma_n \| x_n - x_{n-1} \|)
& \leq \| x_n - p \|^2 + \sigma_n \| x_n - x_{n-1} \| 3M, \tag{38}
\end{align*}
\]
where $\exists M > 0$ s.t. $M \geq \sup_{n \geq 1} \{\|x_n - p\|, \sigma_n\|x_n - x_{n-1}\|\}$. Using Algorithm 1 and the convexity of $\| \cdot \|^2$, we get
\[
\|x_{n+1} - p\|^2 \leq \|\beta_n(v_n - p) + \gamma_n(z_n - p) + \delta_n(T_n z_n - p)\|^2 + 2\gamma_n \sigma_n \|h_n - f(x_n), x_{n+1} - p\|
\leq \beta_n\|v_n - p\|^2 + (1 - \beta_n)\|\frac{1}{1 - \beta_n} \gamma_n(z_n - p) + \delta_n(T_n z_n - p)\|^2 + 2\gamma_n \sigma_n \|h_n - f(x_n), x_{n+1} - p\|
+ 2\gamma_n \sigma_n \|h_n - p, x_{n+1} - p\| + 2\gamma_n \sigma_n \|p - f(x_n), x_{n+1} - p\|,
\]
which leads to
\[
\|x_{n+1} - p\|^2 \leq \beta_n\|v_n - p\|^2 + (1 - \beta_n)\|\|1 - \alpha_n\|\|h_n - p\|^2 + 2\alpha_n(f(x_n) - p, z_n - p)\|
+ \gamma_n \sigma_n \|\|h_n - p\|^2 + \|x_{n+1} - p\|^2\| + 2\gamma_n \sigma_n \|p - f(x_n), x_{n+1} - p\|.
\]
Using (17) and (38) we deduce that $\|h_n - p\|^2 \leq \|v_n - p\|^2 \leq \|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|^3 M$. Hence,
\[
\|x_{n+1} - p\|^2 \leq [1 - \alpha_n(1 - \beta_n)]\|x_{n} - p\|^2 + [1 - \alpha_n(1 - \beta_n)]\sigma_n\|x_{n} - x_{n-1}\|^3 M
+ 2\alpha_n \sigma_n \|f(x_n) - p, z_n - p\| + \gamma_n \sigma_n \|\|h_n - p\|^2 + \|x_{n+1} - p\|^2\| + 1 - \alpha_n(1 - \beta_n)]\|x_{n} - p\|^2 + \sigma_n\|x_{n} - x_{n-1}\|^3 M
+ 2\gamma_n \sigma_n \|f(x_n) - p, z_n - x_{n+1}\|
+ 2\gamma_n \sigma_n \|\|h_n - p\|^2 + \|x_{n+1} - p\|^2\| + \sigma_n\|x_{n} - x_{n-1}\|^3 M
\leq [1 - \alpha_n(1 - \beta_n)]\|x_{n} - p\|^2 + 2\alpha_n \sigma_n \|f(x_n) - p, z_n - x_{n+1}\|
+ 2\gamma_n \sigma_n \|\|h_n - p\|^2 + \|x_{n+1} - p\|^2\| + \sigma_n\|x_{n} - x_{n-1}\|^3 M,
\]
which immediately yields
\[
\|x_{n+1} - p\|^2 \leq [1 - \frac{(1 - \alpha_n)\beta_n - \gamma_n}{\alpha_n}]\|x_{n} - p\|^2 + \frac{(1 - \alpha_n)\beta_n - \gamma_n}{\alpha_n} \sigma_n\|x_{n} - x_{n-1}\|^3 M \frac{2\gamma_n}{\alpha_n} \|f(x_n) - p\| \|z_n - x_{n+1}\|
+ \frac{2\gamma_n}{\alpha_n} \|f(x_n) - p\| \|z_n - x_{n+1}\| + \frac{2\gamma_n}{\alpha_n} \sigma_n\|x_{n} - x_{n-1}\|^3 M.
\] (39)

**Step 4.** In order to show that $x_n \to x^* \in \Omega$, which is the unique solution of (12), we can follow a similar method to that in Step 4 for the proof of Theorem 1.

Finally, we apply our main results to solve the VIP and common fixed point problem (CFPP) in the following illustrating example.

The starting point $x_0 = x_1$ is randomly picked in the real line. Put $f(u) = \frac{1}{8} \sin u, \gamma = l = \mu = \frac{1}{2}, \sigma_n = \alpha_n = \frac{1}{n+1}, \beta_n = \frac{1}{\sqrt{5}}, \gamma_n = \frac{1}{n+1}$ and $\delta_n = \frac{1}{2}$.

We first provide an example of Lipschitzian, pseudomonotone self-mapping $A$ satisfying the boundedness of $A(C)$ and strictly pseudocontractive self-mapping $T_1$ with $\Omega = \text{Fix}(T_1) \cap \text{VI}(C, A) \neq \emptyset$. Let $C = [-1, 2]$ and $H$ be the real line with the inner product $(a, b) = ab$ and induced norm $\| \cdot \| = | \cdot |$. Then $f$ is a $\delta$-contractive map with $\delta = \frac{1}{5} \in [0, \frac{1}{2})$ and $f(H) \subset C$ because $\|f(u) - f(v)\| = \frac{1}{6} |\sin u - \sin v| \leq \frac{1}{6} |u - v|$ for all $u, v \in H$.

Let $A : H \to H$ and $T_1 : H \to H$ be defined as $Au := \frac{1}{1 + |\sin u|} - \frac{1}{1 + |u|}$, and $T_1 u := \frac{1}{2} u - \frac{3}{8} \sin u$ for all $u \in H$. Now, we first show that $A$ is $L$-Lipschitzian, pseudomonotone operator with $L = 2$, such that $A(C)$ is bounded. In fact, for all $u, v \in H$ we get
\[
\|Au - Av\| \leq \left\| \frac{1}{1 + |u|} - \frac{1}{1 + |v|} \right\| + \left\| \frac{1}{1 + |u|} \sin u - \frac{1}{1 + |v|} \sin v \right\|
\leq \left\| \frac{1}{1 + |u|} \right\| \left\| \frac{1 + |v|}{1 + |u|} \right\| + \left\| \frac{1 + |u|}{1 + |v|} \sin v \right\|
\leq 2 \|u - v\|.
\]
This implies that $A$ is 2-Lipschitzian. Next, we show that $A$ is pseudomonotone. For any given $u, v \in H$, it is clear that the relation holds:

$$
\langle Au, u - v \rangle = \left( \frac{1}{1 + |\sin u|} - \frac{1}{1 + |u|} \right) (u - v) \leq 0 \Rightarrow \langle Av, u - v \rangle = \left( \frac{1}{1 + |\sin v|} - \frac{1}{1 + |v|} \right) (u - v) \leq 0.
$$

Furthermore, it is easy to see that $T_1$ is strictly pseudocontractive with constant $\xi_1 = \frac{1}{2}$. In fact, we observe that for all $u, v \in H$,

$$
\|T_1 u - T_1 v\| \leq \frac{1}{2} \|u - v\| + \frac{3}{8} \| u - \sin v \| \leq \| u - v \| + \frac{1}{4} \| (I - T_1) u - (I - T_1) v \|.
$$

It is clear that $(\gamma_n + \delta_n) \xi_1 = (\frac{1}{8} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2}) \leq \frac{1}{8} = \gamma_n < (1 - 2\delta) \xi_1 = (1 - 2 \cdot \frac{1}{8} \cdot \frac{1}{2}) = \frac{3}{8}$ for all $n \geq 1$. In addition, it is clear that $\text{Fix}(T_1) = \{0\}$ and $A0 = 0$ because the derivative $d(T_1u)/du = \frac{1}{2} - \frac{3}{8} \cos u > 0$. Therefore, $\Omega = \{0\} \neq \emptyset$. In this case, Algorithm 1 can be rewritten below:

$$
\begin{align*}
\{v_n\} = x_n &- \frac{1}{n+1} (x_{n-1} - x_n), \\
u_n = P_C(v_n - \ell_n Au_n), \\
z_n = \frac{1}{n+1} f(x_n) + \frac{n}{n+1} P_C(v_n - \ell_n Au_n), \\
x_{n+1} = \frac{1}{2} v_n + \frac{1}{6} P_C(v_n - \ell_n Au_n) + \frac{1}{2} T_1 z_n \quad \forall n \geq 1,
\end{align*}
$$

with $\{C_n\}$ and $\{\ell_n\}$, selected as in Algorithm 1. Then, by Theorem 1, we know that $x_n \to 0 \in \Omega$ iff $x_n - x_{n+1} \to 0 (n \to \infty)$ and $\sup_{n \geq 1} |x_n - \frac{1}{8} \sin x_n| < \infty$.

On the other hand, Algorithm 2 can be rewritten below:

$$
\begin{align*}
\{v_n\} = x_n &- \frac{1}{n+1} (x_{n-1} - x_n), \\
u_n = P_C(v_n - \ell_n Au_n), \\
z_n = \frac{1}{n+1} f(x_n) + \frac{n}{n+1} P_C(v_n - \ell_n Au_n), \\
x_{n+1} = \frac{1}{2} v_n + \frac{1}{6} P_C(v_n - \ell_n Au_n) + \frac{1}{2} T_1 z_n \quad \forall n \geq 1,
\end{align*}
$$

with $\{C_n\}$ and $\{\ell_n\}$, selected as in Algorithm 2. Then, by Theorem 2, we know that $x_n \to 0 \in \Omega$ iff $x_n - x_{n+1} \to 0 (n \to \infty)$ and $\sup_{n \geq 1} |x_n - \frac{1}{8} \sin x_n| < \infty$.

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**References**


