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Global Behavior of a Higher Order Fuzzy Difference Equation

Guangwang Su [1,2], Taixiang Sun [1,2,*] and Bin Qin [2,3]

1 College of Information and Statistics, Guangxi University of Finance and Economics, Nanning 530003, China; s1g6w3@163.com
2 Guangxi Key Laboratory Cultivation Base of Cross-Border E-Commerce Intelligent Information Processing, Nanning 530003, China; q3009b@163.com
3 Guangxi (ASEAN) Research Center of Finance and Economics, Nanning 530003, China
* Correspondence: stxhql@gxu.edu.cn; Tel.: +86-138-7815-8146

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Abstract: Our aim in this paper is to investigate the convergence behavior of the positive solutions of a higher order fuzzy difference equation and show that all positive solutions of this equation converge to its unique positive equilibrium under appropriate assumptions. Furthermore, we give two examples to account for the applicability of the main result of this paper.

Keywords: fuzzy difference equation; fuzzy number; positive solution; positive equilibrium

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1. Introduction

It is well known that nonlinear difference equations and systems of difference equations of order greater than one are of great importance in applied sciences, where the n-st situation depends on the previous k situations because many models in economics, biology, computer sciences, engineering, etc. are represented by these equations naturally. For a detailed study of the theory of difference equations, see the monographs [1–6] and References [7–15]).

Since the data that are observed in relation to a real world phenomenon that can be described by a difference equation may be imprecise, this leads to introducing the fuzzification of the corresponding crisp difference equations. For this reason, studies of linear and nonlinear fuzzy difference equations (see [16–20]), and max-type fuzzy difference equations (see [21–23]), are more interesting as well as complicated.

In 2002, Papaschinopoulos and Papadopoulos [24] investigated global behavior of the following fuzzy difference equation:

$$y_{n+1} = C + \frac{y_n}{y_{n-k}}, \quad n \in \mathbf{N}_0 = \{0, 1, \cdots\} \tag{1}$$

under appropriate assumptions, where $k \in \mathbf{N} \equiv \{1, 2, \cdots\}$, C and the initial values y_i $(i \in [-k, 0]_{\mathbf{Z}})$ are positive fuzzy numbers, where $[u, v]_{\mathbf{Z}} = \{u, \cdots, v\}$ for any integers $u \leq v$.

In 2012, Zhang et al. [25] studied the existence, the boundedness and the asymptotic behavior of the positive solutions of the following fuzzy nonlinear difference equation

$$y_{n+1} = \frac{a + by_n}{A + y_{n-1}}, \quad n \in \mathbf{N}_0, \tag{2}$$

where a, b, A and the initial values y_{-1}, y_0 are positive fuzzy numbers.

Our aim in this paper is to investigate the global behavior of the positive solutions of the following more general fuzzy difference equation:

$$x_n = F(A_1, \cdots, A_s, x_{n-m}, B_1, \cdots, B_t, x_{n-k}), \;\; n \in \mathbf{N}_0, \tag{3}$$

where $m, k \in \mathbf{N}$, A_i $(i \in [1,s]_{\mathbf{Z}})$ and B_j $(j \in [1,t]_{\mathbf{Z}})$ are fuzzy numbers and the initial values $x_i(i \in [-d,-1]_{\mathbf{Z}})$ are positive fuzzy numbers with $d = \max\{m,k\}$ and F is the Zadeh's extension of $f : G \longrightarrow \mathbf{R}_+$, where $G = J_1 \times \cdots \times J_s \times \Re \times K_1 \times \cdots \times K_t \times \Re$, $\mathbf{R}_+ = (0, +\infty)$, and J_i $(i \in [1,s]_{\mathbf{Z}})$ and K_j $(i \in [1,t]_{\mathbf{Z}})$ are connected subsets of $\mathbf{R}_0 = [0, +\infty)$, and $\Re \in \{\mathbf{R}_0, \mathbf{R}_+\}$. In the following, we assume that the following conditions hold:

(H_1) $f \in C(G, \mathbf{R}_+)$.

(H_2) $f(u_1, \cdots, u_s, x, v_1, \cdots, v_t, y)$ is strictly decreasing on x and strictly increasing on y, and is decreasing on every u_i $(i \in [1,s]_{\mathbf{Z}})$ and increasing on every v_j $(j \in [1,t]_{\mathbf{Z}})$.

(H_3) For every $x \in \mathbf{R}_+$, $f(u_1, \cdots, u_s, x, v_1, \cdots, v_t, y)/y$ is decreasing on y in $\mathbf{R}_+$.

This paper is arranged as follows. We give some necessary definitions and preliminary results in Section 2. We show that under some conditions all positive solutions of (3) converge to its unique positive equilibrium in Section 3. Finally, two examples are given to account for the applicability of the main result of this paper.

2. Preliminaries

In this section, some definitions and preliminary results are given, which will be used in this paper, for more details, see [26].

Let J be a connected subset of $\mathbf{R}_0$. If A is a function from J into the interval $[0,1]$, then A is called a fuzzy set (for J). A fuzzy set A is called fuzzy convex if, for every $\lambda \in [0,1]$ and $x, y \in J$, we have $A(\lambda x + (1-\lambda)y) \geq \min\{A(x), A(y)\}$. For a fuzzy set A, $A_\alpha = \{x \in J : A(x) \geq \alpha\}$ is called the α-cuts of A for any $\alpha \in (0,1]$. It is known that the α-cuts determine the fuzzy set A.

Definition 1. *Let J be a connected subset of $\mathbf{R}_0$ and denote by $\overline{K}$ the closure of subset K of $\mathbf{R}_0$. We say that fuzzy set A is a fuzzy number (for J) if the following conditions hold:*

(1) $A_1 \neq \emptyset$.
(2) *A is fuzzy convex.*
(3) *A is upper semi-continuous.*
(4) *The support* $\mathbf{supp}A = \overline{\{x : A(x) > 0\}}$ *of A is compact.*

Denote by $\mathcal{F}(J)$ all fuzzy numbers (for J). If $A \in \mathcal{F}(J)$, then A_α is a closed interval for any $\alpha \in (0,1]$. Write $\mathcal{F}^+(J) = \{A \in \mathcal{F}(J) : \min(\mathbf{supp}A) > 0\}$, which is called the set of positive fuzzy numbers. If $A \in J$, then $A \in \mathcal{F}(J)$ with $A_\alpha = [A, A]$ for any $\alpha \in (0,1]$, which is called a trivial fuzzy number.

Definition 2 (see [24,27])**.** *Let $A, B \in \mathcal{F}(J)$ with $A_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$ and $B_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$ for any $\alpha \in (0,1]$. Then, we define the metric on $\mathcal{F}(J)$ as follows:*

$$D(A,B) = \sup_{\alpha \in (0,1]} \max\{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\}. \tag{4}$$

It is easy to see that $(\mathcal{F}(J), D)$ is a complete metric space.

Let $f \in C(G, \mathbf{R}_+)$ and write $\mathcal{F}(G) = \mathcal{F}(J_1) \times \cdots \times \mathcal{F}(J_s) \times \mathcal{F}(\Re) \times \mathcal{F}(K_1) \times \cdots \times \mathcal{F}(K_t) \times \mathcal{F}(\Re)$. We define a map $F : \mathcal{F}(G) \longrightarrow \mathcal{F}(\mathbf{R}_+)$ by, for any $g = (A_1, \cdots, A_s, u_1, B_1, \cdots, B_t, u_2) \in \mathcal{F}(G)$ and $z \in \mathbf{R}_+$,

$$(F(g))(z) = \sup\{\min\{A_1(a_1), \ldots, A_s(a_s), u_1(x_1), B_1(b_1), \cdots, B_t(b_t), u_2(x_2)\}\}, \tag{5}$$

where the sup is taken for all $a = (a_1, \cdots, a_s, x_1, b_1, \cdots, b_t, x_2) \in G$ such that $f(a) = z$. Thus, F is called the Zadeh's extension of f. By [28], we see that f is continuous if and only if F is continuous and, by [24], we see that, for any $\alpha \in (0, 1]$,

$$[F(g)]_\alpha = f([A_1]_\alpha, \cdots, [A_s]_\alpha, [u_1]_\alpha, [B_1]_\alpha, \cdots, [B_t]_\alpha, [u_2]_\alpha). \tag{6}$$

Definition 3. *We say that a sequence of positive fuzzy numbers $\{x_n\}_{n=-d}^{\infty}$ is a positive solution of (3) if it satisfies (3). We say that $x \in \mathcal{F}^+(\Re)$ is a positive equilibrium of (3) if $x = F(A_1, \cdots, A_s, x, B_1, \cdots, B_t, x)$.*

We say that a sequence of fuzzy numbers $\{x_n\}_{n=0}^{\infty}$ ($x_n \in \mathcal{F}(\Re)$) converges to $x \in \mathcal{F}(\Re)$ with respect to metric D if $\lim_{n\longrightarrow\infty} D(x_n, x) = 0$.

Proposition 1. *Let $g = (A_1, \cdots, A_s, u_1, B_1, \cdots, B_t, u_2) \in \mathcal{F}(G)$ and $u_3 \in \mathcal{F}(\mathbf{R}_+)$, and $[A_i]_\alpha = [A_{i,l,\alpha}, A_{i,r,\alpha}]$ $(i \in [1, s]_\mathbf{Z})$, $[B_j]_\alpha = [B_{j,l,\alpha}, B_{j,r,\alpha}]$ $(j \in [1, t]_\mathbf{Z})$, $[u_\lambda]_\alpha = [u_{\lambda,l,\alpha}, u_{\lambda,r,\alpha}]$ $(\lambda \in [1, 3]_\mathbf{Z})$ for any $\alpha \in (0, 1]$. If $u_3 = F(g)$, then*

$$\begin{cases} u_{3,l,\alpha} = f(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, u_{1,r,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, u_{2,l,\lambda}), \\ u_{3,r,\alpha} = f(A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, u_{1,l,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}, u_{2,r,\alpha}). \end{cases} \tag{7}$$

Proof. It follows from (6) that for any $\alpha \in (0, 1]$, we have

$$[u_{3,l,\alpha}, u_{3,r,\alpha}] = f([A_{1,l,\alpha}, A_{1,r,\alpha}], \cdots, [A_{s,l,\alpha}, A_{s,r,\alpha}],$$

$$[u_{1,l,\alpha}, u_{1,r,\alpha}], [B_{1,l,\alpha}, B_{1,r,\alpha}], \cdots, [B_{t,l,\alpha}, B_{t,r,\alpha}], [u_{2,l,\alpha}, u_{2,r,\alpha}]). \tag{8}$$

Let $a_i, a'_i \in [A_{i,l,\alpha}, A_{i,r,\alpha}]$ $(i \in [1, s]_\mathbf{Z})$, $b_j, b'_j \in [B_{j,l,\alpha}, B_{j,r,\alpha}]$ $(j \in [1, t]_\mathbf{Z})$, $p_\lambda, p'_\lambda \in [u_{\lambda,l,\alpha}, u_{\lambda,r,\alpha}]$ $(\lambda \in [1, 2]_\mathbf{Z})$ such that

$$\begin{cases} u_{3,l,\alpha} = f(a_1, \cdots, a_s, p_1, b_1, \cdots, b_t, p_2), \\ u_{3,r,\alpha} = f(a'_1, \cdots, a'_s, p'_1, b'_1, \cdots, b'_t, p'_2). \end{cases} \tag{9}$$

Then, according to (H_2), we obtain

$$\begin{cases} u_{3,l,\alpha} = f(a_1, \cdots, a_s, p_1, b_1, \cdots, b_t, p_2) \\ \geq f(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, u_{1,r,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, u_{2,l,\alpha}) \geq u_{3,l,\alpha}, \\ u_{3,r,\alpha} = f(a'_1, \cdots, a'_s, p'_1, b'_1, \cdots, b'_t, p'_2) \\ \leq f(A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, u_{1,rl,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}, u_{2,r,\alpha}) \leq u_{3,r,\alpha}, \end{cases} \tag{10}$$

from which it follows that

$$\begin{cases} u_{3,l,\alpha} = f(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, u_{1,r,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, u_{2,l,\alpha}), \\ u_{3,r,\alpha} = f(A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, u_{1,l,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}, u_{2,r,\alpha}). \end{cases} \tag{11}$$

Proposition 1 is proven. □

Proposition 2. *For given $x_{-d}, \cdots, x_{-1} \in \mathcal{F}^+(\Re)$, (3) has one unique positive solution $\{x_n\}_{n=-d}^{\infty}$ with initial values $x_{-d}, \cdots, x_{-1}$.*

Proof. For any $\alpha \in (0, 1]$, write

$$\begin{cases} [x_\lambda]_\alpha = [y_{\lambda,\alpha}, z_{\lambda,\alpha}] \ (\lambda \in [-d, -1]_\mathbf{Z}), \\ [A_i]_\alpha = [A_{i,l,\alpha}, A_{i,r,\alpha}] \ (i \in [1, s]_\mathbf{Z}), \\ [B_j]_\alpha = [B_{j,l,\alpha}, B_{j,r,\alpha}] \ (j \in [1, t]_\mathbf{Z}). \end{cases} \tag{12}$$

Let $\{(y_{n,\alpha}, z_{n,\alpha})\}_{n=-d}^{\infty} (\alpha \in (0,1])$ is the unique solution of the following system of difference equations:

$$\begin{cases} y_{n,\alpha} = f(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, z_{n-m,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, y_{n-k,\alpha}), \\ z_{n,\alpha} = f(A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, y_{n-m,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}, z_{n-k,\alpha}), \end{cases} \tag{13}$$

with initial values $(y_{i,\alpha}, z_{i,\alpha})$ $(i \in [-d, -1]_{\mathbf{Z}})$. Since $(A_1, \cdots, A_s, x_{-m}, B_1, \cdots, B_t, x_{-k}) \in \mathcal{F}(J_1) \times \cdots \times \mathcal{F}(J_s) \times \mathcal{F}^+(\Re) \times \mathcal{F}(K_1) \times \cdots \times \mathcal{F}(K_t) \times \mathcal{F}^+(\Re)$, there exist $0 \leq L_0 \leq M_0$ and $0 < L_0' \leq M_0'$ such that, for any $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 \leq \alpha_2$, we have

$$\begin{cases} L_0 \leq A_{i,l,\alpha_1} \leq A_{i,l,\alpha_2} \leq A_{i,r,\alpha_2} \leq A_{i,r,\alpha_1} \leq M_0 \ \ (i \in [1,s]_{\mathbf{Z}}), \\ L_0 \leq B_{i,l,\alpha_1} \leq B_{i,l,\alpha_2} \leq B_{i,r,\alpha_2} \leq B_{i,r,\alpha_1} \leq M_0 \ \ (i \in [1,t]_{\mathbf{Z}}), \\ 0 < L_0' \leq y_{i,\alpha_1} \leq y_{i,\alpha_2} \leq z_{i,\alpha_2} \leq z_{i,\alpha_1} \leq M_0' < M_0'' \ \ (i = -m, -k). \end{cases} \tag{14}$$

It follows from (H_1) and (H_2) that, for any $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 \leq \alpha_2$, we have

$$\begin{cases} 0 < f(M_0, \cdots, M_0, M_0'', L_0, \cdots, L_0, L_0') < \\ L_1 = f(M_0, \cdots, M_0, M_0', L_0, \cdots, L_0, L_0') \leq \\ y_{0,\alpha_1} = f(A_{1,r,\alpha_1}, \cdots, A_{s,r,\alpha_1}, z_{-m,\alpha_1}, B_{1,l,\alpha_1}, \cdots, B_{t,l,\alpha_1}, y_{-k,\alpha_1}) \leq \\ y_{0,\alpha_2} = f(A_{1,r,\alpha_2}, \cdots, A_{s,r,\alpha_2}, z_{-m,\alpha_2}, B_{1,l,\alpha_2}, \cdots, B_{t,l,\alpha_1 2}, y_{-k,\alpha_2}) \leq \\ z_{0,\alpha_2} = f(A_{1,l,\alpha_2}, \cdots, A_{s,l,\alpha_2}, y_{-m,\alpha_2}, B_{1,r,\alpha_2}, \cdots, B_{t,r,\alpha_2}, z_{-k,\alpha_2}) \leq \\ z_{0,\alpha_1} = f(A_{1,l,\alpha_1}, \cdots, A_{s,l,\alpha_1}, y_{-m,\alpha_1}, B_{1,r,\alpha_1}, \cdots, B_{t,r,\alpha_1}, z_{-k,\alpha_1}) \leq \\ \qquad f(L_0, \cdots, L_0, L_0', M_0, \cdots, M_0, M_0') = M_1. \end{cases} \tag{15}$$

It is easy to see that $y_{0,\alpha}, z_{0,\alpha}$ are left continuous on $\alpha \in (0,1]$ (see [29]) and $\overline{\cup_{\alpha \in (0,1]} [y_{n,\alpha}, z_{n,\alpha}]} \subset [L_1, M_1]$ (i.e., $\overline{\cup_{\alpha \in (0,1]} [y_{0,\alpha}, z_{0,\alpha}]}$ is compact). Hence, $[y_{0,\alpha}, z_{0,\alpha}]$ determines a unique $x_0 \in \mathcal{F}^+(\Re)$ such that $[x_0]_\alpha = [y_{0,\alpha}, z_{0,\alpha}]$ for all $\alpha \in (0,1]$ (see [29]).

Moreover, by mathematical induction, we can prove that:

(i) For every $n \in \mathbf{N}$, there exist $0 \leq L_{n+1} \leq M_{n+1}$ such that, for any $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 \leq \alpha_2$, we have $L_{n+1} \leq y_{n,\alpha_1} \leq y_{n,\alpha_2} \leq z_{n,\alpha_2} \leq z_{n,\alpha_1} \leq M_{n+1}$.

(ii) For every $n \in \mathbf{N}$, $y_{n,\alpha}$ and $z_{n,\alpha}$ are left continuous on $\alpha \in (0,1]$.

Hence, for every $n \in \mathbf{N}$, $\overline{\cup_{\alpha \in (0,1]} [y_{n,\alpha}, z_{n,\alpha}]} \subset [L_{n+1}, M_{n+1}]$ is compact, and $[y_{n,\alpha}, z_{n,\alpha}]$ determines a unique $x_n \in \mathcal{F}^+(\Re)$ such that $[x_n]_\alpha = [y_{n,\alpha}, z_{n,\alpha}]$ for every $\alpha \in (0,1]$, from which it follows that $\{x_n\}_{n=-d}^{\infty}$ is the unique positive solution of (3) with initial values x_i $(i \in [-d, -1]_{\mathbf{Z}})$.

Proposition 2 is proven. □

3. Main Result

In this section, we investigate the convergence behavior of the positive solutions of (3). For any positive solution $\{x_n\}_{n=-d}^{\infty}$ of (3) with initial values $x_i \in \mathcal{F}^+(\Re)$ $(i \in [-d, +\infty)_{\mathbf{Z}})$ and any $\alpha \in (0,1]$, we write

$$\begin{cases} [x_n]_\alpha = [y_{n,\alpha}, z_{n,\alpha}] \ \ (n \in [-d, -1]_{\mathbf{Z}}), \\ [A_i]_\alpha = [A_{i,l,\alpha}, A_{i,r,\alpha}] \ \ (i \in [1,s]_{\mathbf{Z}}), \\ [B_j]_\alpha = [B_{j,l,\alpha}, B_{j,r,\alpha}] \ \ (j \in [1,t]_{\mathbf{Z}}). \end{cases} \tag{16}$$

By Proposition 1, we see that $\{(y_{n,\alpha}, z_{n,\alpha})\}_{n=-d}^{\infty} (\alpha \in (0,1])$ satisfies the following system

$$\begin{cases} y_{n,\alpha} = f(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, z_{n-m,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, y_{n-k,\alpha}), \\ z_{n,\alpha} = f(A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, y_{n-m,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}, z_{n-k,\alpha}), \end{cases} \tag{17}$$

with initial values $(y_{i,\alpha}, z_{i,\alpha})$ $(i \in [-d, -1]_{\mathbf{Z}})$. For the convenience, we write

$$\begin{cases} y_{n,\alpha} = f(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, z_{n-m,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, y_{n-k,\alpha}) = h(z_{n-m,\alpha}, y_{n-k,\alpha}), \\ z_{n,\alpha} = f(A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, y_{n-m,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}, z_{n-k,\alpha}) = g(y_{n-m,\alpha}, z_{n-k,\alpha}). \end{cases} \tag{18}$$

Lemma 1. *Assume that (H_1)–(H_3) hold, and h and g satisfy the following hypotheses:*

(H_4) *System of equations*

$$\begin{cases} y_\alpha = h(z_\alpha, y_\alpha), \\ z_\alpha = g(y_\alpha, z_\alpha) \end{cases} \tag{19}$$

has a unique positive solution

$$\begin{cases} y_\alpha = y(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}), \\ z_\alpha = z(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}), \end{cases} \tag{20}$$

and $y_\alpha, z_\alpha \in C(E \times E, \mathbf{R}_+)$ and $E = J_1 \times \cdots \times J_s \times K_1 \times \cdots \times K_t$.

(H_5)

$$\begin{cases} p_{1,\alpha} = p_1(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}) = \inf_{(z,y)\in\Re\times\Re} h(z,y) \in \Re, \\ p_{2,\alpha} = p_2(A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}) = \inf_{(y,z)\in\Re\times\Re} g(y,z) \in \Re. \end{cases} \tag{21}$$

$h(p_{2,\alpha}, y) = y$ has only one solution $q_{1,\alpha} = q_1(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, p_{2,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha})$ in interval $(p_{1,\alpha}, +\infty)$, and $g(p_{1,\alpha}, z) = z$ has only one solution $q_{2,\alpha} = q_2(A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, p_{1,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha})$ in interval $(p_{2,\alpha}, +\infty)$.

Then,

$$\begin{cases} h(q_{2,\alpha}, p_{1,\alpha}) < y_\alpha < q_{1\alpha}, \\ g(q_{1,\alpha}, p_{2,\alpha}) < z_\alpha < q_{2,\alpha}. \end{cases} \tag{22}$$

Proof. For the convenience, we write $q_{2,\alpha} = q_2, p_{1,\alpha} = p_1, y_\alpha = y, q_{1\alpha} = q_1, p_{2,\alpha} = p_2, z_\alpha = z$. By $(H_2), (H_4)$ and (H_5), we have

$$\begin{cases} y = h(z,y) > h(z+2,y) \geq p_1, \\ z = g(y,z) > h(y+2,z) \geq p_2. \end{cases} \tag{23}$$

Suppose for the sake of contradiction that $y \geq q_1$. Then, it follows from (21) and (H_2)–(H_5), that

$$q_1 = h(p_2, q_1) > h(z, q_1) = q_1 \frac{h(z,q_1)}{q_1} \geq q_1 \frac{h(z,y)}{y} = q_1, \tag{24}$$

which is a contradiction. Therefore, $y < q_1$. In a similar fashion, we can obtain $z < q_2$. Thus, by (H_2),(H_4) and (H_5), we have that

$$\begin{cases} h(q_2, p_1) < h(z,y) = y < q_1, \\ g(q_1, p_2) < g(y,z) = z < q_2. \end{cases} \tag{25}$$

Lemma 1 is proven. □

Lemma 2. *Let $I_\alpha = [p_{1,\alpha}, M_{1,\alpha}]$ and $J_\alpha = [p_{2,\alpha}, M_{2,\alpha}]$ with $M_{1,\alpha} \geq q_{1,\alpha}$ and $M_{2,\alpha} \geq q_{2,\alpha}$. Assume that (H_1)–(H_5) hold. If $(y_{i,\alpha}, z_{i,\alpha}) \in I_\alpha \times J_\alpha$ $(i \in [-d, -1]_{\mathbf{Z}})$, then $(y_{n,\alpha}, z_{n,\alpha}) \in I_\alpha \times J_\alpha$ for any $n \in \mathbf{N}_0$.*

Proof. By (H_2), (H_3) and (H_5), we have

$$\begin{cases} p_{1,\alpha} \le y_{0,\alpha} = h(z_{-m,\alpha}, y_{-k,\alpha}) \le \frac{h(p_{2,\alpha}, M_{1,\alpha})}{M_{1,\alpha}} M_{1,\alpha} \le \frac{h(p_{2,\alpha}, q_{1,\alpha})}{q_{1,\alpha}} M_{1,\alpha} = M_{1,\alpha}, \\ p_{2,\alpha} \le z_{0,\alpha} = g(y_{-m,\alpha}, z_{-k,\alpha}) \le \frac{g(p_{1,\alpha}, M_{2,\alpha})}{M_{2,\alpha}} M_{2,\alpha} \le \frac{g(p_{1,\alpha}, q_{2,\alpha})}{q_{2,\alpha}} M_{2,\alpha} = M_{2,\alpha}. \end{cases} \tag{26}$$

By mathematical induction, we can obtain $(y_{n,\alpha}, z_{n,\alpha}) \in I_\alpha \times J_\alpha$ for any $n \in \mathbf{N}_0$. Lemma 2 is proven. □

Suppose that I_α and J_α are as in Lemma 2. Let $u_{0,\alpha} = p_{1,\alpha}$, $U_{0,\alpha} = M_{1,\alpha}$, $v_{0,\alpha} = p_{2,\alpha}$ and $V_{0,\alpha} = M_{2,\alpha}$, and, for any $n \in \mathbf{N}$,

$$\begin{cases} U_{n,\alpha} = h(v_{n-1,\alpha}, U_{n-1,\alpha}), \;\; u_{n,\alpha} = h(V_{n-1,\alpha}, u_{n-1,\alpha}), \\ V_{n,\alpha} = g(u_{n-1,\alpha}, V_{n-1,\alpha}), \;\; v_{n,\alpha} = g(U_{n-1,\alpha}, v_{n-1,\alpha}). \end{cases} \tag{27}$$

Lemma 3. *Let I_α and J_α be as in Lemma 2. Assume that (H_1)–(H_5) hold. Suppose that h and g satisfy the following hypotheses:*

(H_6) *If $U_\alpha, u_\alpha \in I_\alpha$ with $u_\alpha \le U_\alpha$ and $V_\alpha, v_\alpha \in J_\alpha$ with $v_\alpha \le V_\alpha$ are a solution of the system*

$$\begin{cases} U_\alpha = h(v_\alpha, U_\alpha), \;\; u_\alpha = h(V_\alpha, u_\alpha), \\ V_\alpha = g(u_\alpha, V_\alpha), \;\; v_\alpha = g(U_\alpha, v_\alpha), \end{cases} \tag{28}$$

then $U_\alpha = u_\alpha$ and $V_\alpha = v_\alpha$.

Then, $\lim_{n\longrightarrow\infty} U_{n,\alpha} = \lim_{n\longrightarrow\infty} u_{n,\alpha} = y_\alpha$ and $\lim_{n\longrightarrow\infty} V_{n,\alpha} = \lim_{n\longrightarrow\infty} v_{n,\alpha} = z_\alpha$.

Proof. For the convenience, we write $U_{n,\alpha} = U_n$, $v_{n,\alpha} = v_n$, $u_{n,\alpha} = u_n$, $V_{n,\alpha} = V_n$, $y_\alpha = y$ and $z_\alpha = z$. By Lemmas 1 and 2, we obtain

$$\begin{cases} u_0 \le u_1 = h(V_0, u_0) < h(z, y) = y < h(v_0, U_0) = U_1 \le U_0, \\ v_0 \le v_1 = g(U_0, v_0) < g(y, z) = z < g(u_0, V_0) = V_1 \le V_0, \end{cases} \tag{29}$$

and

$$\begin{cases} u_1 = h(V_0, u_0) \le u_2 = h(V_1, u_1) < h(z, y) = y < h(v_1, U_1) = U_2 \le h(v_0, U_0) = U_1, \\ v_1 = g(U_0, v_0) \le v_2 = g(U_1, v_1) < g(y, z) = z < g(u_1, V_1) = V_2 \le g(u_0, V_0) = V_1. \end{cases} \tag{30}$$

By mathematical induction, we can obtain

$$\begin{cases} u_0 \le u_1 \le \cdots \le u_n \le \cdots < y < \cdots \le U_n \le \cdots \le U_1 \le U_0, \\ v_0 \le v_1 \le \cdots \le v_n \le \cdots < z < \cdots \le V_n \le \cdots \le V_1 \le V_0. \end{cases} \tag{31}$$

Let

$$\begin{cases} \lim_{n\longrightarrow\infty} U_n = U, \;\; \lim_{n\longrightarrow\infty} u_n = u, \\ \lim_{n\longrightarrow\infty} V_n = V, \;\; \lim_{n\longrightarrow\infty} v_n = v. \end{cases} \tag{32}$$

By (27), we have

$$\begin{cases} U = h(v, U), \;\; u = h(V, u), \\ V = g(u, V), \;\; v = g(U, v). \end{cases} \tag{33}$$

It follows from (H_6) that $U = u$ and $V = v$, which with (H_4) implies $U = v = y$ and $V = v = z$. Lemma 3 is proven. □

Lemma 4. *Let $\{(y_{n,\alpha}, z_{n,\alpha})\}_{n=-d}^{\infty}$ be a positive solution of (16). Then, $\lim_{n\longrightarrow\infty}(y_{n,\alpha}, z_{n,\alpha}) = (y_\alpha, z_\alpha)$.*

Proof. Let $\{(y_{n,\alpha}, z_{n,\alpha})\}_{n=-d}^{\infty}$ be a positive solution of (16), and $M_{1,\alpha} = \max\{y_{0,\alpha}, \cdots, y_{d,\alpha}, q_{1,\alpha}\}$, $M_{2,\alpha} = \max\{z_{0,\alpha}, \cdots, z_{d,\alpha}, q_{2,\alpha}\}$ and $u_{n,\alpha}, U_{n,\alpha}, v_{n,\alpha}, V_{n,\alpha}$ be the same as (27) and Lemma 3. By Lemma 2 we have $y_{n,\alpha} \in [u_{0,\alpha}, U_{0,\alpha}] = [p_{1,\alpha}, M_{1,\alpha}]$ and $z_{n,\alpha} \in [v_{0,\alpha}, V_{0,\alpha}] = [p_{2,\alpha}, M_{2,\alpha}]$ for any $n \in \mathbf{N}$. Moreover, we have

$$\begin{cases} u_{1,\alpha} = h(V_{0,\alpha}, u_{0,\alpha}) \leq h(z_{d-m,\alpha}, y_{d-k,\alpha}) = y_{d,\alpha} \leq h(v_{0,\alpha}, U_{0,\alpha}) = U_{1,\alpha}, \\ v_{1,\alpha} = g(U_{0,\alpha}, v_{0,\alpha}) \leq g(y_{d-m,\alpha}, z_{d-k,\alpha}) = z_{d,\alpha} \leq h(u_{0,\alpha}, V_{0,\alpha}) = V_{1,\alpha}. \end{cases} \tag{34}$$

In similar fashion, we may show $y_{n,\alpha} \in [u_{1,\alpha}, U_{1,\alpha}]$ and $z_{n,\alpha} \in [v_{1,\alpha}, V_{1,\alpha}]$ for any $n \geq d$. By mathematical induction, we can show that, for any $n \geq kd$,

$$y_{n,\alpha} \in [u_{k,\alpha}, U_{k,\alpha}], \ \ z_{n,\alpha} \in [v_{k,\alpha}, V_{k,\alpha}]. \tag{35}$$

It follows from Lemma 3 that $\lim_{n\to\infty} y_{n,\alpha} = y_\alpha$ and $\lim_{n\to\infty} z_{n,\alpha} = z_\alpha$. Lemma 4 is proven. □

Now, we state and show the main result of this paper.

Theorem 1. *Assume that* (H_1)*–*(H_6) *hold. Then, every positive solution of (3) converges to the unique positive equilibrium of (3).*

Proof. Let $\{x_n\}_{n=-d}^{\infty}$ be a positive solution of (3) with initial values $x_i \in \mathcal{F}^+(\mathbf{R}_+)$ $(i \in [-d, -1]_{\mathbf{Z}})$. By (16), we see $y_{n,\alpha_1} \leq y_{n,\alpha_2} \leq z_{n,\alpha_2} \leq z_{n,\alpha_1}$ for any $\alpha_1, \alpha_2 \in (0, 1]$ with $\alpha_1 \leq \alpha_2$. Then, it follows from Lemma 4 that

$$\lim_{n\longrightarrow\infty} y_{n,\alpha_1} = y_{\alpha_1} \leq \lim_{n\longrightarrow\infty} y_{n,\alpha_2} = y_{\alpha_2} \leq \lim_{n\longrightarrow\infty} z_{n,\alpha_2} = z_{\alpha_2} \leq \lim_{n\longrightarrow\infty} z_{n,\alpha_1} = z_{\alpha_1}. \tag{36}$$

Let $\mathbf{supp}A_i = [a_i, a_i']$ $(i \in [1, s]_{\mathbf{Z}})$ and $\mathbf{supp}B_j = [b_j, b_j']$ $(j \in [1, t]_{\mathbf{Z}})$. Then, for any $\alpha \in (0, 1]$,

$$\begin{cases} [A_{i,l,\alpha}, A_{i,r,\alpha}] \subset \mathbf{supp}A_i = [a_i, a_i'] \subset J_i \ (i \in [1, s]_{\mathbf{Z}}), \\ [B_{j,l,\alpha}, B_{j,r,\alpha}] \subset \mathbf{supp}B_j = [b_j, b_j'] \subset K_j \ (j \in [1, t]_{\mathbf{Z}}). \end{cases} \tag{37}$$

By (H_4), we see that for any $\alpha \in (0, 1]$, the following system

$$\begin{cases} y_\alpha = f(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, z_\alpha, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, y_\alpha), \\ z_\alpha = f(A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, y_\alpha, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}, z_\alpha) \end{cases} \tag{38}$$

has a unique positive solution

$$\begin{cases} y_\alpha = y(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}), \\ z_\alpha = z(A_{1,r,\alpha}, \cdots, A_{s,r,\alpha}, B_{1,l,\alpha}, \cdots, B_{t,l,\alpha}, A_{1,l,\alpha}, \cdots, A_{s,l,\alpha}, B_{1,r,\alpha}, \cdots, B_{t,r,\alpha}), \end{cases} \tag{39}$$

and $y_\alpha, z_\alpha \in C(S, \mathbf{R}_+)$, where $S = [a_1, a_1'] \times \cdots \times [a_s, a_s'] \times [b_1, b_1'] \times \cdots \times [b_t, b_t'] \times [a_1, a_1'] \times \cdots \times [a_s, a_s'] \times [b_1, b_1'] \times \cdots \times [b_t, b_t']$. Let

$$M = \max_{\alpha\in S} z(\alpha), \quad m = \min_{\alpha\in S} y(\alpha). \tag{40}$$

Then, we see $0 < m \leq M < +\infty$, and that y_α and z_α are left continuous on $\alpha \in (0, 1]$, and $\overline{\cup_{\alpha\in(0,1]}[y_\alpha, z_\alpha]} \subset [m, M]$ (i.e., $\overline{\cup_{\alpha\in(0,1]}[y_\alpha, z_\alpha]}$ is compact). Therefore, (3) has the unique positive equilibrium $x \in \mathcal{F}^+(\mathbf{R}_+)$ such that $x_\alpha = [y_\alpha, z_\alpha]$ for any $\alpha \in (0, 1]$. Furthermore, by Lemma 4, we see

$$\lim_{n\longrightarrow\infty} D(x_n, x) = \lim_{n\longrightarrow\infty} \sup_{\alpha\in(0,1]} \max\{|y_{n,\alpha} - y_\alpha|, |z_{n,\alpha} - z_\alpha|\} = 0. \tag{41}$$

Theorem 1 is proven. □

4. Examples

In this section, we give two examples to account for the applicability of Theorem 1.

Example 1. *Let*

$$x_n = F(A, x_{n-m}, B, C, x_{n-k}) = \frac{B + Cx_{n-k}}{A + x_{n-m}}, \ \ n \in \mathbf{N}_0, \tag{42}$$

where $m, k \in \mathbf{N}$, $A, C \in \mathcal{F}^+(\mathbf{R}_+)$ with $\max(\mathbf{supp}C) < \min(\mathbf{supp}A)$, and $B \in \mathcal{F}(\mathbf{R}_0)$, and the initial values $x_i \in \mathcal{F}^+(\mathbf{R}_+)$ $(i \in [-d, -1]_{\mathbf{Z}})$ with $d = \max\{m, k\}$. Then, all positive solutions of (42) converge to its unique positive equilibrium as $n \longrightarrow \infty$.

Proof. For the convenience, for any $\alpha \in (0, 1]$, we write $A_\alpha = [A_{1,\alpha}, A_{2,\alpha}] = [A_1, A_2], B_\alpha = [B_{1,\alpha}, B_{2,\alpha}] = [B_1, B_2]$ and $C_\alpha = [C_{1,\alpha}, C_{2,\alpha}] = [C_1, C_2]$. Let

$$f(u, x, v, w, y) = \frac{v + wy}{u + x}. \tag{43}$$

In the following, we verify that $(H_1) - (H_6)$ hold.

(1) $f \in C(\mathbf{R}_+ \times \mathbf{R}_0 \times \mathbf{R}_0 \times \mathbf{R}_+ \times \mathbf{R}_0, \mathbf{R}_+)$ is strictly decreasing on every $\mu \in \{u, x\}$, and strictly increasing on every $\nu \in \{v, w, y\}$.

(2) For every $x \in \mathbf{R}_+$, $f(u, x, v, w, y)/y$ is decreasing on y in $\mathbf{R}_+$.

(3) System of equations

$$\begin{cases} y_\alpha = \frac{B_1 + C_1 y_\alpha}{A_2 + z_\alpha}, \\ z_\alpha = \frac{B_2 + C_2 z_\alpha}{A_1 + y_\alpha} \end{cases} \tag{44}$$

has a unique positive solution

$$\begin{cases} y_\alpha = y(A_1, B_1, C_1, A_2, B_2, C_2) \\ = \frac{B_1 - B_2 - (A_1 - C_2)(A_2 - C_1) + \sqrt{(B_1 - B_2)^2 + (A_1 - C_2)^2(A_2 - C_1)^2 + 2(B_1 + B_2)(A_1 - C_2)(A_2 - C_1)}}{2(A_2 - C_1)}, \\ z_\alpha = z(A_1, B_1, C_1, A_2, B_2, C_2) \\ = \frac{B_2 - B_1 - (A_1 - C_2)(A_2 - C_1) + \sqrt{(B_1 - B_2)^2 + (A_1 - C2)^2(A_2 - C_1)^2 + 2(B_1 + B_2)(A_1 - C_2)(A_2 - C_1)}}{2(A_1 - C_2)} \end{cases} \tag{45}$$

and $y_\alpha, z_\alpha \in C(\mathbf{R}_+ \times \mathbf{R}_0 \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_0 \times \mathbf{R}_+, \mathbf{R}_+)$.

(4)

$$\begin{cases} p_{1,\alpha} = p_1(A_1, A_2, B_1, B_2, C_1, C_2) = \inf_{(z,y) \in \mathbf{R}_0 \times \mathbf{R}_0} \frac{B_1 + C_1 y}{A_2 + z} = 0 \in \mathbf{R}_0, \\ p_{2,\alpha} = p_2(A_1, A_2, B_1, B_2, C_1, C_2) = \inf_{(y,z) \in \mathbf{R}_0 \times \mathbf{R}_0} \frac{B_2 + C_2 z}{A_1 + y} = 0 \in \mathbf{R}_0. \end{cases} \tag{46}$$

$h(p_{2,\alpha}, y) = (B_1 + C_1 y)/A_2 = y$ has only one solution $q_{1,\alpha} = q_1(A_1, A_2, B_1, B_2, C_1, C_2) = B_1/(A_2 - C_1)$ in interval $(p_{1,\alpha}, +\infty)$, and $g(p_{1,\alpha}, z) = (B_2 + C_2 z)/A_1 = z$ has only one solution $q_{1,\alpha} = q_1(A_1, A_2, B_1, B_2, C_1, C_2) = B_2/(A_1 - C_2)$ in interval $(p_{2,\alpha}, +\infty)$.

(5) Let $u = u_\alpha \leq U_\alpha = U$ and $v = v_\alpha \leq V_\alpha = V$ be a solution of the system

$$\begin{cases} U = h(v, U) = \frac{B_1 + C_1 U}{A_2 + v}, \\ u = h(V, u) = \frac{B_1 + C_1 u}{A_2 + V}, \\ V = g(u, V) = \frac{B_2 + C_2 V}{A_1 + u}, \\ v = g(U, v) = \frac{B_2 + C_2 v}{A_1 + U}. \end{cases} \tag{47}$$

Then,

$$\begin{cases} U(A_2 + v) = B_1 + C_1 U, \ \ v(A_1 + U) = B_2 + C_2 v, \\ V(A_1 + u) = B_2 + C_2 V, \ \ u(A_2 + V) = B_1 + C_1 u. \end{cases} \tag{48}$$

From the above, we have

$$\begin{cases} UA_2 - vA_1 = B_1 + C_1 U - B_2 - C_2 v, \\ VA_1 - uA_2 = B_2 + C_2 V - B_1 - C_1 u. \end{cases} \tag{49}$$

Thus,

$$0 \leq (U-u)(A_2 - C_1) = (v - V)(A_1 - C_2) \leq 0. \tag{50}$$

Since $0 < C_1 \leq C_2 < A_1 \leq A_2$, we obtain $U = u$ and $V = v$. It follows from Theorem 1 that all positive solutions of (42) converge to its unique positive equilibrium as $n \longrightarrow \infty$.

□

Remark 1. *When $k = 1$ and $m = 2$, the fuzzy difference Equation (42) is the fuzzy difference Equation (4) investigated in [25].*

Example 2. *Let*

$$x_n = F(x_{n-m}, A, B, C, x_{n-k}) = A + \frac{B + Cx_{n-k}}{x_{n-m}}, \quad n \in \mathbf{N}_0, \tag{51}$$

where $m, k \in \mathbf{N}$, $A, C \in \mathcal{F}^+(\mathbf{R}_+)$ with $\max(\mathbf{supp}C) < \min(\mathbf{supp}A)$, and $B \in \mathcal{F}(\mathbf{R}_0)$, and the initial values $x_i \in \mathcal{F}^+(\mathbf{R}_+)$ $(i \in [-d, -1]_{\mathbf{Z}})$ with $d = \max\{m, k\}$. Then, all positive solutions of (51) converge to its unique positive equilibrium as $n \longrightarrow \infty$.

Proof. For the convenience, for any $\alpha \in (0, 1]$, we write $A_\alpha = [A_{1,\alpha}, A_{2,\alpha}] = [A_1, A_2], B_\alpha = [B_{1,\alpha}, B_{2,\alpha}] = [B_1, B_2]$ and $C_\alpha = [C_{1,\alpha}, C_{2,\alpha}] = [C_1, C_2]$. Let

$$f(x, u, v, w, y) = u + \frac{v + wy}{x}. \tag{52}$$

In the following, we verify that (H_1)–(H_6) hold.

(1) $f \in C(\mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_0 \times \mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}_+)$ is strictly decreasing on x and strictly increasing on every $v \in \{u, v, w, y\}$.

(2) For every $x \in \mathbf{R}_+$, $f(x, u, v, w, y)/y$ is decreasing on y in $\mathbf{R}_+$.

(3) System of equations

$$\begin{cases} y_\alpha = A_1 + \frac{B_1 + C_1 y_\alpha}{z_\alpha}, \\ z_\alpha = A_2 + \frac{B_2 + C_2 z_\alpha}{y_\alpha} \end{cases} \tag{53}$$

has a unique positive solution

$$\begin{cases} y_\alpha = y(A_1, B_1, C_1, A_2, B_2, C_2) \\ = \frac{B_1 - B_2 + A_1 A_2 - C_1 C_2 + \sqrt{(B_2 - B_1 + B_1 B_2 - A_1 A_2)^2 + 4(A_2 - C_1)(B_2 A_1 - B_1 C_2)}}{2(A_2 - C_1)}, \\ z_\alpha = z(A_1, B_1, C_1, A_2, B_2, C_2) \\ = \frac{B_2 - B_1 + A_1 A_2 - C_1 C_2 + \sqrt{(B_2 - B_1 + B_1 B_2 - A_1 A_2)^2 + 4(A_2 - C_1)(B_2 A_1 - B_1 C_2)}}{2(A_1 - B_2)} \end{cases} \tag{54}$$

and $y_\alpha, z_\alpha \in C(\mathbf{R}_+ \times \mathbf{R}_0 \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_0 \times \mathbf{R}_+, \mathbf{R}_+)$.

(4)

$$\begin{cases} p_{1,\alpha} = p_1(A_1, A_2, B_1, B_2, C_1, C_2) = \inf_{(z,y) \in \mathbf{R}_+ \times \mathbf{R}_+} (A_1 + \frac{B_1 + C_1 y}{z}) = A_1 \in \mathbf{R}_+, \\ p_{2,\alpha} = p_2(A_1, A_2, B_1, B_2, C_1, C_2) = \inf_{(y,z) \in \mathbf{R}_+ \times \mathbf{R}_+} (A_2 + \frac{B_2 + C_2 z}{y}) = A_2 \in \mathbf{R}_+. \end{cases} \tag{55}$$

$h(p_{2,\alpha}, y) = A_1 + (B_1 + C_1 y)/A_2 = y$ has only one solution $q_{1,\alpha} = q_1(A_1, A_2, B_1, B_2, C_1, C_2) = (A_1 A_2 + B_1)/(A_2 - C_1)$ in interval $(p_{1,\alpha}, +\infty)$, and $g(p_{1,\alpha}, z) = A_2 + (B_2 + C_2 z)/A_1 = z$ has only one solution $q_{1,\alpha} = q_1(A_1, A_2, B_1, B_2, C_1, C_2) = (A_1 A_2 + B_2)/(A_1 - C_2)$ in interval $(p_{2,\alpha}, +\infty)$.

(5) Let $u = u_\alpha \leq U_\alpha = U$ and $v = v_\alpha \leq V_\alpha = V$ be a solution of the system

$$\begin{cases} U = h(v,U) = A_1 + \frac{B_1+C_1U}{v}, \\ u = h(V,u) = A_1 + \frac{B_1+C_1u}{V}, \\ V = g(u,V) = A_2 + \frac{B_2+C_2V}{u}, \\ v = g(U,v) = A_2 + \frac{B_2+C_2v}{U}. \end{cases} \tag{56}$$

Then,

$$\begin{cases} Uv = A_1v + B_1 + C_1U, \quad vU = A_2U + B_2 + C_2v, \\ Vu = A_2u + B_2 + C_2V, \quad uV = A_1V + B_1 + C_1u. \end{cases} \tag{57}$$

From the above, we have

$$\begin{cases} A_1v + B_1 + C_1U = A_2U + B_2 + C_2v, \\ A_2u + B_2 + C_2V = A_1V + B_1 + C_1u. \end{cases} \tag{58}$$

Thus,

$$0 \leq (U-u)(A_2 - C_1) = (v - V)(A_1 - C_2) \leq 0. \tag{59}$$

Since $0 < C_1 \leq C_2 < A_1 \leq A_2$, we obtain $U = u$ and $V = v$. It follows from Theorem 1 that all positive solutions of (51) converge to its unique positive equilibrium as $n \longrightarrow \infty$.

□

Remark 2. *When $B = 0, C = 1$ and $k = 1$, the fuzzy difference Equation (51) is the fuzzy difference Equation (1.3) investigated in [24].*

5. Conclusions

In this study, we investigate the convergence behavior of the positive solutions of the higher order fuzzy difference Equation (3) and show that all positive solutions of (3) converge to its unique positive equilibrium as $n \longrightarrow \infty$ under appropriate assumptions. Finally, two examples are given to account for the applicability of the main result (Theorem 1) of this paper. In the future, we intend to investigate the existence, the boundedness and the asymptotic behavior of the more general fuzzy difference equation $x_n = F(A_1, \cdots, A_s, x_{n-m_1}, \cdots, x_{n-m_\lambda}, B_1, \cdots, B_t, x_{n-k_1}, \cdots, x_{n-k_\mu})$ under appropriate assumptions.

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