Positively Continuum-Wise Expansiveness for $C^1$ Differentiable Maps

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Received: 2 September 2019; Accepted: 14 October 2019; Published: 16 October 2019

Abstract: We show that if a differentiable map $f$ of a compact smooth Riemannian manifold $M$ is $C^1$ robustly positive continuum-wise expansive, then $f$ is expanding. Moreover, $C^1$-generically, if a differentiable map $f$ of a compact smooth Riemannian manifold $M$ is positively continuum-wise expansive, then $f$ is expanding.

Keywords: positively expansive; positively measure expansive; generic; positively continuum-wise expansive; expanding

MSC: 58C25; 37C20; 37D20

1. Introduction and Statements

Starting with Utz [1], expansive dynamical systems have been studied by researchers. Regarding this concept, many researchers suggest various expansivenesses (e.g., N-expansive [2], measure expansive [3] and continuum-wise expansive [4]). These concepts were used to show chaotic systems (see References [3,5–7]) and hyperbolic structures (see References [8–14]).

For chaoticity, Morales and Sirvent proved in Reference [3] that every Li-Yorke chaotic map in the interval or the unit circle are measure-expansive. Kato proved in Reference [7] that, if a homeomorphism $f$ of a compactum $X$ with $\dim X > 0$ is continuum-wise expansive and $Z$ is a chaotic continuum of $f$, then either $f$ or $f^{-1}$ is chaotic in the sense of Li and Yorke on almost all Cantor sets $C \subset Z$. Hertz [5,6] proved that if a homeomorphism $f$ of locally compact metric space $X$ or Polish continua $X$ is expansive or continuum-wise expansive then $f$ is sensitive dependent on the initial conditions.

For hyperbolicity, Mañé proved in Reference [12] that if a diffeomorphism $f$ of a compact smooth Riemannian manifold $M$ is robustly expansive then it is quasi-Anosov. Arbieto proved in Reference [8] that, $C^1$ generically, if a diffeomorphism $f$ of a compact smooth Riemannian manifold $M$ is expansive then it is Axiom A and has no cycles. Sakai proved in Reference [13] that, if a diffeomorphism $f$ of a compact smooth Riemannian manifold $M$ is robustly expansive then it is quasi-Anosov. Lee proved in Reference [9] that, $C^1$ generically, if a diffeomorphism $f$ of a compact smooth Riemannian manifold $M$ is continuum-wise expansive then it is Axiom A and has no cycles.

Through these results, we are interested in general concepts of expansiveness. Actively researching positive expansivities (positively expansive [15], positively measure-expansive [16,17]) is a motivation of this paper. In this paper, we study positively continuum-wise expansiveness, which is the generalized notion of positive expansiveness and positive measure expansiveness.

In this paper, we assume that $M$ is a compact smooth Riemannian manifold. A differentiable map $f : M \to M$ is positively expansive (write $f \in \mathcal{PE}$) if there exists a constant $\delta > 0$ such that for any $x, y \in M$, if $d(f^i(x), f^i(y)) \leq \delta \forall i \geq 0$ then $x = y$. From Reference [18], if a differentiable map $f \in \mathcal{PE}$ then $f$ is open and a local homeomorphism. For any $\delta > 0$, we define a dynamical $\delta$-ball for $x \in M$ such as \{ $y \in M : d(f^i(x), f^i(y)) \leq \delta \forall i \geq 0$ \}. Put $\Gamma^+_{\delta}(x) = \{ y \in M : d(f^i(x), f^i(y)) \leq \delta \forall i \geq 0 \}$.
Note that if a differentiable map \( f \in \mathcal{P}\mathcal{E} \), then \( \Gamma^+_f(x) = \{x\} \) for any \( x \in M \). Here \( \delta > 0 \) is called an expansive constant of \( f \).

Let us introduce a generalization of the positively expansive called the positively measure-expansive (see Reference [3]). Let \( \mathcal{M}(M) \) be the space of a Borel probability measure of \( M \). A measure \( \mu \in \mathcal{M}(M) \) is atomic if \( \mu(\{x\}) \neq 0 \), for some point \( x \in M \). Let \( \mathcal{A}(M) \) be the set of atomic measures of \( M \). Note that \( \mathcal{A}(M) \) is dense in \( \mathcal{M}(M) \). Let \( \mathcal{M}^*(M) = \mathcal{M}(M) \setminus \mathcal{A}(M) \). A differentiable map \( f : M \to M \) is positively measure-expansive (write \( f \in \mathcal{P}\mathcal{M}\mathcal{E} \)) if there exists a constant \( \delta > 0 \) such that \( \mu(\Gamma^+_f(x)) = 0 \) for any \( \mu \in \mathcal{M}^*(M) \), where \( \delta > 0 \) is called a measure expansive constant. In Reference [17], the authors found that there exists a differentiable map \( f : S^1 \to S^1 \) that is positively \( \mu \)-expansive for any \( \mu \in \mathcal{M}^*_f(S^1) \) but not positively expansive where \( \mathcal{M}^*_f(M) \) is the set of non-atomic invariant measures of \( M \).

Now, we introduce another generalization of the positive expansiveness, which is called positively continuum-wise expansiveness (see Reference [4]). We say that \( C \) is a continuum if it is compact and connected.

**Definition 1.** A differentiable map \( f \) is positively continuum-wise expansive (write \( f \in \mathcal{P}\mathcal{C}\mathcal{W}\mathcal{E} \)) if there is a constant \( \varepsilon > 0 \) such that if \( C \subset M \) is a non-trivial continuum, then there is \( n \geq 0 \) such that \( \text{diam} f^n(C) > \varepsilon \), where if \( C \) is a trivial, then \( C \) is a one point set.

Note that \( f \in \mathcal{P}\mathcal{C}\mathcal{W}\mathcal{E} \) if and only if \( f^n \in \mathcal{P}\mathcal{C}\mathcal{W}\mathcal{E} \ \forall n \geq 1 \). We say that \( f \) is countably expansive (write \( f \in \mathcal{C}\mathcal{E} \)) if there is a constant \( \delta > 0 \) such that for all \( x \in M \), \( \Gamma^+_f(x) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta \ \forall i \in \mathbb{Z}\} \) is countable. In Reference [19], the authors showed that if a homeomorphism \( f : M \to M \) is measure expansive then \( f \) is countably expansive. Moreover, the converse is true. Then, as in the proof of Theorem 2.1 in Reference [19], it is easy to show that \( f \) is positively countable-expansive if and only if \( f \) is positively measure expansive. In this paper, we consider the relationship between the positively measure-expansive and the positively continuum-wise expansive (see Lemma 1). We can know that if \( f \) is positively measure-expansive then it is not positively continuum-wise expansive because a continuum is not countable, in general.

**Definition 2.** A differentiable map \( f : M \to M \) is expanding if there exist constants \( C > 0 \) and \( \lambda > 1 \) such that

\[
\|D_x f^n(v)\| \geq C\lambda^n \|v\|,
\]

for any vector \( v \in T_x M(x \in M) \) and any \( n \geq 0 \).

Note that a positively measure-expansive differentiable map is not necessarily expanding. However, under the \( C^1 \) robust or \( C^1 \) generic condition, it is true.

A differentiable map \( f \) is \( C^1 \) robustly positive \( \Phi \) if there exists a \( C^1 \) neighborhood \( \mathcal{U}(f) \) of \( f \) such that for any \( g \in \mathcal{U}(f) \), \( g \) is positive \( \Psi \).

A point \( x \in M \) is a singular if \( D_x f : T_x M \to T_{f(x)} M \) is not injective. Denoted by \( S_f \) the set of singular points of \( f \).

Sakai proved in Reference [15] that if a differentiable map \( f \) is \( C^1 \) robustly positive expansive then \( S_f = \emptyset \) and it is an expanding map. Lee et al. [17] proved that if \( f \) is \( C^1 \) robustly positive measure-expansive, then \( S_f = \emptyset \) and it is expanding. Note that if a differentiable map \( f \) is expanding then it is expansive. According to these facts, we prove the following.

**Theorem A** If a differentiable map \( f : M \to M \) is \( C^1 \) robustly positive continuum-wise expansive (write \( f \in \mathcal{R}\mathcal{P}\mathcal{C}\mathcal{W}\mathcal{E} \)) then \( S_f = \emptyset \) and it is expanding.

Let \( D^1(M) \) be the set of differentiable maps \( f : M \to M \). Note that \( D^1(M) \) contains the set of diffeomorphisms \( \text{Diff}^1(M) \) on \( M \) and \( \text{Diff}^1(M) \) is open in \( D^1(M) \). We say that a subset
\( \mathcal{G} \subset D^1(M) \) is residual if it contains a countable intersection of open and dense subsets of \( D^1(M) \). Note that the countable intersection of residual subsets is a residual subset of \( D^1(M) \). A property “P” holds generically if there exists a residual subset \( \mathcal{G} \subset D^1(M) \) such that for any \( f \in \mathcal{G} \), \( f \) has the “P” property. Some times we write for \( C^1 \) generic \( f \in D^1(M) \) which means that there exists a residual set \( \mathcal{G} \subset D^1(M) \) such that for any \( f \in \mathcal{G} \). Arbieto [8] and Sakai [15] proved that, \( C^1 \) generically, a positively expansive map is expanding. Ahn et al. [16] proved that for a \( \lambda > 0 \) such that \( \lambda \) belongs to \( \Gamma_f^+ \), the “P”. Some times we write for \( \mathcal{G} \subset D^1(M) \) such that for any \( f \in \mathcal{G} \).

Theorem B For \( C^1 \) generic \( f \in D^1(M) \), if \( f \) is positively continuum-wise expansive then \( S_f = \emptyset \) and it is expanding.

2. The Proof of Theorem A

The following proof is similar to Lemma 2.2 in Reference [19].

Lemma 1. Let \( C \subset M \) be compact and connected. A differentiable map \( f \in PCWE \) if and only if there is a constant \( \delta > 0 \) such that for all \( x \in M \), if a continuum \( C \subset \Gamma_f^+ (x) \) then \( C \) is a trivial continuum set.

Proof. Let \( \delta > 0 \) be a continuum-wise expansive constant and \( C \) be compact and connected (that is, a continuum). Take \( c = \delta / 2 \). We assume that for any \( x \in M \), if \( C \subset \Gamma_f^+ (x) \) then \( \text{diam } f^n(C) \leq 2c \) for all \( n \geq 0 \). Since \( f \) is positively continuum-wise expansive, \( C \) should be a trivial continuum set. Thus, if \( f \in PCWE \), then for all \( x \in M \), if a continuum \( C \subset \Gamma_f^+ (x) \), then \( C \) is a trivial continuum set.

For the converse part, suppose that \( f \in PCWE \). Then, there is a constant \( c > 0 \) such that \( \text{diam } f^n(C) \leq c \) for all \( n \geq 0 \), where \( C \) is a continuum. Let \( x \in C \) be given. Since \( \text{diam } f^n(C) \leq c \), for all \( y \in C \) we have

\[
\text{d}(f^n(x), f^n(y)) \leq c \forall n \geq 0.
\]

Thus, we know \( y \in \Gamma_f(x) \). Since \( y \in C \) and \( y \) is arbitrary, we have \( C \subset \Gamma_f(x) \). Since a continuum \( C \subset \Gamma_f(x) \), we have that \( C \) is a trivial continuum set.

A periodic point \( p \in P(f) \) is hyperbolic if \( D_p f^{\pi(p)} : T_p M \to T_p M \) has no eigenvalue with a modulus equal to 0 or 1, where \( \pi(p) \) is the period of \( p \). Then, \( T_p M = E_{p}^{\sigma} \oplus E_{p}^{\mu} \) of subspaces such that

\[
\begin{align*}
(1) & \quad D_p f^{\pi(p)}(E_{p}^{\sigma}) = E_{p}^{\sigma} (\sigma = s, u), \\
(2) & \quad \text{there exist constants } C > 0, \text{ and } \lambda \in (0, 1) \text{ satisfies for all positive integer } n \in \mathbb{N},
\end{align*}
\]

- \( \| D_p f^n(v) \| \leq C \lambda^n \| v \| \text{ for any } v \in E_{p}^{\sigma} \text{ and} \)
- \( \| D_p f^{-n}(v) \| \leq C \lambda^n \| v \| \text{ for any } v \in E_{p}^{\mu} \)

A hyperbolic point \( p \in P(f) \) is a sink if \( E_{p}^{\sigma} = \{0\} \), a source if \( E_{p}^{\mu} = \{0\} \), and a saddle if \( E_{p}^{\sigma} \neq \{0\} \) and \( E_{p}^{\mu} \neq \{0\} \). Let \( P_i(f) \) be the set of hyperbolic periodic points of \( f \). The dimension of the stable manifold \( W^s(p) = \{ x \in M : d(f^n(x), f^n(p)) \to 0 \text{ as } n \to \infty \} \) is written by the index of \( p_i \) and denoted by \( \text{ind}(p) \). Then, we know \( 0 \leq \text{ind}(p) \leq \text{dim } M \). Let \( P_i(f) \) be the set of all \( p \in P_i(f) \) with \( \text{ind}(p) = i \).

Lemma 2. If a differentiable map \( f \in PCWE \) then \( P_i(f) = \emptyset \) for \( 1 \leq i \leq \text{dim } M \).

Proof. By contradiction, we assume that there is \( i \in [1, \text{dim } M] \) such that \( P_i(f) \neq \emptyset \). Take \( p \in P_i(f) \) and \( \delta > 0 \). Then, we can find a local stable manifold \( W_i^s(p) \) of \( p \) such that \( W_i^s(p) \neq \emptyset \). We can construct a continuum \( J_p \subset W_i^s(p) \) centered at \( p \) such that \( \text{diam } J_p = \delta/4 \). Let \( \Gamma_{i/2}^+ \) be \( \{ y \in M : \)
If a differentiable map \( f : S^1 \to S^1 \) such that \( S_f \neq \emptyset \). Thus, if \( f \) is positively measure-expansive then \( S_f \neq \emptyset \). But if \( f \) is \( C^1 \) robustly positive measure-expansive then \( S_f = \emptyset \). For that, we consider that \( f \) is \( C^1 \) robustly positive continuum-wise expansive.

The following is a version of differentiable maps of Franks’ lemma (see Lemma 2.1 in Reference [8]).

**Lemma 3** ([20]). Let \( f : M \to M \) be a differentiable map and let \( U(f) \) be a \( C^1 \) neighborhood of \( f \). Then, there exists \( \delta > 0 \) such that for a finite set \( A = \{ x_1, x_2, \ldots, x_n \} \subset M \), a neighborhood \( U \) of \( A \) and a linear map \( L_i : T_{x_i}M \to T_{f(x_i)}M \) satisfying \( \| L_i - D_{x_i}f \| < \delta \) for \( 1 \leq i \leq n \), there exist \( \epsilon_0 > 0 \) and \( g \in U(f) \) having the following properties:

(a) \( g(x) = f(x) \) if \( x \in A \), and
(b) \( g(x) = \exp_{f(x)} \circ L_i \circ \exp_{x_i}^{-1}(x) \) if \( x \in B_{\epsilon_0}(x_i) \) and \( \forall i \in \{1, \ldots, n\} \).

It is clear that assertion (b) implies that

\[ g(x) = f(x) \quad \text{if} \quad x \in A \]

and that \( D_{x_i}g = L_i, \forall i \in \{1, \ldots, n\} \).

**Theorem 1.** If a differentiable map \( f \in \mathcal{R}PCWE \) then \( S_f = \emptyset \).

**Proof.** Suppose that there is \( x \in S_f \). Then, by Lemma 3, we can take \( g \) \( C^1 \) close to \( f \) such that \( g \) has a closed connected small arc \( B_\epsilon(x) \) centered at \( x \) with radius \( \epsilon > 0 \), such that \( \dim B_\epsilon(x) = 1 \) and \( g(B_\epsilon(x)) \) is one point. Take \( \delta = 2\epsilon \). Let \( \Gamma_\delta^+ (x) = \{ y : d(g^i(x), g^i(y)) \leq \delta \ \forall i \geq 0 \} \). It is clear \( B_\epsilon(x) \subset \Gamma_\delta^+ (x) \).

Since \( g(B_\epsilon(x)) \) is one point, for any \( y \in B_\epsilon(x) \), we know that \( \dim g^i(B_\epsilon(x)) \leq \delta \) for all \( i \geq 0 \). However, \( B_\epsilon(x) \) is not a trivial continuum set, by Lemma 1 this is a contradiction. \( \square \)

Recall that a differentiable map \( f : M \to M \) is star if every periodic point of \( g(C^1 \text{ nearby } f) \) is hyperbolic.

**Lemma 4.** If a differentiable map \( f \in \mathcal{R}PCWE \) then \( f \) is star.

**Proof.** Suppose that \( f \) is not star. Then, we can take \( g \) \( C^1 \) close to \( f \) such that \( g \) has a non-hyperbolic \( p \in P(g) \). As Lemma 3, we can find \( g_1 \) \( C^1 \) close to \( g \) \((g_1 \text{ close to } f)\) such that \( D_p g_1 \) has an eigenvalue \( \lambda \) with \(|\lambda| = 1 \). For simplicity, we assume that \( g_1^n(p) = p \). Let \( E^c_p \) be associated with \( \lambda \). If \( \lambda \in \mathbb{R} \) then \( \dim E^c_p = 1 \), and if \( \lambda \in \mathbb{C} \) then \( \dim E^c_p = 2 \).

First, we consider \( \dim E^c_p = 1 \). Then, we assume that \( \lambda = 1 \) (the other case can be proved similarly). By Lemma 3, there are \( \epsilon > 0 \) and \( h \) \( C^1 \) close to \( g_1 \) (also, \( C^1 \) close to \( f \)), having the following properties:

- \( h(p) = g_1(p) = p \),
- \( h(x) = \exp_p \circ D_p g_1 \circ \exp_p^{-1}(x) \) if \( x \in B_\epsilon(p) \), and
- \( h(x) = g_1(x) \) if \( x \notin B_\epsilon(p) \).

Since \( \lambda = 1 \), we can construct a closed connected small arc \( I_p \subset B_\epsilon(p) \cap \exp_p(E^c_p(\epsilon)) \) with its center at \( p \) such that

- \( \dim I_p = \epsilon/4 \),
- \( h(I_p) = I_p \), and
- the map \( h|_{I_p} : I_p \to I_p \) which is the identity.
Take $\delta = \epsilon/2$. Let $\Gamma^+_\delta(p) = \{x \in M : d(h^i(x), h^i(p)) \leq \delta \forall i \geq 0\}$. Then, it is clear $\mathcal{I}_p \subset \Gamma^+_\delta(p)$, and $\text{diam} h^i(\mathcal{I}_p) = \text{diam} \mathcal{I}_p$ for all $i \geq 0$. Since $f \in \text{RPCWE}$, according to Lemma 1, $\mathcal{I}_p$ has to be just a trivial continuum set. This is a contradiction since $\mathcal{I}_p$ is not a trivial continuum set.

Finally, we consider $\dim E^c_p = 2$. For convenience, we assume that $g^{\pi(p)}(p) = g(p) = p$. As Lemma 3, we can find $\|i\| > 0$ and $g_1 \in \mathcal{U}(f)$, which has the following properties:

- $g_1(p) = g(p) = p$,
- $g_1(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x)$ if $x \in B_c(p)$, and
- $g_1(x) = g(x)$ if $x \notin B_c(p)$.

For any $v \in E^c_p(e)$, there is $l > 0$ such that $D_p g^l(v) = v$. Take $u \in E^c_p(e)$ such that $\|u\| = \epsilon/2$. As in the previous arguments, we can construct a closed connected small arc $\mathcal{J}_p \subset B_c(p) \cap \exp_u(E^c_p(e))$ such that

- $\text{diam} \mathcal{J}_p = \epsilon/4$,
- $g_1^l(\mathcal{J}_p) = \mathcal{J}_p$, and
- $g_1^l|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map.

As in the proof of the first case, take $\delta = \epsilon/2$. Let $\Gamma^+_\delta(p) = \{x \in M : d(g_1^l(x), g_1^l(p)) \leq \delta \forall i \geq 0\}$. It is clear that $\mathcal{J}_p \subset \Gamma^+_\delta(p)$. Then, by Lemma 1, $\mathcal{J}_p$ must be a trivial continuum set but it is not possible since $\mathcal{J}_p$ is a closed connected small arc. Thus, if $f \in \text{RPCWE}$ then $f$ is star.

The differentiable maps $f, g : M \to M$ are conjugate if there is a homeomorphism $h : M \to M$ such that $f \circ h = h \circ g$. We say that a differentiable map $f$ is structurally stable if there is a $C^1$ neighborhood $\mathcal{U}(f)$ of $f \in D^1(M)$ such that for any $g \in \mathcal{U}(f)$, $g$ is conjugate to $f$. A differentiable map $f$ is $\Omega$ stable if there is a $C^1$ neighborhood $\mathcal{U}(f)$ of $f \in D^1(M)$ such that for any $g \in \mathcal{U}(f)$, $g|_{\Omega(f)}$ is conjugate to $f|_{\Omega(f)}$, where $\Omega(f)$ denotes the nonwandering points of $f$. Przytycki proved in Reference [21] that if $f$ is an Anosov differentiable map then it is not an Anosov diffeomorphism or expanding which are not structurally stable. Moreover, assume that $f$ is Axiom A (i.e., $\overline{\mathcal{P}(f)} = \Omega(f)$ is hyperbolic) and has no singular points in the nonwandering set $\Omega(f)$. Then $f$ is $\Omega$ stable if and only if $f$ is strong Axiom A and has no cycles (see Reference [22]). Here, $f$ is strong Axiom A means that $f$ is Axiom A and $\Omega(f)$ is the disjoint union $\Lambda_1 \cup \Lambda_2$ of two closed $f$ invariant sets.

According to the above results of a diffeomorphism $f \in \text{Diff}^1(M)$, one can consider the case of a differentiable $f \in D^1(M)$ which is an extension of a diffeomorphism. For instance, a diffeomorphism $f \in \text{Diff}(M)$ is said to be star if we can choose a $C^1$ neighborhood $\mathcal{U}(f)$ of $f$ such that every periodic point of $g$ is hyperbolic, for all $g \in \mathcal{U}(f)$.

If a diffeomorphism $f$ is star then $f$ is Axiom A and has no cycles (see References [23,24]). Aoki et al. Theorem A in Reference [25] proved that if a differentiable map $f$ is star and the nonwandering set $\Omega(f) \cap S_f \subset \{p \in P(f) : p$ is a sink $\}$ then $f$ is Axiom A and has no cycles.

**Theorem 2.** Let $f \in D^1(M)$. If $f \in \text{RPCWE}$ then $f$ is Axiom A and has no cycles.

**Proof.** Suppose that $f \in \text{RPCWE}$. As Lemma 4, $f$ is star. By Theorem 1, we know $S_f = \emptyset$, and so, $\Omega(f) \cap S_f = \emptyset$. By Lemma 2, there do not exist sinks in $P(f)$, that is, $\{p \in P(f) : p$ is a sink $\} = \emptyset$. Thus, by Theorem A in Reference [25], $f$ is Axiom A and has no cycles.

**Proof of Theorem A.** Suppose that $f \in \text{RPCWE}$. Then, by Lemma 2, Theorem 2 and Proposition 2.7 in [17], $\Omega(f) = \widehat{P_0(\mathcal{E})}$ is hyperbolic and $\widehat{P_0(\mathcal{E})}$ is expanding. Then, by Lemma 2.8 in Reference [17], $M = \widehat{P_0(\mathcal{E})}$. Thus, $f$ is expanding.

**3. The Proof of Theorem B**

Denote by $\mathcal{K}\mathcal{S}$ the set of Kupka–Smale $C^1$ maps of $M$. By Shub [26], $\mathcal{K}\mathcal{S}$ is a residual set of $D^1(M)$. If $f \in \mathcal{K}\mathcal{S}$ then every $p \in P(f)$ is hyperbolic. Then, we can see the following.
Lemma 5. Let \( f \in K.S \). If \( f \in \mathcal{PCWE} \) then \( P(f) = P_0(f) \).

Proof. Let \( f \in \mathcal{PCWE} \). Suppose, by contradiction, that \( P(f) \neq \emptyset \) for some \( 1 \leq i \leq \dim M \). Take \( p \in P_i(f) \) and \( \delta > 0 \). Then, we can define a local stable manifold \( W^s_p(f) \) of \( p \) such that \( W^s_p(f) \neq \emptyset \). We can construct a closed connected small arc \( \mathcal{J}_p \subset W^s_p(f) \) with its center at \( p \) such that \( \text{diam} \mathcal{J}_p = \delta / 4 \). Let \( \Gamma^s_\delta(p) = \{ x \in M : d(f^n(x), f^n(p)) \leq \delta \text{ for all } n \geq 0 \} \). Then, it is clear \( \mathcal{J}_p \subset \Gamma^s_\delta(p) \). Since \( f \in \mathcal{PCWE} \), by Lemma 1, \( \mathcal{J}_p \) must be a trivial continuum set. This is a contradiction since \( \mathcal{J}_p \) is not a trivial continuum set. Thus, every \( p \in P(f) \) is a source so that \( P(f) = P_0(f) \). 

Lemma 6. Lemma 8 in [15]. There exists a residual subset \( G_1 \subset D^1(M) \) such that for a given \( f \in G_1 \), if for any \( C^1 \) neighborhood \( U(f) \) of \( f \) there exist \( g \in U(f) \) and \( p \in P_h(g) \) with \( \text{ind}(p) = i(0 \leq i \leq \dim M) \), then there is \( p' \in P_h(f) \) with \( \text{ind}(p') = i \).

Lemma 7. There exists a residual subset \( G_2 \subset D^1(M) \) such that for a given \( f \in G_2 \), if \( f \in \mathcal{PCWE} \) then \( S_f \cap \overline{P_0(f)} = \emptyset \).

Proof. Let \( f \in G_2 = K.S \cap G_1 \) and \( f \in \mathcal{PCWE} \). Suppose, by contradiction, that \( S_f \cap \overline{P_0(f)} \neq \emptyset \). Since \( S_f \cap \overline{P_0(f)} \neq \emptyset \), we can choose a point \( x \in S_f \cap \overline{P_0(f)} \). Then, we can find a sequence of periodic points \( \{ p_n \} \subset P_0(f) \) with period \( \pi(p_n) \) such that \( p_n \to x \) as \( n \to \infty \). As Lemma 3, there exists \( g \in C^1 \) close to \( f \) such that \( g^{\pi(p_n)}(p_n) = p_n \) and \( p_n \in S_\delta \). Again using Lemma 3, there exists \( g_1 \in C^1 \) closed to \( g \) such that \( g_1 \in C^1 \) is close to \( f \), \( g_1^{\pi(p_n)}(p_n) = p_n \), and \( \pi(p_n) = i(1 \leq i \leq \dim M) \). Since \( f \in G_1 \), by Lemma 6, \( f \) has a hyperbolic saddle periodic point \( q \) with \( \text{ind}(q) = i(1 \leq i \leq \dim M) \). This is a contradiction by Lemma 2.

For a \( \delta > 0 \), a point \( p \in P(f)(f^{\pi(p)}(p) = p) \) said to be a \( \delta \)-hyperbolic (see Reference [27]) if for an eigenvalue of \( DF^{\pi(p)}(p) \), we can take an eigenvalue \( \lambda \) of \( DF^{\pi(p)}(p) \) such that

\[
(1 - \delta)^{\pi(p)} < |\lambda| < (1 + \delta)^{\pi(p)}.
\]

Lemma 8. There exists a residual subset \( G_3 \subset D^1(M) \) such that for a given \( f \in G_3 \), if \( f \in \mathcal{PCWE} \), then we can take \( \delta > 0 \) such that \( f \) has no \( \delta \)-hyperbolic.

Proof. Let \( f \in G_3 = K.S \cap G_1 \cap G_2 \), and let \( f \in \mathcal{PCWE} \). Since \( f \in K.S \cap G_1 \cap G_2 \), by Lemma 2 and Lemma 7, we know \( S_f \cap \overline{P_0(f)} = \emptyset \). Assume that for any \( \delta > 0 \), there is a \( p \in P_h(f) \) with a \( \delta \)-hyperbolic. By Lemma 3, we can take \( g \in C^1 \) close to \( f \) such that \( p \) has an eigenvalue with modulus one. Again using Lemma 3, there exists \( g_1 \in C^1 \) close to \( g \) such that \( g_1 \) has a saddle \( q \in P_h(g_1) \) with \( \text{ind}(q) = i(1 \leq i \leq \dim M) \), where \( P_h(g_1) \) is the set of all hyperbolic periodic points of \( g_1 \). Since \( f \in G_1 \), \( f \) has a saddle \( q' \in P_h(f) \) with \( \text{ind}(q') = i(1 \leq i \leq \dim M) \). This is a contradiction by Lemma 2.

Lemma 9. Lemma 7 in Reference [15]. There exists a residual subset \( G_4 \subset D^1(M) \) such that for a given \( f \in G_4 \) and \( \delta > 0 \), if any \( C^1 \) neighborhood \( U(f) \) of \( f \) there exist \( g \in U(f) \) and \( p \in P_h(g) \) with a \( \delta \)-hyperbolic, then we can find \( p' \in P_h(f) \) with a \( 2\delta \)-hyperbolic.

Lemma 10. There exists a residual subset \( G_5 \subset D^1(M) \) such that for a given \( f \in G_5 \), if \( f \in \mathcal{PCWE} \) then \( f \) is star.

Proof. Let \( f \in G_5 = G_3 \cap G_4 \) and \( f \in \mathcal{PCWE} \). Suppose that \( f \) is not star. Then, as Lemma 3, we can take \( g \in C^1 \) close to \( f \) such that \( g \) has a \( q \in P_h(g) \) with a \( \delta / 2 \)-hyperbolic for some \( \delta > 0 \). Since \( f \in G_4 \), \( f \) has a hyperbolic periodic point \( p' \) with a \( \delta \)-hyperbolic. This is a contradiction by Lemma 8.
\( \Omega(f) = \overline{P(f)} \).

**Proof of Theorem B.** Let \( f \in \mathcal{G} = \mathcal{G}_S \cap \mathcal{CL} \) and \( f \in \mathcal{PCWE} \). It is enough to show that \( M = \overline{P_0(f)} \).

By Lemmas 5 and 7, \( P(f) = P_0(f) \) and \( S_f \cap P_0(f) = \emptyset \). Since \( f \in \mathcal{CL} \), \( \Omega(f) = P(f) \). According to Lemma 10, \( f \) is star, and so \( \{ \Omega(f) \setminus P(f) \} \cap S_f = \emptyset \). Thus we have \( \Omega(f) = \overline{P(f)} = \overline{P_0(f)} \) is hyperbolic. As Proposition 2.7 in Reference [17], we have that \( P_0(f) \) is expanding. Then, as in the proof of Lemma 3.8 in Reference [17], we have \( M = \overline{P_0(f)} \). \( \square \)

**Funding:** This work is supported by the National Research Foundation of Korea (NRF) of the Korea government (MSIP) (No. NRF-2017R1A2B4001892).

**Acknowledgments:** The author would like to thank the referee for valuable help in improving the presentation of this article.

**Conflicts of Interest:** The author declares no conflict of interest.

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